ERLANGIAN APPROXIMATIONS FOR FINITE-HORIZON RUIN PROBABILITIES

BY

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ABSTRACT

For the Cramér-Lundberg risk model with phase-type claims, it is shown that the probability of ruin before an independent phase-type time H coincides with the ruin probability in a certain Markovian fluid model and therefore has an matrix-exponential form. When H is exponential, this yields in particular a probabilistic interpretation of a recent result of Avram & Usabel. When H is Erlang, the matrix algebra takes a simple recursive form, and fixing the mean of H at T and letting the number of stages go to infinity yields a quick approximation procedure for the probability of ruin before time T. Numerical examples are given, including a combination with extrapolation.

KEYWORDS

Erlang expiration, extrapolation, finite time ruin probabilities, fluid model, phase-type distributions, semi-Markov embedding.

1. INTRODUCTION

Consider the reserve $R_t = u - S_t$ at time t of an insurance company where S is a Lévy process of the form

$$S_t = \sum_{k=1}^{N_t} Z_k - ct \tag{1}$$

$Z_1, Z_2, \ldots$ are i.i.d. random variables with common distribution F concentrated on $(0, \infty)$ and N is a Poisson processes with arrival rate λ ($N, Z_1, Z_2, \ldots$ are independent). We will assume w.l.o.g. (by scaling time) that $c = 1$.

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The time of ruin is

$$\tau = \tau(u) = \inf\{t > 0 : R_t < 0\} = \inf\{t > 0 : S_t > u\},$$

the infinite horizon ruin probability is $\psi(u) = \mathbf{P}(\tau(u) < \infty)$ and the probability of ruin before time T is

$$\psi(u, T) = \mathbf{P}(\tau(u) \le T)$$

(the probability and expectation are taken with respect to the process S_t).

Typically, the study of $\psi(u, T)$ is much harder than that of $\psi(u)$, and even $\psi(u)$ can only be found explicitly in a few cases, mainly when F is *phase-type* (see, e.g., [20], [21], [3], [17] or [7] for background and terminology). With α the initial row vector and $\boldsymbol{G}$ the phase generator, it then holds that

$$\psi(u) = \boldsymbol{\eta}\, e^{(\boldsymbol{G}+\boldsymbol{g}Bf\boldsymbol{\eta})u}\, \mathbf{1} \tag{2}$$

where $\boldsymbol{g} = -\boldsymbol{G}\mathbf{1}$ with $\mathbf{1} = (1 \ldots 1)'$ and $\boldsymbol{\eta} = -(\lambda/c)\alpha\boldsymbol{G}^{-1}$. This was first pointed out by Asmussen & Rolski [10] in the risk theoretic setting, but in fact formula (2) is identical to the representation of the M/PH/1 waiting time distribution given in Neuts [20]. In the finite horizon case, the phase-type assumption did so far not appear to provide any substantial simplication (but see Stanford & Stroinski [23] for an attempt). However, recently Avram & Usabel [11] found a formula of similar form as (2) for the probability of ruin before an exponential time H_a independent of the risk process: if $H = H_a$ is exponential with rate a and

$$\psi_a(u) = \psi(u, H_a) = \mathbf{P}(\tau(u) \le H_a) = \mathbf{E}\, e^{-a\tau(u)} \tag{3}$$

(where the last equality holds by integration by parts), then

$$\psi(u, H_a) = \boldsymbol{\eta}_a\, e^{(\boldsymbol{G}+\boldsymbol{g}\boldsymbol{\eta}_a)u}\, \mathbf{1} \tag{4}$$

where $\boldsymbol{\eta}_a = (\lambda/c)\,\alpha(s_a\boldsymbol{I} - \boldsymbol{G})^{-1}$ and s_a is the unique positive root of the Cramér Lundberg equation (12). Their result dealt more generally also with the deficit at ruin, which is in fact always automatic for phase type jumps – see Corollary 4. The approach was algebraic, with a key step in the proof being a linear algebra identity similar to that used in Asmussen & Bladt [8].

In the present paper we generalize the results of [11] to a random horizon H with a general phase-type distribution, using a different approach proposed in Asmussen [7]. This approach solves perpetual first time passage problems for Levy processes with phase-type jumps by embedding them first into an equivalent continuous Markov modulated diffusion with a finite order Markovian environment (also called **fluid model** in the absence of a Brownian component). This has the analytic advantage of replacing the original Markovian integro-differential generator by a simpler first order ordinary-differential generator for the fluid model (see [13] and the proof of Theorem 2 below), and reducing ultimately (at least for constant premiums) this type of problems to obtaining the Jordan decomposition of certain matrices. The end result is that formulas like (2), (4), are available under a wide variety of models, like for example that of phase-type renewal arrivals [13]. Moreover, as shown below, we may also incorporate a phase-type horizon H merely by increasing the dimension of the modulating environment (and of $\boldsymbol{\eta}$).

The case of an Erlang H is of particular interest. Such an H with a large number L of stages provides a very good approximation for its mean since, as shown by Aldous & Shepp [1], the Erlang distribution is the phase-type distribution with minimal variance for a given number of phases. Correspondingly, we find in Theorem 6 that if we fix the mean T and let L go to infinity, the corresponding $\psi(u, H)$ converges to $\psi(u, T)$ with a rate of convergence which is optimal. Furthermore, the Erlangian case turns out to be simpler computationally, since it leads to an explicit recursion for $\boldsymbol{U}$. We thereby provide a quick route to approximations of $\psi(u, T)$ (a topic which is in general not easy at all – see the survey in [7] Ch. IV). We illustrate the approach with a couple of numerical examples which show that indeed a good fit is obtained with relatively few stages L, and that the efficiency of the numerical scheme is much improved when combined with extrapolation.

