NULL 2-TYPE CHEN SURFACES

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1. Introduction. Let M be an n-dimensional connected submanifold in an m-dimensional Euclidean space E^m. Denote by Δ the Laplacian of M associated with the induced metric. Then the position vector x and the mean curvature vector H of M in E^m satisfy

$$\Delta x = -nH. \tag{1.1}$$

This yields the following fact: a submanifold M in E^m is minimal if and only if all coordinate functions of E^m, restricted to M, are harmonic functions. In other words, minimal submanifolds in E^m are constructed from eigenfunctions of Δ with one eigenvalue 0. By using (1.1), T. Takahashi proved that minimal submanifolds of a hypersphere of E^m are constructed from eigenfunctions of Δ with one eigenvalue λ ($\neq 0$). In [**3, 4**], Chen initiated the study of submanifolds in E^m which are constructed from harmonic functions and eigenfunctions of Δ with a nonzero eigenvalue. The position vector x of such a submanifold admits the following simple spectral decomposition:

$$x = x_0 + x_q, \quad \text{with} \quad \Delta x_0 = 0 \quad \text{and} \quad \Delta x_q = \lambda x_q \tag{1.2}$$

for some non-constant maps x_0 and x_q, where λ is a nonzero constant. He simply calls such a submanifold a submanifold of null 2-type.

Chen has proved in [**3, 4**] that the only null 2-type surfaces in E^3 are open portions of circular cylinders and the only null 2-type surfaces in E^4 with constant mean curvature are open portions of helical cylinders, by which we mean the product surfaces of a straight line and a circular helix. In this paper we study Chen surfaces of null 2-type and obtain the following result.

THEOREM. *Let M be a (connected) non pseudo-umbilical Chen surface in E^m with constant mean curvature. If M is of null 2-type, then M is flat and lies fully in an affine subspace E^3, E^4, E^5 or E^6 of E^m.*

Since circular cylinders in E^3 and helical cylinders in E^4 all are Chen surfaces, the theorem is a generalization of Chen's results (for the definition of Chen surfaces please see [**2, 7**] or Section 2). Moreover, some examples will be given in Section 4 for null 2-type Chen surfaces fully in E^5 and E^6.

2. Preliminaries. Let M be an n-dimensional submanifold in an m-dimensional Euclidean space E^m. We denote by h, A, H, ∇ and D the second fundamental form, the Weingarten map, the mean curvature vector, the Riemannian connection and the normal connection of M in E^m. We choose an orthonormal local frame $\{e_1, \ldots, e_m\}$ on M such that $e_1, \ldots, e_n$ are tangent to M and e_{n+1} is in the direction of H. Denote by $\{\omega^1, \ldots, \omega^m\}$ the dual frame and ω_B^A $(A, B = 1, 2, \ldots, m)$ the connection forms associated with

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$\{e_1, \ldots, e_m\}$. Then we have the following useful Chen's formula (cf. [2, p.271]):

$$\Delta H = \Delta^D H + \sum_{i=1}^{n} h(e_i, A_H e_i) + 2\,\mathrm{tr}(A_{DH}) + \tfrac{1}{2} n \nabla\alpha^2, \tag{2.1}$$

where $\Delta^D H$ is the Laplacian of H with respect to the normal connection D, $\alpha = |H|$ the mean curvature of M in E^m and $\nabla\alpha^2$ the gradient of α^2. Moreover, for the terms in (2.1) we have

$$\sum_{i=1}^{n} h(e_i, A_H e_i) = \|A_{n+1}\|^2 H + a(H), \tag{2.2}$$

$$\mathrm{tr}(A_{DH}) = \sum_{i=1}^{n} A_{D_i H} e_i = A_{n+1} \nabla\alpha + \alpha \sum_{r=n+2}^{m} A_r(\omega^r_{n+1})_{\#}, \tag{2.3}$$

where we put $A_r = A_{e_r}$, $D_i H = D_{e_i} H$, $\|A_{n+1}\|^2 = \mathrm{tr}\, A_{n+1} A_{n+1}$, $(\omega^r_{n+1})_{\#} = \sum_{i=1}^{n} \omega^r_{n+1}(e_i) e_i$ and $a(H) = \sum_{r=n+2}^{m} \mathrm{tr}(A_{n+1} A_r) e_r$. A submanifold M in E^m is called a Chen submanifold if $a(H) = 0$ identically. Minimal submanifolds, pseudo-umbilical submanifolds and hypersurfaces are trivial examples of Chen submanifolds. Non-trivial examples can be found in [7]. For Chen surfaces of null 2-type, we have the following result.

