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THE SPHERICAL BUILDING AND REGULAR SEMISIMPLE ELEMENTS

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Let G be a connected reductive algebraic group defined over a finite field k. The finite group $G(k)$ of k-rational points of G acts on the spherical building $\mathcal{B}(G)$, a polyhedron which is functorially associated with G. We identify the subspace of points of $\mathcal{B}(G)$ fixed by a regular semisimple element s of $G(k)$ topologically as a subspace of a sphere (apartment) in $\mathcal{B}(G)$ which depends on an element of the Weyl group which is determined by s. Applications include the derivation of the values of certain characters of $G(k)$ at s by means of Lefschetz theory. The characters considered arise from the action of $G(k)$ on the cohomology of equivariant sheaves over $\mathcal{B}(G)$.

Let k be the finite field $\mathbb{F}_q$ of q elements and G a connected reductive group defined over k. In [2] there was constructed a certain topological space $\mathcal{B}(G)$ (the construction in [2] applied for an arbitrary field k) which is associated with G functorially. The (metric) space $\mathcal{B}(G)$ is a union of spheres $\mathcal{B}(S)$ as S runs over the maximal k-split tori of G, and has a "rational subspace" which may be roughly thought of as the space of one parameter subgroups of G, suitably topologized. In [2] the construction was applied to the derivation of a character formula for the group $G(k)$ of rational points $G(k)$ acting on the homology of $\mathcal{B}(G)$; this formula follows from the identification of the fixed-point set

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of $g \in G(k)$ on $\mathcal{B}(G)$, via Lefschetz theory.

In the present work we prove a general result concerning the fixed point set $\mathcal{B}(G)^s$ of a regular semisimple element $s \in G(k)$ acting on $\mathcal{B}(G)$. This relates $\mathcal{B}(G)^s$ to the set of points of an "apartment" $\mathcal{B}(S)$ (S a maximal split torus) which are fixed by an element w_s of the Weyl group of G which corresponds to s (see §2 below). As an application, we use Lefschetz theory applied to certain subspaces $\mathcal{B}_J(G)$ (J a subset of the set Π of simple roots) of $\mathcal{B}(G)$ to deduce the values of certain principal series characters of $G(k)$ at s. One consequence of this is that in the split case, we have $1_{P_J(k)}^{G(k)}(s) = 1_{W_J}^{W}(w_s)$ $(J \subset \Pi)$. This result is of course not new. Lusztig has proved vastly more general results ([8], [9], [10]) giving the values of all irreducible unipotent characters of G on regular semisimple elements for most groups G and the result has been dealt with explicitly (by different methods) by Deligne and Lusztig [4, §§7, 8], Kawanaka [7] and Surowski [12]. However it seems useful to put it into the present geometric context, at least partly because similar results can be obtained for any representation of $G(k)$ which is realized on the cohomology of an equivariant sheaf over $\mathcal{B}(G)$ (see §6 below).

Section 1 is devoted to the recollection of the main properties of $\mathcal{B}(G)$ and Section 2 to the statement and proof of the main theorem. In Section 3 we introduce certain closed subspaces $\mathcal{B}_J(G)$ of $\mathcal{B}(G)$ and give them a simplicial structure. In Section 4 the homology of the spaces $\mathcal{B}_J(G)$ is computed simplicially, and in Section 5 the characters of $G(k)$ on the homology groups is studied via the fixed point theory of Section 2. Finally, in Section 6, we discuss some special cases, and give a general formulation for $G(k)$-equivariant sheaves over $\mathcal{B}(G)$.

1. The spherical building $\mathcal{B}(G)$ of a reductive k-group

Let G be a reductive k-group (k *any* field). For any maximal k-split torus S of G, $\mathcal{B}(S)$ is defined as the sphere whose points are the rays (half-lines) of $Y(S) \otimes \mathbb{R}$ $\left(Y(S) = \mathrm{Hom}(G_m, S)\right)$, G_m being the

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multiplicative group of $\bar{k}$). To every point b of $\mathcal{B}(S)$ we associate a parabolic k-subgroup $P(b) = P_G(b)$ of G as in [2, §1]: $P(b)$ is the unique parabolic k-subgroup of G which contains S and which corresponds to the region of $Y(S) \otimes \mathbb{R}$ in which b lies (these regions are the simplexes of the Coxeter complex in the semisimple case).