2. The probability of ruin at a phase-type time

The connection between Levy processes with phase-type jumps (or, more generally, Markov modulated Levy processes with phase-type jumps) and Markov-modulated diffusions follows from an observation which seems to have been first exploited in Asmussen [5] (see also [6], [7]) and Asmussen & Kella [9]). The general trick is to level out the positive jumps to sample path segments with slope +1 and the negative jumps to sample path segments with slope –1 and add an extra phase say ζ (or more if necessary), for the "regular time" when the process drifts. This embeds the process with phase-type jumps S_t in a continuous Markov additive process (J, V), where the Markov component J_t (called environment) is ζ at a regular time and gives the current phase of the jump otherwise.

In this way, one obtains a semi-Markov (or Markov modulated) risk process of the kind studied in [2], but much simpler, in that there are no jumps, but only Markov modulated deterministic drifts (or "premiums"). This approach has technical advantages, like removing certain integrability difficulties arising due to the jumps (see Asmussen & Kella [9] Theorem 2.1.4); more importantly, it replaces the original problem involving an integro-differential equation by a related simpler problem involving a first order differential system for the resulting fluid model.

In our case of a phase-type horizon H, we have to keep track also of the phase of the horizon, and therefore the imbedding of S_t in a Markovian fluid $\{(J_t, V_t)\}$ is slightly more complicated than the one described before. Assume that the set of phases for the jump distribution F is E_F with p_F elements and that H is phase-type with set of phases E_H with p_H elements, phase generator $\boldsymbol{H}$

and initial vector $\boldsymbol{\beta}$. We denote by $\boldsymbol{h} = -\boldsymbol{H}\mathbf{1}$ the rates of absorbtion of the phase process into its absorption "cemetery" state C (note that $\mathbf{1}$ and similarly the identity matrix $\boldsymbol{I}$ and zero vectors/matrices $\mathbf{0}$ will have a dimension varying on the context throughout the paper). The background Markov process $\{J_t\}$ has state space $E = E_0 \cup E_+ \times E_-$ where $E_0 = \{C\}$ is the absorbing state, $E_- = E_H$, $E_+ = E_H \times E_F$ (matrices and vectors are written in block-partitioned form corresponding to this ordering, and we write $p_- = p_H$, $p_+ = p_H + p_F$). The fluid model $\{V_t\}$ moves linearly at a rate r_i when $J_t = i$; more precisely, it is defined as follows:

- $J_t = C$ corresponds to the final segment of the risk process after time H and here $r_C = 0$.
- $J_t \in E_- = E_F$ corresponds to a segment of the risk process S_t without jumps, i.e., downwards motion at the linear rate $r_i = -1$ and the state for $\{J_t\}$ is the current phase of the phase process for H.
- $J_t \in E_+ = E_H \times E_F$ corresponds to a jump of the risk process. The first component of $\{J_t\}$ is fixed at the current phase of the phase process for H during a segment corresponding to a fixed jump, and the second goes through the phases of the phase process for the jump. Further, $r_i = 1$ for $i \in E_H \times E_F$.

It follows that the initial distribution for $\{J_t\}$ is $(0\ \boldsymbol{\beta}\ \mathbf{0})$ and that its transitions intensity matrix is

$$\boldsymbol{\Lambda} = \begin{pmatrix} 0 & \mathbf{0} & \mathbf{0} \\ \boldsymbol{h} & \boldsymbol{H}-\lambda\boldsymbol{I} & \lambda\boldsymbol{I}\otimes\boldsymbol{\alpha} \\ \mathbf{0} & \boldsymbol{I}\otimes\boldsymbol{g} & \boldsymbol{I}\otimes\boldsymbol{G} \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{0} & \mathbf{0} \\ \boldsymbol{h} & \boldsymbol{\Lambda}^{--} & \boldsymbol{\Lambda}^{-+} \\ \mathbf{0} & \boldsymbol{\Lambda}^{+-} & \boldsymbol{\Lambda}^{++} \end{pmatrix}$$

(we used notation like $\boldsymbol{\Lambda}^{+-} = \boldsymbol{I}\otimes\boldsymbol{g},\ldots$; for Kronecker products $\otimes$, see e.g. Graham [16]. For example, if H is exponential with rate a, we have

$$\boldsymbol{\Lambda} = \begin{pmatrix} 0 & 0 & \mathbf{0} \\ a & -a-\lambda & \lambda\boldsymbol{\alpha} \\ \mathbf{0} & \boldsymbol{g} & \boldsymbol{G} \end{pmatrix},$$

and if H is Erlang(2) with rate a, then

$$\boldsymbol{\Lambda} = \left(\begin{array}{c|cc|cc} 0 & 0 & 0 & \mathbf{0} & \mathbf{0} \\ \hline 0 & -a-\lambda & a & \lambda\boldsymbol{\alpha} & \mathbf{0} \\ a & 0 & -a-\lambda & \mathbf{0} & \lambda\boldsymbol{\alpha} \\ \hline \mathbf{0} & \boldsymbol{g} & \mathbf{0} & \boldsymbol{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{g} & \mathbf{0} & \boldsymbol{G} \end{array}\right)$$

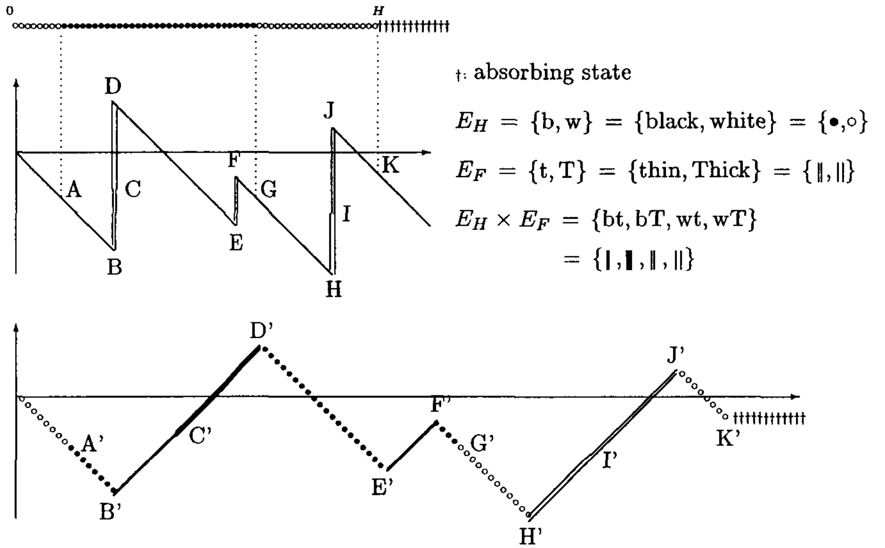

Figure 1

Fig. 1 illustrates the connection between S_t (upper part) and (J_t, V_t) (lower part). The following result follows immediately by a sample path comparison.