LEMMA 1. *Let M be a Chen surface in E^m with constant mean curvature such that M is not of 1-type. Them M is of null 2-type if and only if the following hold*:

$$\sum_{r=4}^{m} A_r(\omega^r_3)_{\#} = 0; \tag{2.4}$$

$$\mathrm{tr}(D\omega^r_3) = \langle De_3, De_r \rangle, \qquad r = 4, \ldots, m; \tag{2.5}$$

$$\|A_3\|^2 + \langle De_3, De_3 \rangle = c, \quad \textit{for some nonzero constant } c. \tag{2.6}$$

Proof. Let M be a surface in E^m of null 2-type. From (1.1) and (1.2), we have

$$\Delta H = \lambda H, \quad \text{for some nonzero constant } \lambda. \tag{2.7}$$

On the other hand, if M is a Chen surface in E^m with constant mean curvature, then (2.1) becomes

$$\Delta H = \Delta^D H + \|A_3\|^2 H + 2\alpha \sum_{r=4}^{m} A_r(\omega^r_3)_{\#}, \tag{2.8}$$

where $\Delta^D H = \langle De_3, De_3 \rangle H + \sum_{r=4}^{m} \alpha\{\langle De_3, De_r \rangle - \mathrm{tr}(\nabla\omega^r_3)\} e_r$. Combining (2.7) and (2.8), we obtain (2.4)–(2.6) immediately. The converse is clear. □

We also need the following, which is a straightforward generalization of Chen's result in [3].

LEMMA 2. [8]. *A surface M in E^m with parallel normalized mean curvature vector is of null 2-type if and only if M is an open portion of a circular cylinder.*

Here the normalized mean curvature vector means the unit vector field in the direction of the mean curvature vector.

3. Proof of the theorem. In this section, we shall prove the theorem by the methods used in [**1**] and [**6**].

Let M be a Chen surface of E^m. We choose $\{e_1, e_2\}$ which diagonalizes A_H and $H=\alpha e_3$. Suppose that M is not pseudo-umbilical, i.e., the Weingarten map A_H is not proportional to the identity map. Then we have $h_{11}^4=\ldots=h_{11}^m=0$. If the normalized mean curvature vector e_3 is not parallel, i.e., $De_3 \neq 0$, since M is 2-dimensional, we may assume that De_3 lies in the normal subspace spanned by e_4 and e_5. Then by a suitable choice of $e_1, \ldots, e_m$ we have

$$A_3=\begin{pmatrix}\beta & 0\\ 0 & \gamma\end{pmatrix},\qquad A_4=\begin{pmatrix}0 & \delta\\ \delta & 0\end{pmatrix},\qquad A_5=\begin{pmatrix}0 & \varepsilon\\ \varepsilon & 0\end{pmatrix},\qquad A_6=\begin{pmatrix}0 & \zeta\\ \zeta & 0\end{pmatrix},$$
$$A_7=\ldots=A_m=0,\qquad De_3=\omega_3^4e_4+\omega_3^5e_5. \tag{3.1}$$

If M is of null 2-type and the mean curvature α of M is constant, then by Lemma 1 we have (2.4), which implies, with the help of (3.1),

$$\delta\omega_3^4(e_i)+\varepsilon\omega_3^5(e_i)=0,\qquad i=1,2. \tag{3.2}$$

This means that

$$\langle h(e_1,e_2),\xi_i\rangle=0,\qquad i=1,2, \tag{3.3}$$

where we put $\xi_i=\omega_3^4(e_i)e_4+\omega_3^5(e_i)e_5$, $i=1,2$. Consequently, if $\xi_1\wedge\xi_2=0$, then $D_1e_3\wedge D_2e_3=0$, and we may let $De_3=\omega_3^5e_5$ and choose e_4 so that $h(e_1,e_2)=\delta e_4$. If $\xi_1\wedge\xi_2\neq 0$, then $D_1e_3\wedge D_2e_3\neq 0$, and we may let $h(e_1,e_2)=\zeta e_6$. Summarising, we have the following.