If $\mathcal{B}_1$ is the disjoint union of the spheres $\mathcal{B}(S)$ (over all maximal k-split tori S of G) then we define an equivalence relation on $\mathcal{B}_1$ by $b \sim c$ if $c = adg.b$ for some $g \in P(b)(k)$. The building $\mathcal{B}(G)$ is then defined as

(1.1) $\mathcal{B}(G) = \mathcal{B}_1(G)/\sim$.

Clearly $G(k)$ acts on $\mathcal{B}(G)$. Some of its principal properties are as follows (these may be found in [2]). Let S be a maximal k-split torus of G .

(1.2) The projection $\mathcal{B}_1(G) \to \mathcal{B}(G) = \mathcal{B}_1(G)/\sim$ restricts to an injection of $\mathcal{B}(S)$ into $\mathcal{B}(G)$.

(1.3) The point b $(\in \mathcal{B}(G))$ is in $\mathcal{B}(S)$ if and only if $S \subset P(b)$.

(1.4) The isotropy group of b in $G(k)$ is $P(b)(k)$.

(1.5) Let s be a semisimple element of $G(k)$. Then $\mathcal{B}\left(Z_G(s)^0\right)$ is the fixed point set $\mathcal{B}(G)^s$ of s acting on $\mathcal{B}(G)$.

Suppose now that k is a finite field (say $k = \mathbb{F}_q$ as in the introduction). Then (*cf.* Lusztig [10]) associated with the k-structure of G , we have a Frobenius endomorphism $F : G \to G$ which satisfies

(1.6) $G(k) = G^F = \{g \in G \mid F(g) = g\}$;

(1.7) an algebraic subset $H \leq G$ is defined over k if and only if $F(H) = H$;

(1.8) for any k-subgroup H of G we have $H(k) = H^F$.

2. Fixed points of regular semisimple elements

Let s be a regular semisimple element of $G(k)$. Then s lies in a

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unique maximal torus T, which is defined over k since $T = Z_G(s)$ (*cf.* [*1*, §§10.3, 10.5]).

(2.1) **LEMMA.** *Let T be the maximal torus of G which contains the regular semisimple element $s \in G(k)$. The fixed-point set $\mathcal{B}(G)^s$ of s acting on $\mathcal{B}(G)$ is $\mathcal{B}(T_d)$ where T_d is the maximal k-split subtorus of T.*

Proof. From (1.5) we have that $\mathcal{B}(G)^s = \mathcal{B}\left(Z_G(s)^0\right)$. But $Z_G(s)^0 = T$, whence $\mathcal{B}(G)^s = \mathcal{B}(T)$. From [*2*, Lemma (7.1) (i)] and [*1*, §1.4] we have that $\mathcal{B}(T) = \mathcal{B}(T_d)$, whence the result. □

The above result applies when k is any field. Henceforth, we take $k = \mathbb{F}_q$.

(2.2) **LEMMA.** *A k-torus R of G is k-split if and only if F acts on R by taking elements to their qth power.*

Proof. R is split precisely when there is a k-isomorphism $\phi : R \to D$ where D is a group of diagonal matrices. The Frobenius map on D is given by taking qth powers. Since ϕ is a k-morphism, it commutes with Frobenius ("transports the k-structure"), whence F is also the q-power map on R. Conversely, if F consists of taking qth powers, F acts on $Y(R)$ as multiplication by q. Hence $Y(R) = Y(R)_k$, whence R is split [*1*, §1.3]. □

(2.2.1) **COROLLARY.** *For any k-torus R of G, the split part R_d of R is given by $R_d = \{r \in R \mid F(r) = r^q\}$.*

We now fix the following notation:

S a maximal k-split torus of G;

$\bar{S}$ a maximal k-torus of G containing S (so $(\bar{S})_d = S$);

$W = N_G(\bar{S})/\bar{S}$ (the Weyl group of G with respect to $\bar{S}$);

${}_kW = N_G(S)/Z_G(S)$;

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$\dot{w}$ is a representative in $N_G(\bar{S})$ for $w \in W$.

We give a proof of the following result of Springer and Steinberg [13, II.1.2] for the reader's convenience as well as for the purpose of establishing notation.