Theorem 1. *$\psi(u, H)$ coincides with the infinite horizon ruin probability for $\{V_t\}$ (i.e. with the probability that $\{V_t\}$ will ever exceed level u).*

We state now a general result concerning the upcrossing probabilities of fluid models due to [14] and [18] and generalized by [4] to the case of Markov-modulated diffusions, following Theorem 3.2 of [4]. Here, the special structure of the matrices Λ^{--} etc plays no role; we will see however in the next section that if H is Erlang, then a considerably simpler recursive scheme is available.

Theorem 2. *Let $\boldsymbol{\eta}$ denote the $p_- \times p_+$ matrix of "upcrossing phase probabilities" at the completion of a downwards excursion for a fluid model $\{V_t\}$, conditioned on starting in a decreasing state $i \in E_-$, let $\boldsymbol{\psi}(u) = (\psi_i(u), i = 1,,,,p_-)$ denote the infinite horizon phase-distribution at u (starting at 0), conditioned on starting in a decreasing state, and let $\boldsymbol{\phi}(u) = (\psi_i(u), i = 1,,,,p_+)$ denote the infinite horizon phase-distribution at u, conditioned on starting in an increasing state. Then,*

a) $\boldsymbol{\eta}$ satisfies the Riccati equation:

$$\boldsymbol{\eta} \boldsymbol{U} + \boldsymbol{\Lambda}^{--} \boldsymbol{\eta} + \boldsymbol{\Lambda}^{-+} = \boldsymbol{0} \tag{5}$$

and

$$\boldsymbol{\psi}(u) = \boldsymbol{\eta} \boldsymbol{\phi}(u) = \boldsymbol{\eta} e^{\boldsymbol{U}_u} \boldsymbol{1} \tag{6}$$

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where $\boldsymbol{U}$ *is the* $p_+ \times p_+$ *matrix*

$$\boldsymbol{U} = \boldsymbol{\Lambda}^{++} + \boldsymbol{\Lambda}^{+-}\,\boldsymbol{\eta} \tag{7}$$

b) Let now $\delta > 0$ *be a "convergence improving parameter", satisfying* $\delta > -h_{ii} + \lambda$ *for all* $i \in E_H$. *Then, the solution* $\boldsymbol{\eta}$ *of the Riccati equation* (5) *can be computed by the iteration scheme* $\boldsymbol{\eta} = \lim_{n\to\infty} \boldsymbol{\eta}^{(n)}$ *where* $\boldsymbol{\eta}^{(0)} = 0$,

$$\boldsymbol{\eta}^{(n+1)} = ((\delta \boldsymbol{I} + \boldsymbol{\Lambda}^{--})\boldsymbol{\eta}^{(n)} + \boldsymbol{\Lambda}^{-+})\,(\delta \boldsymbol{I} - (\boldsymbol{\Lambda}^{++} + \boldsymbol{\Lambda}^{+-}\boldsymbol{\eta}^{(n)})^{-1}$$

c) Alternatively, let $\boldsymbol{K}[s]$ *be the matrix of dimension* $p_- + p_+$ *given by*

$$\boldsymbol{K}[s] = \begin{pmatrix} \boldsymbol{\Lambda}^{--} - s\boldsymbol{I} & \boldsymbol{\Lambda}^{-+} \\ \boldsymbol{\Lambda}^{+-} & \boldsymbol{\Lambda}^{++} + s\boldsymbol{I} \end{pmatrix} = \begin{pmatrix} \boldsymbol{H} - \lambda\boldsymbol{I} - s\boldsymbol{I} & \lambda\boldsymbol{I}\otimes\boldsymbol{\alpha} \\ \boldsymbol{I}\otimes\boldsymbol{g} & \boldsymbol{I}\otimes\boldsymbol{G} + s\boldsymbol{I} \end{pmatrix}$$

and assume that the equation $\boldsymbol{K}[s] = 0$ *has* p_+ *distinct generalized eigenvalues* $s_1, \ldots, s_{p_+}$ *with* $\Re(s_i) < 0$ *and let* $\boldsymbol{k}_i = (\boldsymbol{k}_i^{(-)}\boldsymbol{k}_i^{(+)})$, $i = 1, \ldots, p_+$ *be right (generalized) eigenvectors of* $\boldsymbol{K}[s_i]$ *corresponding to the eigenvalue* s_i. *Then* $\boldsymbol{\eta}$ *is the solution of the linear equations*

$$\boldsymbol{\eta}\left(\boldsymbol{k}_1^{(+)} \ldots \boldsymbol{k}_{p_+}^{(+)}\right) = \left(\boldsymbol{k}_1^{(-)} \ldots \boldsymbol{k}_{p_+}^{(-)}\right), \tag{8}$$

and further

$$\boldsymbol{U} = \left(s_1 \boldsymbol{k}_1^{(+)} \ldots s_{p_+} \boldsymbol{k}_{p_+}^{(+)}\right) = \left(\boldsymbol{k}_1^{(+)} \ldots \boldsymbol{k}_{p_+}^{(+)}\right)^{-1}.$$

