DAVIS, C., GRÜNBAUM, B. and SHERK, F.A. (eds.) *The geometric vein:* the Coxeter Festchrift (Springer-Verlag, Berlin-Heidelberg-New York, 1982), 598 pp. DM 122.

Sing a song of sixpence,
A pocketful of rye;
Four and twenty blackbirds
Baked in a pie.

When the pie was opened
The birds began to sing!
Wasn't that a dainty dish
To set before the king?

There are one and forty essays in this handsomely-printed volume. Its three editors convened a symposium at Toronto in May 1979 to honour H. S. M. Coxeter whose close association with Toronto then spanned over 40 years. Special invitations were sent to fourteen mathematicians and contributions from all but one of them are in this book. If one may be so perverse as to open the review of a book by referring to something that is not in it perhaps it is permissible to record disappointment that the material of an exciting lecture involving an icosian algebra does not appear, and to be puzzled over possible explanations of this defection. But there is no other disappointment; rather a chorus of jubilation.

The members of the symposium covered, by their association with him, different stages of Coxeter's career and included fellow students from his youthful years at Cambridge. These were accorded a specially warm welcome in Toronto and have contributed to this book. They are still, as in the far-off days at Cambridge, concerned with projective geometry in higher space and will be intrigued to see that an essay on convex bodies culminates with a mention of the Veronese surface—though forbearing to disclose its name—and that the cubic generated by its chords, as also by its tangent planes, is called (for the first time?) its convex cover.

A reviewer of 41 essays covering 600 pages is forced to be selective, and can hardly do better than begin with the masterful essay chosen by the editors to open the proceedings. Its three authors' claim that the uniform tilings with hollow tiles are visually attractive is confirmed by their diagrams. There are three sets of 25, 26, 33 tilings and only seven regular polygons occur, five convex and two stellated; all vertices in any one tiling are to have congruent vertex figures, all of these being single finite polygons. In the first set all tiles must be regular polygons; in the second apeirogons are admitted—collinear non-overlapping congruent segments end to end, infinite in both directions, so functioning as regular polygons with an infinite number of sides; in the third zig-zags can also occur. The authors' attitudes as to the completeness of the three sets are: moral conviction for the 25, confident conjecture for the 26, honest doubt for the 33. The historical background is described from Kepler onwards, and further problems are suggested for investigation.

Any suggestion of randomness in selecting an essay from the 41 may be countered by choosing the longest; it covers pp. 379–442 and is on Inversive Geometry. Save for an algebraic treatment with $n=2$ on pp. 414–420 it is concerned throughout with inversion in an n-sphere Σ in Euclidean space $\mathbb{R}^{n+1}$ and follows Coxeter in using a unified definition of inversion and reflection. The bibliography lists 100 items; of these some 30 are decorated by an asterisk: this means that, in the author's opinion, they could have benefited, had the present essay been available, both by enlarging their meaning and simplifying their proofs.

The inversions in Σ generate the Möbius group M_n and the author linearises it, showing it to be a subgroup of the group of linear self-transformations of a certain bilinear form. He uses caps, bounded by $(n-1)$-spheres, on Σ and calls a cluster a set of $n+2$ caps any two of which are externally tangent: that there are such clusters is clear by taking any regular simplex to which Σ is circumscribed and centering caps at its vertices. It transpires that M_n is sharply transitive on clusters. (These ideas are used to advantage in another essay: pp. 243–250.) The conjugacy classes of M_1, M_2, M_3 are found without any algebraic manipulations.

The writer insists that inversive geometry can serve as a foundation for all three classical geometries in n-space—Euclidean, spherical, hyperbolic—and stakes his claim on pp. 420–427.

The whole essay is carefully planned and attractively presented, generously provided with diagrams at exactly the right places and built on sound scholarship. There are teasing glimpses of deeper waters such as Hopf fibration, and not merely glimpses but actual involvement in the argument of the contraction mapping theorem and Brouwer's fixed point theorem.

If, however, one wants inversion without tears one can sample it in the immediately preceding essay where the inversions are all in the plane though the circles and lines are now oriented. But while the inversions themselves are in the plane they are, following Coolidge, mapped by the points of a quadric in four-dimensional space. The treatment is less algebraic than with Coolidge and results are obtained that Coolidge did not notice. The first half of this essay is one of the most satisfying pieces of elementary geometry it has ever been my good fortune to encounter. The second half is less elementary, generalising what was itself a generalisation of inversion by Laguerre. After reading this essay it seemed natural to turn to the conspectus (pp. 253–269) on elementary geometry. It runs easily; and if the petulant outburst attributed to Dieudonné should induce the guilty feeling that enjoyment of inversion without tears was inordinate, even illicit, the scruple is easily dismissed. This same writer also contributes a piece (pp. 345–353) about inversion that covers not only Laguerre's extension but also certain of Lie's contact transformations.

It seemed fitting, as an aid to selecting the next essay for cogitation, to cast one's mind back to the Toronto gathering and thereby light on some lectures that seemed at the time to be markedly significant. Different people might choose differently, but my own thoughts soon homed on a lecture about cubature. The corresponding essay occupies pp. 203–218 and is a quite dazzling display of mathematical virtuosity. This question of averaging a polynomial over a unit sphere S by a linear combination of the values it assumes at a finite number of points of S, the combination giving the correct average for all polynomials of degree $\leqq t$ is, as Sobolev remarked twenty years ago, eased when the points compose the orbit of any one of them under a subgroup of the orthogonal group. This essay includes a description of a technique for increasing t by "killing" invariants of the group. Several examples are given, all fascinating, culminating with the 23-sphere and the Leech lattice.

This mention of groups is a sharp reminder that one has not yet alluded to any of the 80 pages devoted to groups and their presentations. They open with an essay on the generation of linear groups; this is a mine of information and gives an ample bibliography; stress is laid upon generation by homologies or elations. There is a tribute to H. H. Mitchell, who "solved problems which Jordan, Valentiner, Burnside and Dickson could not". The writer is at pains to emphasise, when listing the discoveries made by contributors to the theory, whether their proofs are, as with Mitchell, geometric or whether they rely on "very deep simple group classification theorems" or on representation theory, ordinary or modular. But he grants that group theory may discover results otherwise difficult to prove, so that these may, once known, be obtained in a more gentlemanly fashion or, as he phrases it, more elegantly. But should anyone prefer powerful group-theoretic arguments to geometry he will appreciate the essay immediately following. It was shown by Hurwitz that the order of the group of automorphisms of an algebraic curve, or Riemann surface, of genus g cannot exceed $84(g-1)$. That this bound is attained when $g=3$ was already known, and Fricke showed later that it was also attained when $g=7$. In this essay an infinite sequence of Hurwitz groups is found; for these the lowest value of g seems to be 687.

The next essay, an impressive account of a theory initiated with a view to treating exceptional Lie groups geometrically, is by Tits; it would indeed be a formidable task to do justice to this authoritative survey. In any event the review may be adjudged long enough already. But there are other items in the book just as important as those here recognised, and handled with no less expertise; so one hopes that there will be many readers eager to sample them. After all there is a classic precedent: little Jack Horner in the nursery rhyme pulled out a plum from his Christmas pie.

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