(2.3) LEMMA. *The* $G(k) = G^F$ *-conjugacy classes of* F*-stable maximal tori of* G *are in bijective correspondence with the* F*-classes of* W, *where* F*-equivalence is defined by* $w_1 \sim_F w_2$ *if* $w_2 = vw_1F(v)^{-1}$, *some* $v \in W$.

Proof. Let S_1 be any F-stable maximal torus in G. Then $S_1 = g\bar{S}g^{-1}$, some $g \in G$; since S_1 and $\bar{S}$ are F-stable, we have $F(g)\bar{S}F(g)^{-1} = g\bar{S}g^{-1}$ whence $g^{-1}F(g) = \dot{w} \in N(\bar{S})$. It is easily checked that replacing g by gn $(g \in N(\bar{S}))$ or S_1 by a G^F-conjugate has the effect of replacing w by an F-equivalent element of W. □

Now suppose that s is a regular semisimple element of $G(k)$ and that T is the unique maximal torus of G such that $s \in T$. Then T is F-stable (see above) and so by (2.3), T corresponds to an element $w \in W$, which is determined to within F-conjugacy. Replacing s by a $G(k)$-conjugate clearly gives the same F-class. We have shown:

(2.4) PROPOSITION. *To each* $G(k)$*-conjugacy class of regular semisimple elements of* $G(k)$ *there corresponds a unique* F*-conjugacy class in* W.

(2.5) THEOREM. *Let* s *be a regular semisimple element of* $G(k)$ *and let* c_s *be the corresponding (cf. (2.4))* F*-class in* W. *Then the fixed-point set* $\mathcal{B}(G)^s$ *is* $G(k)$*-conjugate to* $\mathcal{B}(Z_S(w_s))$ *for some element* $w_s \in c_s$, *where* $Z_S(w_s) = \{t \in S \mid t\dot{w}_s = \dot{w}_s t\}$ *and* S *is the maximal* k*-split torus of* G *fixed above.*

Proof. If T is the unique maximal torus of G which contains s, and $T = g\bar{S}g^{-1}$, then $g^{-1}F(g) = \dot{w}$ and $w \in c_s$ (see (2.3)).

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Now $\mathcal{B}(G)^s = \mathcal{B}(T_d)$ by (2.1), and since T_d is k-split, there is an element x of $G(k)$ such that ${}^xT_d \subset S$ (all maximal k-split tori are $G(k)$-conjugate by [1, §4.21]). But $\mathcal{B}(G)^{xsx^{-1}} = adx.\mathcal{B}(G)^s$ for any $x \in G(k)$, whence replacing s (and therefore T) by a $G(k)$-conjugate replaces $\mathcal{B}(G)^s$ by a $G(k)$-conjugate and fixes the F-class c_s. Thus we may assume that $T_d \subset S$.

After this reduction, we have that $\bar{S}$ and T are both maximal k-tori in $Z_G(T_d)$, which is a connected reductive k-subgroup of G [1, (2.15) (d)]. Thus there is an element $z \in Z_G(T_d)$ such that $T = z\bar{S}z^{-1}$ and the element $w_s \in W$ defined by $\dot{w}_s = z^{-1}F(z)$ is in the F-class c_s of W.

Moreover since z centralizes T_d, so does $F(z)$ and hence so does $\dot{w}_s = z^{-1}F(z)$. Thus $T_d \leq Z_S(w_s)$. Conversely, suppose that $x \in S$ is fixed by w_s. Then $z^{-1}F(z)xF(z)^{-1}z = x$, whence $zxz^{-1} = F(z)xF(z)^{-1}$. Taking qth powers of both sides, we see that

$$(zxz^{-1})^q = F(z)x^qF(z)^{-1} = F(zxz^{-1})$$

since by (2.2) we have that $x^q = F(x)$.