Case (1)

$$A_3=\begin{pmatrix}\beta & 0\\ 0 & \gamma\end{pmatrix},\qquad A_4=\begin{pmatrix}0 & \delta\\ \delta & 0\end{pmatrix},\qquad A_5=\ldots=A_m=0,\qquad De_3=\omega_3^5e_5; \tag{3.4}$$

Case (2)

$$A_3=\begin{pmatrix}\beta & 0\\ 0 & \gamma\end{pmatrix},\qquad A_6=\begin{pmatrix}0 & \zeta\\ \zeta & 0\end{pmatrix},$$
$$A_4=A_5=A_7=\ldots=A_m=0,\qquad De_3=\omega_3^4e_4+\omega_3^5e_5. \tag{3.5}$$

But Case (1) can be regarded as Case (2) with $\omega_3^4=0$. Thus we have the following lemma.

LEMMA 3. *Let M be a non pseudo-umbilical Chen surface of E^m with constant mean curvature. If M is of null 2-type and the normalized mean curvature vector is not parallel, then on M, with respect to a suitable frame field, we have* (3.5).

For convenience in the following we will investigate Case (1) and (2) separately.

LEMMA 4. *Under the hypothesis of Lemma 3, if Case (1) holds, then M is flat and lies fully in an affine surface E^4 or E^5 of E^m.*

Proof. Suppose that Case (1) holds. Then we have

$$\omega_1^3=\beta\omega^1,\qquad \omega_2^3=\gamma\omega^2,\qquad \omega_1^4=\delta\omega^2,\qquad \omega_2^4=\delta\omega^1,$$
$$\omega_i^r=0,\quad \text{for}\quad i=1,2,\qquad r=5,\ldots,m. \tag{3.6}$$
$$De_3=\omega_3^5e_5,\qquad \omega_3^r=0,\qquad r=4,6,\ldots,m.$$

Differentiating $\omega_i^5 = 0$, we have

$$0 = d\omega_i^5 = -\omega_3^5 \wedge \omega_i^3 - \omega_4^5 \wedge \omega_i^4, \qquad i = 1, 2. \tag{3.7}$$

If $\delta = 0$, (3.7) implies $K = \beta\gamma = 0$, i.e., M is flat, since $De_3 \neq 0$. Further by taking exterior differentiation of $\omega_3^r = 0$, $r = 4, 6, \ldots, m$, we have

$$0 = d\omega_3^r = -\omega_5^r \wedge \omega_3^5, \qquad r = 4, 6, \ldots, m. \tag{3.8}$$

On the other hand, we have from (2.5)

$$0 = \mathrm{tr}(\nabla\omega_3^r) = \langle De_3, De_r \rangle = \sum_{i=1}^{2} \omega_3^5(e_i)\omega_r^5(e_i), \qquad r = 4, 6, \ldots, m. \tag{3.9}$$

Combining (3.8) and (3.9), we find that $\omega_5^r = 0$ for $r = 4, 6, \ldots, m$. Since we have known in (3.6) that $\omega_3^r = 0$ for $r = 4, 6, \ldots, m$, the normal subspace ν spanned by $\{e_3, e_5\}$ is parallel with respect to the normal connection D. Thus by a reduction theorem of submanifolds [**5**], we may conclude that in fact M lies in an affine subspace E^4 of E^m, since the first normal space is just spanned by $\{e_3\}$ and contained in ν. Moreover, from the connection form (ω_B^A), $A, B = 1, 2, 3, 5$ of M in E^m, we may also conclude that M lies fully in E^4.