Proof. a,b) This is a reformulation of Theorem 3.2 of Asmussen [4], with the following small amendments. First, it is assumed there that $E_0 = \emptyset$, but the separate treatment in [4] of the case $E_0 \neq \emptyset$ is not needed here because E_0 is absorbing. Second, [4] gives

$$\boldsymbol{\eta} = ((\delta \boldsymbol{I} + \boldsymbol{\Lambda}^{--})\boldsymbol{\eta} + \boldsymbol{\Lambda}^{-+})\,(\delta \boldsymbol{I} - (\boldsymbol{\Lambda}^{++} + \boldsymbol{\Lambda}^{+-}\boldsymbol{\eta})^{-1}$$

from which (5) is obtained by multiplying by $\delta\boldsymbol{I} - \boldsymbol{U}$ to the right and eliminating δ.

c) The proof is an immediate application of Section 5 of [4]. Note that the fact that (8) satisfies the Riccati equation *for any choice of eigenvectors* $\boldsymbol{k}_i$, $i = 1, \ldots, p_+$ is easily checked, but the fact that we need the negative real part eigenvalues requires some probabilistic argument – see also [9] for an optimal stopping approach. QED

Notes: 1) Here is a brief outline of some of the ideas behind the proof of Theorem 2: Fixing the barrier at 0 and letting the starting point u vary, we find that $\boldsymbol{\psi}(u)$, $\boldsymbol{\phi}(u)$ satisfy the Feynman-Kac equation:

$$\left(\begin{array}{c|c} \boldsymbol{I} & \boldsymbol{0} \\ \hline \boldsymbol{0} & -\boldsymbol{I} \end{array}\right) \begin{pmatrix} \boldsymbol{\psi}'(u) \\ \boldsymbol{\phi}'(u) \end{pmatrix} + \left(\begin{array}{c|c} \boldsymbol{\Lambda}^{--} & \boldsymbol{\Lambda}^{-+} \\ \hline \boldsymbol{\Lambda}^{+-} & \boldsymbol{\Lambda}^{++} \end{array}\right) \begin{pmatrix} \boldsymbol{\psi}(u) \\ \boldsymbol{\phi}(u) \end{pmatrix} = 0, \quad \begin{pmatrix} \boldsymbol{\psi}(\infty) \\ \boldsymbol{\phi}(0) \end{pmatrix} = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{1} \end{pmatrix}$$

or

$$\psi'(u) = -\Lambda^{--}\psi(u) - \Lambda^{-+}\phi(u) \qquad \psi(\infty) = 0$$
$$\phi'(u) = \Lambda^{+-}\psi(u) + \Lambda^{++}\phi(u) \qquad \phi(0) = \mathbf{1}$$

Substituting now the (probabilistically obvious) first equality in (6) $\psi(u) = \boldsymbol{\eta}\phi(u)$ transforms the second equation to the homogeneous form:

$$\phi'(u) = \boldsymbol{U}\,\phi(u)$$

where $\boldsymbol{U} = \Lambda^{++} + \Lambda^{+-}\boldsymbol{\eta}$, and thus $\psi(u) = e^{\boldsymbol{U}u}\mathbf{1}$.

Also, this substitution transforms the first equation (after factoring $\phi(u)$) into the algebraic Riccati equation (5).

2) It is interesting to investigate, in the spirit of [8], whether this result holds also under the less restrictive assumption of jumps with a rational Laplace transform.

Finally, using the fact that in our case the horizon phases start with probabilities $\boldsymbol{\beta}$, we get:

Corollary 3. *One has* $\psi(u,H) = \boldsymbol{\beta\eta}e^{\boldsymbol{U}u}\mathbf{1}$ *where* $\boldsymbol{\eta}$ *is a* $E_- \times E_+$ *matrix and* $\boldsymbol{U}$ *a* $E_+ \times E_+$ *matrix such that*

$$\boldsymbol{U} = \boldsymbol{I} \otimes \boldsymbol{G} + (\boldsymbol{I} \otimes \boldsymbol{g})\boldsymbol{\eta}, \tag{9}$$
$$\mathbf{0} = \boldsymbol{\eta U} + \boldsymbol{H\eta} - \lambda\boldsymbol{\eta} + \lambda\boldsymbol{I} \otimes \alpha. \tag{10}$$

The deficit at the time of ruin $Y = S_\tau$ has also been studied by many authors, see e.g. [15]. An important advantage of phase-type modeling is that it renders the distribution of the deficit at ruin automatically. Indeed, it is easy to see from the probabilistic interpretation of Theorem 2 that this deficit is phase-type on E_+ with initial vector $\boldsymbol{\beta\eta}e^{\boldsymbol{U}u}$ and phase generator $\boldsymbol{I} \otimes \boldsymbol{G}$, which yields the corollary:

Corollary 4. *Let*

$$\psi(u, H, Y) = \mathbf{P}_u\{\tau < H;\, Y > y\} \tag{11}$$

denote the "finite time bivariate probability of ruin" with time span H and severity of ruin bigger than y. Then,

$$\psi(u, H, Y) = \boldsymbol{\beta\eta}e^{\boldsymbol{U}u}e^{\boldsymbol{G}y}\mathbf{1}$$

3. The Erlang case

We first verify that Theorem 2 gives the result of [11] for the exponential case. Here E_- consists of a single point so that $\boldsymbol{\eta}$ is a $E_+ = E_F$ row vector and $\boldsymbol{U}$ a $E_F \times E_F$ matrix, and (9), (10) take the form $\boldsymbol{U} = \boldsymbol{G} + \boldsymbol{g\eta}$,

$$\mathbf{0} = \boldsymbol{\eta U} - (a + \lambda)\boldsymbol{\eta} + \lambda\alpha = \boldsymbol{\eta G} + \boldsymbol{\eta g \eta} - (a + \lambda)\boldsymbol{\eta} + \lambda\alpha.$$