But $zxz^{-1} \in zSz^{-1} \subset z\bar{S}z^{-1} = T$. It follows from (2.2.1) (since $F(zxz^{-1}) = (zxz^{-1})^q$), that zxz^{-1} is in the split part of T, that is $zxz^{-1} \in T_d$. Since z centralizes T_d, it follows that $x \in T_d$ and so $Z_S(w_s) \leq T_d$. Hence $Z_S(w_s) = T_d$ and the theorem is proved. □

(2.6) **COROLLARY.** *Suppose G is k-split and that s is a regular semisimple semisimple element of $G(k)$. Then $\mathcal{B}(G)^s$ is $G(k)$-conjugate to $\mathcal{B}(S)^{w_s}$, for any element $w_s \in c_s$, where c_s is the conjugacy class of W corresponding to s.*

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Proof. In this case $S = \bar{S}$ and F acts trivially on W. Thus the F-conjugacy classes of W are simply the conjugacy classes and the statement follows directly from (2.5). □

We conclude this section with the following characterization of $\mathcal{B}(Z_S(w))$, for $w \in W$.

(2.7) PROPOSITION. *Let* $w \in W$. *A point* b *of* $\mathcal{B}(S)$ *is in* $\mathcal{B}(Z_S(w))$ *if and only if* $\dot{w} \in P(b)$.

Proof. Write $S' = Z_S(w)$. This is a k-split torus of G [1, §1.6] and from [2, Lemma (1.2) (ii)] we have that $b \in \mathcal{B}(S')$ implies that $P(b) \supset Z_G(S')$, so that $\dot{w} \in P(b)$.

Conversely, suppose that $b \in \mathcal{B}(S)$ and $\dot{w} \in P(b)$. Then $S \cap \text{Rad } P(b)$ centralizes $\dot{w}$ and from [2, Lemma (1.2) (ii)] we have $b \in \mathcal{B}(S \cap \text{Rad } P(b))$. Since $S \cap \text{Rad } P(b) \subset Z_G(\dot{w})$, we have $b \in \mathcal{B}(S \cap Z_G(\dot{w})) = \mathcal{B}(Z_S(\dot{w}))$ as required. □

3. The subspaces $\mathcal{B}_J$

Let K be an algebraic closure of k; we write Φ, Φ^+, Π for the set of roots, positive roots and simple roots of G determined by $\bar{S}$ and a Borel subgroup $B \supset \bar{S}$ (B may be assumed to be a k-group). The corresponding Weyl group is $W = W(\bar{S}, G)$. Following Borel and Tits [2, §5], the corresponding data for the k-structure will be denoted ${}_k\Phi = \Phi(S, G)$, ${}_k\Phi^+$, ${}_k\Pi$ and ${}_kW$.

The following facts will be of importance later.

(3.1) The parabolic k-subgroups of G which contain B are the ${}_kP_J$ for the various subsets $J \subset {}_k\Pi$ [1, §5.12].

(3.2) The parabolic k-subgroups of G which contain S are $\{{}^w_kP_J \mid J \subset {}_k\Pi, w \in {}_kW\}$. Moreover ${}^w_kP_J = {}^{w'}_kP_J$, if and only if $J = J'$ and ${}^wW_J = {}^{w'}W_J$ (here ${}^w_kP_J = w({}_kP_J)w^{-1}$, and so on).

This follows from [1, §§5.9 and 5.15].

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(3.3) For any subset $J \subset {}_k\Pi$, there is a unique F-invariant subset $\bar{J} \subset \Pi$ such that ${}_kP_J = P_{\bar{J}}$.

Since ${}_kP_J$ contains B , clearly ${}_kP_J = P_{\bar{J}}$ for some $\bar{J} \subset \Delta$. Moreover, since ${}_kP_J$ is defined over k , it is F-invariant, whence it follows that $W_{\bar{J}}$ is F-invariant. But F maps Π to itself, whence $F(\bar{J}) = \bar{J}$ [13, II.14].

For each subset $J \subset {}_k\Pi$ we define a $G(k)$-invariant subspace $\mathcal{B}_J(G)$ as follows.

(3.4) DEFINITION.

(i) $\mathcal{B}_J(G) = \left\{b \in \mathcal{B}(G) \mid P(b) \geq g\left({}_kP_J\right)g^{-1} \text{ for some } g \in G(k)\right\}$.

(ii) For any reductive k-subgroup $H \leq G$, define $\mathcal{B}_J(H) = \mathcal{B}(H) \cap \mathcal{B}_J(G)$.

Theorem (2.5) may be restated for $\mathcal{B}_J$ as follows.