If $\delta \neq 0$, then (3.7) implies

$$\omega_4^5(e_1) = \frac{\beta}{\delta}\omega_3^5(e_2), \qquad \omega_4^5(e_2) = \frac{\gamma}{\delta}\omega_3^5(e_1). \tag{3.10}$$

Since $\omega_3^4 = 0$, (2.5) gives

$$0 = \mathrm{tr}(\nabla\omega_3^4) = \langle De_3, De_4 \rangle = \omega_3^5(e_1)\omega_4^5(e_1) + \omega_3^5(e_2)\omega_4^5(e_2). \tag{3.11}$$

Substituting (3.10) into (3.11), we find that $\omega_3^5(e_1)\omega_3^5(e_2) = 0$. Without loss of generality, we may choose that $\omega_3^5(e_2) = 0$ and $\omega_3^5(e_1) \neq 0$ since $De_3 \neq 0$. Then the exterior differentiation of $\omega_3^4 = 0$ gives

$$\omega_4^5(e_2)\omega_3^5(e_1) = (\gamma - \beta)\delta, \tag{3.12}$$

and (2.6) becomes

$$(\omega_3^5(e_1))^2 = c - \beta^2 - \gamma^2. \tag{3.13}$$

Combining (3.12) and (3.13), we obtain, with the help of (3.10),

$$(\gamma - \beta)\delta^2 = (c - \beta^2 - \gamma^2)\gamma. \tag{3.14}$$

Furthermore, taking differentiation of $\omega_1^3 = \beta\omega^1$, $\omega_2^3 = \gamma\omega^2$, $\omega_1^4 = \delta\omega^2$ and $\omega_2^4 = \delta\omega^1$, we have

$$\begin{aligned} \omega_1^2 \wedge \omega^1 &= \frac{1}{\gamma - \beta} dr \wedge \omega^2 = \frac{1}{2\delta} d\delta \wedge \omega^2, \\ \omega_1^2 \wedge \omega^2 &= \frac{1}{\gamma - \beta} d\beta \wedge \omega^1 = -\frac{1}{2\delta} d\delta \wedge \omega^1, \end{aligned} \tag{3.15}$$

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and since M is non pseudo-umbilical and α is constant, we have $\gamma - \beta \neq 0$ and $d\beta = -d\gamma$, and thus from (3.15) we may deduce

$$\frac{d(\gamma-\beta)}{\gamma-\beta} = \frac{d\delta}{\delta}, \tag{3.16}$$

which implies

$$\gamma - \beta = k_1 \delta, \tag{3.17}$$

where k_1 is some constant. Now from (3.14), (3.17) and the fact $\beta + \gamma = 2\alpha = \text{constant}$, we may conclude that β, γ and δ are all constant. Consequently, it follows from (3.15) that $\omega_1^2 = 0$. Thus M is flat.

Differentiating $\omega_i^r = 0$, for $i = 1, 2$, $r = 6, \ldots, m$, we find, with the help of (3.6), $\omega_4^r = 0$, $r = 6, \ldots, m$. Applying (2.5) and (3.8) to ω_3^r, $r = 6, \ldots, m$, we may obtain $\omega_5^r = 0$, $r = 6, \ldots, m$. Since in (3.6) we have already known $\omega_3^r = 0$, $r = 6, \ldots, m$, we may conclude that the normal subspace ν spanned by $\{e_3, e_4, e_5\}$ is parallel. Then similar to the case of $\delta = 0$, we may obtain that M lies fully in a E^5 of E^m. The proof of the lemma is completed. □

LEMMA 5. *Under the hypothesis of Lemma* 3, *if Case* (2) *holds*, *then M is flat and lies fully in an affine subspace E^4, E^5 or E^6 of E^m.*

Proof. If Case (2) holds, we have

$$\begin{gathered}\omega_1^3 = \beta\omega^1, \qquad \omega_2^3 = \gamma\omega^2, \qquad \omega_1^6 = \zeta\omega^2, \qquad \omega_2^6 = \zeta\omega^1, \\ \omega_i^r = 0, \qquad r = 4, 5, 7, \ldots, m, \\ De_3 = \omega_3^4 e_4 + \omega_3^5 e_5, \qquad \omega_3^r = 0, \qquad r = 6, \ldots, m.\end{gathered} \tag{3.18}$$