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Defining $s = a + \lambda - \boldsymbol{\eta g}$, this means $\mathbf{0} = \boldsymbol{\eta G} - s\boldsymbol{\eta} + \lambda\alpha$ so that $\boldsymbol{\eta} = \lambda\alpha(s\boldsymbol{I} - \boldsymbol{G})^{-1}$. It only remains to compute s but multiplying by $\boldsymbol{g}$ to the right and appealing to the standard formula for the m.g.f. $f^*(s)$ of a phase -type distribution F, we get

$$-a + s - \lambda + \lambda f^*(s) = \kappa(s) - a = 0 \tag{12}$$

where $\kappa(s) = \log \mathbf{E} e^{s(R_1 - u)} = s + \lambda(f^*(s) - 1)$ denotes the Levy exponent of the process R_t. We note now, as in [11], that since the Laplace transform of the ruin function must be well defined over the positive numbers, s must be the unique positive root of this equation[4].

Next consider the Erlang(2) case. The probabilistic interpretation of $\boldsymbol{\eta}$ given in [4] is that η_{ij} is the probability that when $J_0 = i \in E_-$, then the first upcrossing of level 0 of the fluid model will occur at a time t with $J_t = j \in E_+$. Taking into account the specific structure of the Erlang case, it follows that we can write

$$\boldsymbol{\eta} = \begin{pmatrix} \boldsymbol{\eta}_1 & \boldsymbol{\eta}_2 \\ 0 & \boldsymbol{\eta}_1 \end{pmatrix}$$

where $\boldsymbol{\eta}_1$ coincides with the $\boldsymbol{\eta} = \lambda\alpha(s\boldsymbol{I} - \boldsymbol{G})^{-1}$ just computed for the exponential case. By (9), (10)

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\eta}_1 & \boldsymbol{\eta}_2 \\ \mathbf{0} & \boldsymbol{\eta}_1 \end{pmatrix} \left[\begin{pmatrix} \boldsymbol{G} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{G} \end{pmatrix} + \begin{pmatrix} \boldsymbol{g} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{g} \end{pmatrix} \begin{pmatrix} \boldsymbol{\eta}_1 & \boldsymbol{\eta}_2 \\ \mathbf{0} & \boldsymbol{\eta}_1 \end{pmatrix} \right] +$$
$$\begin{pmatrix} -\lambda - a & a \\ 0 & -\lambda - a \end{pmatrix} \begin{pmatrix} \boldsymbol{\eta}_1 & \boldsymbol{\eta}_2 \\ \mathbf{0} & \boldsymbol{\eta}_1 \end{pmatrix} + \begin{pmatrix} \lambda\boldsymbol{\alpha} & \mathbf{0} \\ \mathbf{0} & \lambda\boldsymbol{\alpha} \end{pmatrix}.$$

Entry by entry (in lexicographical order) this means

$$\begin{aligned} 0 &= \boldsymbol{\eta}_1\boldsymbol{G} + \boldsymbol{\eta}_1\boldsymbol{g}\boldsymbol{\eta}_1 - (\lambda + a)\boldsymbol{\eta}_1 + \lambda\alpha \\ 0 &= \boldsymbol{\eta}_2\boldsymbol{G} + \boldsymbol{\eta}_1\boldsymbol{g}\boldsymbol{\eta}_2 + \boldsymbol{\eta}_2\boldsymbol{g}\boldsymbol{\eta}_1 - (\lambda + a)\boldsymbol{\eta}_2 + a\boldsymbol{\eta}_1 \\ 0 &= \mathbf{0} \\ 0 &= \boldsymbol{\eta}_1\boldsymbol{G} + \boldsymbol{\eta}_1\boldsymbol{g}\boldsymbol{\eta}_1 - (\lambda + a)\boldsymbol{\eta}_1 + \lambda\alpha \end{aligned}$$

It is seen that only the second equation contains information on $\boldsymbol{\eta}_2$, and recalling that $\boldsymbol{\eta}_1\boldsymbol{g} = \lambda + a - s$, it yields

$$\boldsymbol{\eta}_2 = a\boldsymbol{\eta}_1(s\boldsymbol{I} - \boldsymbol{G} - \boldsymbol{g}\boldsymbol{\eta}_1)^{-1}. \tag{13}$$

[4] To see that for $a > 0$ the equation $\kappa(\theta) - a = 0$ has always exactly one positive solution, note that since the Levy exponent κ is convex over $\mathbb{R}^+$, the equation may have at most 2 nonegative solutions. Furthemore, when $a = 0$, $\theta = 0$ is always a solution; let θ_0 denote the largest solution in this case. Note that $\theta_0 > 0$ if and only if $\kappa'(0) = \mathbf{E}(R_1) < 0$. Also, $\kappa(\theta) \leq 0$ for $\theta \in [0, \theta_0]$ while κ is increasing over the interval $[\theta_0, \infty)$; therefore, $\kappa(\theta)$ has a unique continuous inverse $\gamma(a) \geq \theta_0$ which is defined for $a \geq 0$; therefore, $\kappa(\theta) - a = 0$ has always a unique positive solution.