(3.5) PROPOSITION. *Let s be a regular semisimple element in $G(k)$, with corresponding F-conjugacy class c_s in W . There exists $w_s \in c_s$ such that*

$$\mathcal{B}_J(G)^s = \mathcal{B}_J(G) \cap \mathcal{B}\left(Z_S\left(w_s\right)\right) .$$

To investigate the topological nature of $\mathcal{B}_J(G)$ and $\mathcal{B}_J(S)$ we introduce the following finite simplicial complexes.

$\Delta_J = \Delta_J(G, k)$ is the subcomplex of the combinatorial building (see [2, §6]) consisting of the following simplexes:

(3.6) $$\Delta_J = \left\{P \mid P \text{ is a parabolic } k\text{-subgroup of } G,\ P \supset {}_k^gP_J \text{ for some } g \in G(k)\right\} .$$

Analogously, we introduce the subcomplex Γ_J of the Coxeter complex Γ of ${}_kW$ as follows.

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(3.7) $\Gamma_J = \{{}_kW_L w \mid L \supset J,\ w \in {}_kW\}$.

(3.8) **PROPOSITION.** *If* G *is semisimple, then there is a* $G(k)$*-equivariant homeomorphism* $\tau : |\Delta_J(G)| \to B_J(G)$ *satisfying*

(i) for $b \in |\Delta_J(G)|$, $P(\tau(b))$ *is the parabolic subgroup corresponding to the simplex of* Δ_J *containing* b *in its interior;*

(ii) if S *is a maximal* k*-split torus and* $\Delta_J(S)$ *is the subcomplex of* $\Delta_J(G)$ *whose simplexes correspond to the parabolic* k*-subgroups containing* S *, then* τ *restricted to* $|\Delta_J(S)|$ *is a triangulation of* $B_J(S)$.

The proof consists of the observation that the map τ of [2, Proposition (6.1)] takes $|\Delta_J|$ to B_J and $|\Delta_J(S)|$ to $B_J(S)$.

In addition we have

(3.9) **PROPOSITION.** *Let* G' *be the derived group of* G *, and let* d *be the* k*-rank of the connected centre of* G *. Then* $B_J(G)$ *may be identified in a* $G(k)$*-equivariant way with the* d*-fold suspension of* $B_J(G')$ *. This identification maps* $B_J(S)$ *(*S *a maximal* k*-split torus of* G *) to the* d*-fold suspension of* $B_J(S')$ *, where* S' *is a maximal* k*-split torus of* G' *, which is contained in* S $(S' = S \cap G')$.

The proof is an immediate consequence of the proof of (7.1) and (7.2) in [2].

Combining (3.9) and (3.8), we obtain

(3.10) **COROLLARY.** *(i)* $B_J(G)$ *is homeomorphic in a* $G(k)$*-equivariant way with the* d*-fold suspension of* $|\Delta_J|$.

(ii) $B_J(S)$ *is homeomorphic in a* ${}_kW$*-equivariant way to the* d*-fold suspension of* $|\Gamma_J|$.

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4. The simplicial complexes Γ_J and Δ_J

Although similar results are fairly well known (see, for example, [16]), we include the proof of the following result for completeness. The proof follows ideas of Solomon [12] who treated the case $J = \emptyset$.

(4.1) PROPOSITION. *Suppose* $|_k\Pi - J| > 1$. *Then the complex* Γ_J *has rational homology given by*

$$H_i(\Gamma_J) = \begin{cases} \mathbb{Q} & (i = 0) , \\ 0 & (0 < i < |_k\Pi - J - 1|) , \\ \mathbb{Q}^{|Y_J|} , & i = |_k\Pi - J - 1| , \end{cases}$$

where $Y_J = \left\{w \in {}_kW \mid w^{-1}(J) \subset {}_k\Phi^+, w^{-1}({}_k\Pi - J) \subset {}_k\Phi^-\right\}$.

We use the following elementary result, which may be found in Solomon [12].

(4.1.1) LEMMA. *Let* K *be a simplicial complex and* $L, L_1, \ldots, L_n$ *be subcomplexes such that*

(i) $K = L \cup L_1 \cup \ldots \cup L_n$,

(ii) L_i *has the homology of a point (each* i *),*

(iii) $L \cap L_i$ *has the homology of a point (each* i *),*

(iv) $L_i \cap L_j \subset L$ *if* $i \neq j$.

Then $H_*(K) \cong H_*(L)$.