Differentiating $\omega_i^4 = 0$ and $\omega_i^5 = 0$, $i = 1, 2$, we have

$$\beta\omega_3^r \wedge \omega^1 + \zeta\omega_6^r \wedge \omega^2 = 0, \qquad \gamma\omega_3^r \wedge \omega^2 + \zeta\omega_6^r \wedge \omega^1 = 0, \quad \text{for} \quad r = 4, 5. \tag{3.19}$$

If $\zeta = 0$, (3.19) implies that $\beta\gamma = 0$, since $De_3 \neq 0$. Thus by taking notice of $\beta + \gamma = 2\alpha = \text{constant}$, we find that β and γ are both constant. Then differentiating $\omega_1^3 = \beta\omega^1$ and $\omega_2^3 = \gamma\omega^2$, we get that $\omega_1^2 = 0$. Thus M is flat. Moreover, without loss of generality we may choose $\gamma \neq 0$ and $\beta = 0$; then (3.19) implies that $D_1 e_3 = 0$ and $D_2 e_3 = \omega_3^4(e_2)e_4 + \omega_3^5(e_2)e_5$. After rechoosing the normal frame field we have $De_3 = \omega_3^5 e_5$ and the situation turns into Case (1) with $\delta = 0$. Thus M lies fully in a E^4 of E^m.

If $\zeta \neq 0$, (3.19) gives

$$\omega_6^r(e_1) = \frac{\beta}{\zeta}\omega_3^r(e_2), \qquad \omega_6^r(e_2) = \frac{\gamma}{\zeta}\omega_3^r(e_1), \qquad r = 4, 5. \tag{3.20}$$

Since $\omega_3^6 = 0$, we have from (2.5)

$$0 = \operatorname{tr}(\nabla\omega_3^6) = \langle De_3, De_6 \rangle = \sum_{i=1}^{2} \omega_3^4(e_i)\omega_6^4(e_i) + \sum_{i=1}^{2} \omega_3^5(e_i)\omega_6^5(e_i). \tag{3.21}$$

Substituting (3.20) into (3.21), we obtain

$$\omega_3^4(e_1)\omega_3^4(e_2) + \omega_3^5(e_1)\omega_3^5(e_2) = 0. \tag{3.22}$$

It follows that if we choose e_4 such that $D_1e_3 = \omega_3^4(e_1)e_4$, then $D_2e_3 = \omega_3^5(e_2)e_5$. If one of $\omega_3^4(e_1)$ and $\omega_3^5(e_2)$ vanishes, then the situation turns into Case (1) with $\delta \neq 0$, and M is flat and lies fully in a E^5 of E^m.

Now suppose that $\omega_3^4(e_1)\omega_3^5(e_2) \neq 0$. Differentiating $\omega_1^3 = \beta\omega^1$, $\omega_2^3 = \gamma\omega^3$, $\omega_1^6 = \zeta\omega^2$ and $\omega_2^6 = \zeta\omega^1$, and comparing the results, we may obtain, with the help of $d\beta = -d\gamma$,

$$\gamma - \beta = k_2\zeta, \tag{3.23}$$

where k_2 is some constant. Applying (2.5) to $\omega_3^4 = \omega_3^4(e_1)\omega^1$ and $\omega_3^5 = \omega_3^5(e_2)\omega^2$, we have

$$\begin{aligned} &e_1(\omega_3^4(e_1)) - \omega_3^4(e_1)\omega_2^1(e_2) - \omega_3^5(e_2)\omega_4^5(e_2) = 0, \\ &e_2(\omega_3^5(e_2)) - \omega_3^5(e_2)\omega_1^2(e_1) - \omega_3^4(e_1)\omega_5^4(e_1) = 0. \end{aligned} \tag{3.24}$$