In the general case:

Theorem 5. *Assume that H is Erlang with L stages. Then* $\boldsymbol{\eta}$ *is an upper triangular block Toeplitz matrix, with entries* $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \ldots, \boldsymbol{\eta}_L$ *given by the recursion*

$$\boldsymbol{\eta}_L = \left(a\boldsymbol{\eta}_{L-1} + \sum_{j=2}^{L-1} \boldsymbol{\eta}_{L-j+1}\boldsymbol{g}\boldsymbol{\eta}_j\right)(s\boldsymbol{I} - \boldsymbol{G} - \boldsymbol{g}\boldsymbol{\eta}_1)^{-1} \tag{14}$$

and $\boldsymbol{\eta}_1 = \lambda\alpha(s\boldsymbol{I} - \boldsymbol{G})^{-1}$. *In particular,* $\boldsymbol{\eta}_2$ *is given by (13) and*

$$\begin{aligned}\boldsymbol{\eta}_3 &= (a\boldsymbol{\eta}_2 + \boldsymbol{\eta}_2\boldsymbol{g}\boldsymbol{\eta}_2)(s\boldsymbol{I} - \boldsymbol{G} - \boldsymbol{g}\boldsymbol{\eta}_1)^{-1},\\ \boldsymbol{\eta}_4 &= (a\boldsymbol{\eta}_3 + \boldsymbol{\eta}_3\boldsymbol{g}\boldsymbol{\eta}_2 + \boldsymbol{\eta}_2\boldsymbol{g}\boldsymbol{\eta}_3)(s\boldsymbol{I} - \boldsymbol{G} - \boldsymbol{g}\boldsymbol{\eta}_1)^{-1}.\end{aligned}$$

Proof. The form of $\boldsymbol{\eta}$ is seen exactly as when $L = 2$. Considering the upper right block, (9), (10) yield

$$0 = \boldsymbol{\eta}_L\boldsymbol{G} + \sum_{j=1}^{L} \boldsymbol{\eta}_{L-j+1}\boldsymbol{g}\boldsymbol{\eta}_j - (\lambda + a)\boldsymbol{\eta}_{L-1},$$

and from this (14) follows.

Theorem 6. *Let* $T > 0$ *be a fixed time and let* H_L *denote the Erlang distribution with L stages and mean T (i.e.,* $a = L/T$*). Then* $\psi(u, H_L) \to \psi(u, T)$ *as* $L \to \infty$*. More precisely, for some constant D*

$$\psi(u, H_L) = \psi(u, T) + \frac{D}{L} + O\left(L^{-2}\right). \tag{15}$$

Proof. Let $0 < \epsilon < T$. Then it is immediately checked that for all k, $\mathbf{E}[H_L^k; |H_L - T|] > \epsilon$ is of order say $O(e^{-\epsilon_\kappa L})$ (with $\epsilon_\kappa > 0$) as $L \to \infty$. Since it is readily checked that the kth derivative $\psi^{(\kappa)}(u, t)$ of $\psi(u, t)$ w.r.t. t exists and is continuous, in particular bounded on $[T - \epsilon, T + \epsilon]$, it follows (using $\mathbf{E}(H_L - T)^4 = 3/L^2 + 6/L^3$) that

$$\begin{aligned}\psi(u, H_L) &= \mathbf{P}(\tau(u) < H_L; H_L \in [T - \epsilon, T + \epsilon]) + O\left(e^{-\epsilon_0 L}\right)\\ &= \psi(u, T) + \sum_{k=1}^{3} \mathbf{E}\left[\frac{(H_L - T)^k}{k!}; H_L \in [T - \epsilon, T + \epsilon]\right]\psi^{(k)}(u, T)\\ &\quad + O\left(E(H_L - T)^4\right) + O\left(e^{-\epsilon_0 L}\right)\\ &= \psi(u, T) + \sum_{k=1}^{3} \psi^{(k)}(u, T)\frac{\mathbf{E}(H_L - T)^k}{k!} + O\left(3/L^2 + 6/L^3\right)\\ &= \psi(u, T) + \psi^{(2)}(u, T)\frac{1/L}{2!} + \psi^{(3)}(u, T)\frac{2/L^2}{3!} + O(L^{-2})\end{aligned}$$

4. Numerical examples

We implement now the Erlang method in practical calculations of ruin probabilities using the phase-type formula $\psi(u, H) = (\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \ldots, \boldsymbol{\eta}_L)e^{Uu}\mathbf{1}$ and the simple recursive algorithm presented in the former section. The density of the claim sizes is $f(x) = \alpha e^{Gx}\boldsymbol{g}$ and the relative security loading considered in the illustrations below is $\theta = 0.1$. In order to compare the Erlang approximations with the exact values ($L = \infty$) of the ruin probabilities, the Gaver-Stehfest method of inverting the Laplace transform was used, see, for instance, Usabel [24].

We will first consider the hyperexponential distribution used by Wikstad [25] as an example of a highly skewed distribution; in matrix notation

$$\alpha_1 = \begin{pmatrix} 0.0039793 & 0.1078392 & 0.8881815 \end{pmatrix}$$

$$\boldsymbol{G}_1 = \begin{pmatrix} -0.014631 & 0 & 0 \\ 0 & -0.190206 & 0 \\ 0 & 0 & -5.514588 \end{pmatrix}$$

Erlang Approximations $\psi(u, H_L)$ to finite time ruin probabilities

Claim Size $f(x) = \alpha_1 e^{G_1 x} g_1$ (Wikstad [25]) $\theta = 0.1$

$T = 1$		u			
		0	1	10	100
	1	$1.986\,10^{-1}$	$7.587\,10^{-2}$	$1.827\,10^{-2}$	$9.260\,10^{-4}$
	3	$2.190\,10^{-1}$	$8.075\,10^{-2}$	$1.869\,10^{-2}$	$9.249\,10^{-4}$
L	5	$2.228\,10^{-1}$	$8.186\,10^{-2}$	$1.877\,10^{-2}$	$9.246\,10^{-4}$
	7	$2.243\,10^{-1}$	$8.235\,10^{-2}$	$1.881\,10^{-2}$	$9.245\,10^{-4}$
	∞	$2.277\,10^{-1}$	$8.362\,10^{-2}$	$1.891\,10^{-2}$	$9.242\,10^{-4}$

$T = 10$		u			
		0	1	10	100
	1	$4.406\ 10^{-1}$	$3.129\ 10^{-1}$	$1.195\ 10^{-1}$	$9.315\ 10^{-3}$
	3	$4.874\ 10^{-1}$	$3.587\ 10^{-1}$	$1.324\ 10^{-1}$	$9.344\ 10^{-3}$
L	5	$4.980\ 10^{-1}$	$3.697\ 10^{-1}$	$1.355\ 10^{-1}$	$9.348\ 10^{-3}$
	7	$5.027\ 10^{-1}$	$3.747\ 10^{-1}$	$1.370\ 10^{-1}$	$9.349\ 10^{-3}$
	∞	$5.148\ 10^{-1}$	$3.874\ 10^{-1}$	$1.408\ 10^{-1}$	$9.351\ 10^{-3}$