In the proof of (4.1) we shall use the following notation: for any simplex σ , we denote by $[\sigma]$ the complex consisting of σ together with its faces; a chamber of a simplicial complex K is a simplex of maximal dimension; for any subset $L \subset {}_k\Pi$, X_L denotes the set of shortest right coset representatives of ${}_kW_L$ in ${}_kW$, that is, $X_L = \left\{w \in W \mid w^{-1}(L) \subset {}_k\Phi^+\right\}$. Clearly X_J is a disjoint union: $X_J = \dot{\bigcup}_{L \supseteq J} Y_L$.

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For any simplex $\sigma \in \Gamma$, define the distance $d(\sigma)$ of σ from the fundamental chamber [1] by $d(\sigma) = l(w)$, where w is the shortest element in σ. Write $\Gamma_h = \{\sigma \in \Gamma \mid d(\sigma) \le h\}$ (h a positive integer), and correspondingly, write $\Gamma_{J,h} = \Gamma_J \cap \Gamma_h$.

(4.1.2) **LEMMA.** *Let* $\sigma = {}_kW_Lw$ *be a simplex of* Γ, $w \in X_L$. *Then*

(i) the faces of σ *of codimension* 1 *are*

$$\{{}_kW_{L\cup\{r\}}w \mid r = r_\alpha,\ \alpha \in {}_k\Pi - L\} :$$

for any face σ' *of* σ, *we have* $d(\sigma') \le d(\sigma)$;

(ii) the face ${}_kW_{L'}w = \sigma'$ *of* $({}_k\Pi \supset L' \supset L)$ *satisfies* $d(\sigma') < d(\sigma)$ *if and only if there is an element* $\alpha \in L'-L$ *such that* $l(r_\alpha w) < l(w)$;

(iii) if $d(\sigma) > 0$, *then* σ *has a face* σ' *of codimension* 1 *with* $d(\sigma') < d(\sigma)$.

The proof of (4.1.2) is an easy exercise in Weyl groups. As an immediate consequence, we see that Γ_h and $\Gamma_{J,h}$ are subcomplexes of Γ and Γ_J respectively.

(4.1.3) **COROLLARY.** *Let* $\sigma = {}_kW_Lw$ *be a simplex of* Γ, *and assume* $w \in X_L$. *Then* σ *has a proper face* σ' *with* $d(\sigma') = d(\sigma)$ *if and only if* $w \notin Y_L$.

Proof. If $w \notin Y_L$ then there is an element $\alpha \in {}_k\Pi - L$ such that if $r = r_\alpha$ is the corresponding reflection in ${}_kW$, then $l(rw) > l(w)$. By (4.1.2) *(ii)*, the face $\sigma_r = {}_kW_{L\cup\{r\}}w$ then satisfies $d(\sigma_r) = d(\sigma)$.

The converse follows similarly. □

(4.1.4) **LEMMA.** *With notation as in* (4.1.3), *suppose that* $d(\sigma) = h$. *Then*

$$[\sigma] \cap \Gamma_{h-1} = \bigcup_{i=1}^{p} [\sigma_i]$$

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where $\sigma_1, \ldots, \sigma_p$ *are the faces of* σ *which have codimension* 1 *and satisfy* $d(\sigma_i) < h$.

This follows easily from (4.1.2) *(ii)*.

Let $C(\Gamma_J)$ be the collection of simplexes of Γ_J of maximal dimension $(= \{_kW_Jv \mid v \in {}_kW\})$. Write $C^0(\Gamma_J)$ for the set of simplexes σ in $C(\Gamma_J)$ which have a (proper) face σ' such that $d(\sigma') = d(\sigma)$. By (4.1.3), we have that $C^0(\Gamma_J) = \{_kW_Jv \mid v \in X_J - Y_J\}$. Correspondingly, we write $\Gamma_J^0 = \bigcup_{\sigma \in C^0(\Gamma_J)} [\sigma]$.