On the other hand, differentiating them, we have

$$\begin{aligned} &e_2(\omega_3^4(e_1)) - \omega_3^4(e_1)\omega_1^2(e_1) = \omega_3^5(e_2)\omega_5^4(e_1), \\ &e_1(\omega_3^5(e_2)) + \omega_3^5(e_2)\omega_1^2(e_2) = \omega_3^4(e_1)\omega_4^5(e_2). \end{aligned} \tag{3.25}$$

Then (3.24) and (3.25) give, with the help of differentiating $\omega_1^6 = \zeta\omega^2$ and $\omega_2^6 = \zeta\omega^1$,

$$e_i\{[(\omega_3^4(e_1))^2 - (\omega_3^5(e_2))^2]/\zeta\} = 0, \qquad i = 1, 2, \tag{3.26}$$

which implies

$$(\omega_3^4(e_1))^2 - (\omega_3^5(e_2))^2 = k_3\zeta, \tag{3.27}$$

where k_3 is some constant. Differentiating $\omega_3^6 = 0$, we have

$$\beta(\omega_3^5(e_2))^2 - \gamma(\omega_3^4(e_1))^2 = \zeta^2(\gamma - \beta). \tag{3.28}$$

Moreover, from (2.6) we have

$$(\omega_3^4(e_1))^2 + (\omega_3^5(e_2))^2 = c - \beta^2 - \gamma^2. \tag{3.29}$$

Putting (3.23) and (3.27)–(3.29) together and taking notice of $\beta + \gamma = 2\alpha =$ constant, we may conclude that $\beta, \gamma, \zeta, \omega_3^4(e_1)$ and $\omega_3^5(e_2)$ are all constant from which we may deduce that $\omega_1^2 = 0$ as in the proof of Lemma 4. Thus M is flat.

Consequently, applying (2.5) to ω_3^4 and ω_3^5, we have

$$\omega_3^5(e_2)\omega_4^5(e_2) = 0, \qquad \omega_3^4(e_1)\omega_5^4(e_1) = 0. \tag{3.30}$$

Since $\omega_3^4(e_1)\omega_3^5(e_2) \neq 0$, (3.30) implies

$$\omega_4^5 = 0. \tag{3.31}$$

Differentiating $\omega_i^r = 0$, $i = 1, 2$, $r = 7, \ldots, m$, we have

$$\omega_6^r = 0, \qquad r = 7, \ldots, m. \tag{3.32}$$

Then differentiating (3.31) and using (3.20), we have

$$\gamma\omega_3^4(e_1)\omega_4^r(e_1) - \beta\omega_3^5(e_2)\omega_5^r(e_2) = 0, \qquad r = 7, \ldots, m; \tag{3.33}$$

again from (2.5) we have

$$\begin{aligned} 0 = \operatorname{tr}(\nabla\omega_3^r) &= \langle De_3, De_r\rangle \\ &= \omega_3^4(e_1)\omega_r^4(e_1) + \omega_3^5(e_2)\omega_r^5(e_2), \quad r = 6, \ldots, m. \end{aligned} \tag{3.34}$$

(3.33) and (3.34) give

$$\omega_4^r(e_1) = 0, \qquad \omega_5^r(e_2) = 0, \qquad r = 7, \ldots, m. \tag{3.35}$$

Thus from (3.20), (3.21) and (3.35) we have

$$D_1e_4 = \omega_4^3(e_1)e_3, \qquad D_2e_4 = \sum_{r=6}^{m} \omega_4^r(e_2)e_r. \tag{3.36}$$

In (3.36) we see that D_2e_4 has no component in Span$\{e_3, e_4, e_5\}$ and we may choose e_7 in such a way that

$$D_2e_4 = \omega_4^6(e_2)e_6 + \omega_4^7(e_2)e_7. \tag{3.37}$$

It follows that

$$\omega_4^r = 0, \qquad r = 8, \ldots, m. \tag{3.38}$$

On differentiating $\omega_3^r = 0$, $r = 7, \ldots, m$, we have

$$0 = d\omega_3^r = [\omega_4^r(e_2)\omega_3^4(e_1) - \omega_5^r(e_1)\omega_3^5(e_2)]\omega_1 \wedge \omega_2, \qquad (r = 7, \ldots, m). \tag{3.39}$$