$T = 100$		u			
		0	1	10	100
	1	$6.786\ 10^{-1}$	$5.901\ 10^{-1}$	$3.684\ 10^{-1}$	$7.713\ 10^{-2}$
	3	$7.218\ 10^{-1}$	$6.409\ 10^{-1}$	$4.143\ 10^{-1}$	$8.284\ 10^{-2}$
L	5	$7.286\ 10^{-1}$	$6.494\ 10^{-1}$	$4.240\ 10^{-1}$	$8.417\ 10^{-2}$
	7	$7.313\ 10^{-1}$	$6.527\ 10^{-1}$	$4.282\ 10^{-1}$	$8.476\ 10^{-2}$
	∞	$7.375\ 10^{-1}$	$6.605\ 10^{-1}$	$4.384\ 10^{-1}$	$8.632\ 10^{-2}$

T = 1000		u			
		0	1	10	100
	1	$8.290\ 10^{-1}$	$7.792\ 10^{-1}$	$6.339\ 10^{-1}$	$2.922\ 10^{-1}$
	3	$8.569\ 10^{-1}$	$8.146\ 10^{-1}$	$6.860\ 10^{-1}$	$3.357\ 10^{-1}$
L	5	$8.613\ 10^{-1}$	$8.203\ 10^{-1}$	$6.953\ 10^{-1}$	$3.459\ 10^{-1}$
	7	$8.631\ 10^{-1}$	$8.226\ 10^{-1}$	$6.990\ 10^{-1}$	$3.504\ 10^{-1}$
	∞	$8.672\ 10^{-1}$	$8.278\ 10^{-1}$	$7.077\ 10^{-1}$	$3.618\ 10^{-1}$

Next we consider a more regular claim size distribution, an Erlang(3) with mean 1, in matrix notation

$$\boldsymbol{\alpha}_2 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{G}_2 = \begin{pmatrix} -3 & 3 & 0 \\ 0 & -3 & 3 \\ 0 & 0 & -3 \end{pmatrix}$$

Erlang Approximations $\psi(u, H_L)$ to finite time ruin probabilities

Claim Size $f(x) = \alpha_2 e^{G_2 x} g_2$ (Erlang(3)) $\theta = 0.1$

T = 1		u		
		0	1	10
	1	$4.169\ 10^{-1}$	$2.081\ 10^{-1}$	$5.159\ 10^{-5}$
	3	$4.853\ 10^{-1}$	$2.339\ 10^{-1}$	$1.324\ 10^{-5}$
L	5	$5.029\ 10^{-1}$	$2.403\ 10^{-1}$	$8.348\ 10^{-6}$
	7	$5.109\ 10^{-1}$	$2.432\ 10^{-1}$	$6.594\ 10^{-6}$
	∞	$5.323\ 10^{-1}$	$2.508\ 10^{-1}$	$3.146\ 10^{-6}$

T = 10		u		
		0	1	10
	1	$7.350\ 10^{-1}$	$5.206\ 10^{-1}$	$1.355\ 10^{-2}$
	3	$7.949\ 10^{-1}$	$5.837\ 10^{-1}$	$1.112\ 10^{-2}$
L	5	$8.037\ 10^{-1}$	$5.963\ 10^{-1}$	$1.033\ 10^{-2}$
	7	$8.071\ 10^{-1}$	$6.016\ 10^{-1}$	$9.943\ 10^{-3}$
	∞	$8.148\ 10^{-1}$	$6.142\ 10^{-1}$	$8.797\ 10^{-3}$

T = 100		u		
		0	1	10
	1	$8.687\ 10^{-1}$	$7.289\ 10^{-1}$	$1.203\ 10^{-1}$
	3	$8.915\ 10^{-1}$	$7.675\ 10^{-1}$	$1.439\ 10^{-1}$
L	5	$8.941\ 10^{-1}$	$7.728\ 10^{-1}$	$1.502\ 10^{-1}$
	7	$8.951\ 10^{-1}$	$7.748\ 10^{-1}$	$1.532\ 10^{-1}$
	∞	$8.973\ 10^{-1}$	$7.793\ 10^{-1}$	$1.608\ 10^{-1}$

T = 1000		u: 0	1	10	100
L	1	$9.034\ 10^{-1}$	$7.934\ 10^{-1}$	$2.110\ 10^{-1}$	$3.721\ 10^{-7}$
	3	$9.086\ 10^{-1}$	$8.033\ 10^{-1}$	$2.279\ 10^{-1}$	$4.510\ 10^{-7}$
	5	$9.089\ 10^{-1}$	$8.039\ 10^{-1}$	$2.296\ 10^{-1}$	$4.813\ 10^{-7}$
	7	$9.089\ 10^{-1}$	$8.041\ 10^{-1}$	$2.302\ 10^{-1}$	$4.989\ 10^{-7}$
	∞	$9.091\ 10^{-1}$	$8.043\ 10^{-1}$	$2.310\ 10^{-1}$	$5.737\ 10^{-7}$

The reader can see at a glance how quickly the approximations yield at least one or two significant digits even for very small values of the ruin probabilities. Furthermore, motivated by the linear rate of convergence (15), we implemented next the Richardson extrapolation (see e.g. [22]), leading to the improved estimate

$$\psi_1(u,T) \approx (L+1)\,\psi(u,H_{L+1}) - L\psi(u,H_L) \tag{16}$$

with error going to 0 at rate $1/L^2$.

Improved Erlang Approximations $\psi_1(u,H_L)$ to finite time ruin probabilities

Claim Size $f(x) = \alpha_1 e^{G_1 x} g_1$ (Wikstad(1971)) $\theta = 0.1$

T = 1		u: 0	1	10	100
L	1	$2.291\ 10^{-1}$	$8.300\ 10^{-2}$	$1.888\ 10^{-2}$	$9.243\ 10^{-4}$
	3	$2.287\ 10^{-1}$	$8.350\ 10^{-2}$	$1.890\ 10^{-2}$	$9.242\ 10^{-4}$
	5	$2.282\ 10^{-1}$	$8.357\ 10^{-2}$	$1.890\ 10^{-2}$	$9.242\ 10^{-4}$
	7	$2.280\ 10^{-1}$	$8.359\ 10^{-2}$	$1.891\ 10^{-2}$	$9.242\ 10^{-4}$
	∞	$2.277\ 10^{-1}$	$8.362\ 10^{-2}$	$1.891\ 10^{-2}$	$9.242\ 10^{-4}$

T = 10		u: 0	1	10	100
L	1	$5.090\ 10^{-1}$	$3.786\ 10^{-1}$	$1.379\ 10^{-1}$	$9.362\ 10^{-3}$
	3	$5.137\ 10^{-1}$	$3.860\ 10^{-1}$	$1.402\ 10^{-1}$	$9.353\ 10^{-3}$
	5	$5.144\ 10^{-1}$	$3.869\ 10^{-1}$	$1.406\ 10^{-1}$	$9.352\ 10^{-3}$
	7	$5.146\ 10^{-1}$	$3.872\ 10^{-1}$	$1.407\ 10^{-1}$	$9.351\ 10^{-3}$
	∞	$5.148\ 10^{-1}$	$3.874\ 10^{-1}$	$1.408\ 10^{-1}$	$9.351\ 10^{-3}$

T = 100		u: 0	1	10	100
L	1	$7.456\ 10^{-1}$	$6.684\ 10^{-1}$	$4.360\ 10^{-1}$	$8.542\ 10^{-2}$
	3	$7.392\ 10^{-1}$	$6.624\ 10^{-1}$	$4.387\ 10^{-1}$	$8.612\ 10^{-2}$
	5	$7.381\ 10^{-1}$	$6.613\ 10^{-1}$	$4.386\ 10^{-1}$	$8.623\ 10^{-2}$
	7	$7.378\ 10^{-1}$	$6.609\ 10^{-1}$	$4.386\ 10^{-1}$	$8.628\ 10^{-2}$
	∞	$7.375\ 10^{-1}$	$6.605\ 10^{-1}$	$4.384\ 10^{-1}$	$8.632\ 10^{-2}$

T = 1000		*u*			
		0	1	10	100
	1	8.728 10^{-1}	8.345 10^{-1}	7.136 10^{-1}	3.551 10^{-1}
	3	8.681 10^{-1}	8.290 10^{-1}	7.094 10^{-1}	3.610 10^{-1}
L	5	8.676 10^{-1}	8.283 10^{-1}	7.085 10^{-1}	3.616 10^{-1}
	7	8.674 10^{-1}	8.281 10^{-1}	7.081 10^{-1}	3.617 10^{-1}
	∞	8.672 10^{-1}	8.278 10^{-}	7.077 10^{-1}	3.618 10^{-1}

As expected, the extrapolation clearly improves the convergence, yielding at least three relative significant digits in most cases. Of course, for practical purposes, most often even just one or two significant digits will suffice.

The same conclusions are apparent from the extrapolated approximations for the Erlang(3) claim sizes:

Improved Erlang Approximations $\psi_1(u, H_L)$ to finite time ruin probabilities

Claim Size $f(x) = \alpha_2 e^{G_2 x} g_2$ (Erlang(3)) $\theta = 0.1$

T = 1		*u*		
		0	1	10
	1	5.139 10^{-1}	2.451 10^{-1}	–9.963 10^{-6}
	3	5.285 10^{-1}	2.496 10^{-1}	5.008 10^{-7}
L	5	5.308 10^{-1}	2.503 10^{-1}	2.057 10^{-6}
	7	5.315 10^{-1}	2.505 10^{-1}	2.557 10^{-6}
	∞	5.323 10^{-1}	2.508 10^{-1}	3.146 10^{-6}

T = 10		*u*		
		0	1	10
	1	8.289 10^{-1}	6.145 10^{-1}	1.029 10^{-2}
	3	8.176 10^{-1}	6.155 10^{-1}	9.224 10^{-3}
L	5	8.158 10^{-1}	6.148 10^{-1}	8.999 10^{-3}
	7	8.153 10^{-1}	6.145 10^{-1}	8.914 10^{-3}
	∞	8.148 10^{-1}	6.142 10^{-1}	8.797 10^{-3}

T = 100		*u*		
		0	1	10
	1	9.064 10^{-1}	7.904 10^{-1}	1.533 10^{-1}
	3	8.983 10^{-1}	7.811 10^{-1}	1.595 10^{-1}
L	5	8.976 10^{-1}	7.800 10^{-1}	1.604 10^{-1}
	7	8.975 10^{-1}	7.797 10^{-1}	1.606 10^{-1}
	∞	8.973 10^{-1}	7.793 10^{-1}	1.608 10^{-1}

T = 1000		u			
		0	1	10	100
	1	9.125 10^{-1}	8.107 10^{-1}	2.383 10^{-1}	4.759 10^{-7}
	3	9.093 10^{-1}	8.050 10^{-1}	2.326 10^{-1}	5.214 10^{-7}
L	5	9.092 10^{-1}	8.045 10^{-1}	2.316 10^{-1}	5.400 10^{-7}
	7	9.091 10^{-1}	8.044 10^{-1}	2.313 10^{-1}	5.501 10^{-7}
	∞	9.091 10^{-1}	8.043 10^{-1}	2.310 10^{-1}	5.737 10^{-7}

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