In particular, we have, with the help of (3.38),

$$\omega_5^r(e_1) = 0, \qquad r = 8, \ldots, m, \tag{3.40}$$

which with (3.35) gives

$$\omega_5^r = 0, \qquad r = 8, \ldots, m. \tag{3.41}$$

Differentiating $\omega_4^7 = \omega_4^7(e_2)\omega^2$ and $\omega_5^7 = \omega_5^7(e_1)\omega^1$, we find

$$e_1(\omega_7^4(e_2)) = 0, \qquad e_2(\omega_5^7(e_1)) = 0. \tag{3.42}$$

On the other hand, (3.39) gives for $r = 7$

$$\omega_4^7(e_2)\omega_3^4(e_1) - \omega_5^7(e_1)\omega_3^5(e_2) = 0, \tag{3.43}$$

from which we may deduce, by differentiation,

$$e_2(\omega_4^7(e_2)) = 0, \qquad e_1(\omega_5^7(e_1)) = 0. \tag{3.44}$$

Hence we obtain that $\omega_4^7(e_2)$ and $\omega_5^7(e_1)$ are both constant, and (3.43) implies that they either both vanish or both do not. Furthermore, differentiating (3.31), we have, with the help of (3.18), (3.20) and (3.38),

$$\begin{aligned} 0 = d\omega_4^5 &= -\omega_3^5 \wedge \omega_4^3 - \omega_6^5 \wedge \omega_4^6 - \omega_7^5 \wedge \omega_4^7 \\ &= \{\omega_3^5(e_2)\omega_4^3(e_1) - \omega_6^5(e_1)\omega_4^6(e_2) - \omega_7^5(e_1)\omega_4^7(e_2)\}\omega^1 \wedge \omega^2 \\ &= \{[(\delta^2 - \beta\gamma)/\delta^2]\omega_3^5(e_2)\omega_4^3(e_1) - \omega_7^5(e_1)\omega_4^7(e_2)\}\omega^1 \wedge \omega^2. \end{aligned} \tag{3.45}$$

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Then since we know that M is flat, which means that $K=\beta\gamma-\delta^2=0$, we may deduce from (3.45) that $\omega_4^7(e_2)\omega_5^7(e_1)=0$. Consequently, both of $\omega_4^7(e_2)$ and $\omega_5^7(e_1)$ must vanish.

Finally, from (3.18), (3.31), (3.32), (3.35)–(3.37) and (3.41), we have

$$\begin{aligned} De_3&=\omega_3^4e_4+\omega_3^5e_5, \qquad De_4=\omega_4^3e_3+\omega_4^6e_6,\\ De_5&=\omega_5^3e_3+\omega_5^6e_6, \qquad De_6=\omega_6^4e_4+\omega_6^5e_5, \end{aligned} \tag{3.46}$$

which implies that $\mathrm{Span}\{e_3,\ldots,e_6\}$ is parallel. Then we may prove that M lies fully in a E^6 of E^m. This completes the proof of the lemma. □

Proof of the theorem. Combining Lemmas 2–5, we obtain the theorem immediately. □

4. Examples. In this section we will give some examples of surfaces for the theorem; each one is flat and fully in E^3, E^4, E^5 or E^6.

(a) Circular cylinders. Chen **[3]** proved that circular cylinders are the only null 2-type surfaces in E^3. It is clear that they are non pseudo-umbilical Chen surface in E^3 with constant mean curvature. Thus circular cylinders are also the only examples for the theorem in E^3.

(b) Helical cylinders. Chen **[4]** proved that helical cylinders are the only null 2-type surfaces in E^4 with constant mean curvature. For a helical cylinder M in E^4, by a suitable choice of the Euclidean coordinates, its equation takes the following form:

$$x(u,v)=(u,a\cos v,a\sin v,bv), \tag{4.1}$$

for some constants a and b. By direct computation, we may confirm that M is a non pseudo-umbilical Chen surface in E^4 with constant mean curvature which is flat and lies fully in E^4. Thus helical cylinders are also the only examples for the theorem in E^4.

(c) Let M be a surface in E^5 which takes the following form:

$$x(u,v)=(au,b\cos u\cos v,b\cos u\sin v,b\sin u\cos v,b\sin u\sin v) \tag{4.2}$$

for some constant a and b ($ab\neq0$). By direct computation, we see that the Laplacian Δ of M is given by

$$\Delta=\frac{1}{a^2+b^2}\frac{\partial^2}{\partial u^2}-\frac{1}{b^2}\frac{\partial^2}{\partial v^2}. \tag{4.3}$$

We put

$$\begin{aligned} x_0&=(au,0,0,0,0),\\ x_q&=(0,b\cos u\cos v,b\cos u\sin v,b\sin u\cos v,b\sin u\sin v). \end{aligned} \tag{4.4}$$

Then we have

$$\Delta x_0=0,\qquad \Delta x_q=\lambda x_q,\qquad \lambda=\frac{1}{a^2+b^2}+\frac{1}{b^2}. \tag{4.5}$$

This shows that M is of null 2-type. Furthermore, we may choose orthonormal frame field $\{e_1,\ldots,e_5\}$ on M in such a way that

$$e_1=x_u/\sqrt{a^2+b^2},\qquad e_2=x_v/b,\qquad e_3=x_{uu}/b,\qquad e_4=x_{uv}/b. \tag{4.6}$$

Then, by direct computation, we may obtain

$$A_3=\begin{pmatrix} b/(a^2+b^2) & 0 \\ 0 & 1/b \end{pmatrix}, \qquad A_4=\begin{pmatrix} 0 & 1/\sqrt{a^2+b^2} \\ 1/\sqrt{a^2+b^2} & 0 \end{pmatrix}, \qquad A_5=0. \tag{4.7}$$

From (4.7) it is easy to see that M is a non pseudo-umbilical Chen surface in E^5 with constant mean curvature which is flat and lies fully in E^5. Thus M can serve as an example for the theorem in E^5.

(d) Let M be a surface in E^6 which takes the following form:

$$x(u,v)=(au, cv, b\cos u\cos v, b\cos u\sin v, b\sin u\cos v, b\sin u\sin v), \tag{4.8}$$

for some constant a, b, and c ($abc\neq 0$). By direct computation, we see that the Laplacian Δ of M is given by

$$\Delta=-\frac{1}{a^2+b^2}\frac{\partial^2}{\partial u^2}-\frac{1}{c^2+b^2}\frac{\partial^2}{\partial v^2}. \tag{4.9}$$

We put

$$\begin{aligned} x_0&=(au, cv, 0,0,0,0),\\ x_q&=(0,0,b\cos u\cos v, b\cos u\sin v, b\operatorname{con} u\cos v, b\sin u\sin v). \end{aligned} \tag{4.10}$$

Then we have

$$\Delta x_0=0, \qquad \Delta x_q=\lambda x_q, \qquad \lambda=\frac{1}{a^2+b^2}+\frac{1}{c^2+b^2}. \tag{4.11}$$

This shows that M is of null 2-type. We choose an orthonormal frame field $\{e_1,\dots,e_6\}$ on M in such a way that

$$e_1=x_u/\sqrt{a^2+b^2}, \qquad e_2=x_v/\sqrt{c^2+b^2}, \qquad e_3=x_{uu}/b, \qquad e_6=x_{uv}/b. \tag{4.12}$$

Then, by direct computation, we obtain

$$\begin{aligned} A_3&=\begin{pmatrix} b/(a^2+b^2) & 0 \\ 0 & b/(c^2+b^2) \end{pmatrix}, \qquad A_4=A_5=0,\\ A_6&=\begin{pmatrix} 0 & b/\sqrt{(a^2+b^2)(c^2+b^2)} \\ b/\sqrt{(a^2+b^2)(c^2+b^2)} & 0 \end{pmatrix}. \end{aligned} \tag{4.13}$$

Now it is clear that if $a\neq c$, then M is a non pseudo-umbilical Chen surface in E^6 with constant mean curvature which is flat and lies fully in E^6. Thus if $a\neq c$, M is an example for the theorem in E^6.

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