(4.1.5) **LEMMA.** *Suppose* $|_k\Pi - J| > 1$. *Then* Γ_J^0 *has the homology of a point.*

Proof. Write $\Gamma_{J,h}^0 = \Gamma_J^0 \cap \Gamma_h$ $(h = 0, 1, 2, \ldots)$. Clearly $\Gamma_{J,0}^0 = [_kW_J]$ is contractible. We show by induction on h that $\Gamma_{J,h}^0$ has the homology of a point for all h . Now

$$\Gamma_{J,h+1} = \Gamma_{J,h}^0 \cup [_kW_Jw_1] \cup \ldots \cup [_kW_Jw_m]$$

where $\{_kW_Jw_i \mid i = 1, \ldots, m\} = \left\{\sigma \in C^0(\Gamma_J) \mid d(\sigma) = h+1\right\}$. One now checks the conditions of Lemma (4.1.1): *(i)* and *(ii)* are trivial; thus it remains to verify

(iii) $[\sigma_i] \cap \Gamma_{J,h}^0$ has the homology of a point $(\sigma_i = {}_kW_Jw_i)$ and

(iv) $[\sigma_i] \cap [\sigma_j] \subset \Gamma_{J,h}^0$ if $i \neq j$.

For *(iii)*, we have from (4.1.4) that $[\sigma_i] \cap \Gamma_{J,h}^0 = \bigcup_{j=1}^{p} [\tau_{ij}]$ where $\tau_{i_1}, \ldots, \tau_{i_p}$ are the faces of σ_i which have codimension 1 and satisfy $d(\tau_{ij}) \leq h$. Now by (4.1.2) *(iii)*, σ_i has a face τ_{i1} with

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$d(\tau_{i1}) \leq h$. Further, since $\sigma_i \in \Gamma_J^0$, σ_i also has a face τ with $d(\tau) = h+1$. Thus $\bigcap_{j=1}^{p} [\tau_{ij}] \neq \emptyset$ $(\supset {}_kW_{\Pi_k-\{r\}}w_i$, where $\tau = {}_kW_{J\cup\{r\}}w_i$ is any face satisfying $d(\tau) = d(\sigma_i)$), and it follows that $\bigcup_{j=1}^{p} [\tau_{ij}]$ is contractible. Condition *(iv)* is easily verified. Thus Lemma (4.1.1) applies, and we deduce that $\Gamma_{J,h}^0$ and $\Gamma_{J,h+1}^0$ have the same homology. The result follows. □

(4.1.6) **LEMMA.** *The complexes* Γ_J *and* Γ_J^0 *have the same* $(d-1)$-*skeleton, where* $d = \dim \Gamma_J = |{}_k\Pi - J| - 1$.

Proof. Let ${}_kW_{J\cup\{r\}}w$ be a $(d-1)$-simplex of Γ_J ; we may assume w to be in $X_{J\cup\{\alpha\}}$, where $r = r_\alpha$. Then *a fortiori*, $w \in X_J$ and it follows that the simplex $\sigma = {}_kW_Jw$ lies in $C^0(\Gamma_J)$ (it has the face $\sigma' = {}_kW_{J\cup\{r\}}w$ with $d(\sigma') = d(\sigma)$). Thus $\sigma' \in [\sigma] \subset \Gamma_J^0$. □

Proof of (4.1). From (4.1.6), we have

$$\Gamma_J = \Gamma_J^0 \cup \{\omega_1\} \cup \{\omega_2\} \cup \ldots \cup \{\omega_t\}$$

where $\omega_i = {}_kW_Jy_i$ and $Y_J = \{y_1, \ldots, y_t\}$. Thus

$$H_J(\Gamma_J) = \begin{cases} H_p(\Gamma_J^0) & (p < d) , \\ H_p(\Gamma_J^0) \oplus \mathbb{Q}^{|Y_J|} & (p = d) . \end{cases}$$

The result now follows immediately from (4.1.5). □

From (4.1) we quickly deduce the homology of Δ_J using

(4.2) **LEMMA.** *The partially ordered sets* ${}_kW\backslash(\Gamma_J \times \Gamma_J)$ *and* $G(k)\backslash(\Delta_J \times \Delta_J)$ *are isomorphic.*

Proof. The proof is exactly the same as that of the corresponding

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result for ${}_kW\backslash(\Gamma \times \Gamma)$ and $G(k)\backslash(\Delta \times \Delta)$. It may be found in [3].

(4.3) **COROLLARY.** *Suppose* $|{}_k\Pi-J| > 1$. *Then then homology of* Δ_J *is given by*

$$H_i(\Delta_J) = \begin{cases} \mathbb{Q} & (i = 0) , \\ 0 , & 0 < i < |{}_k\Pi-J-1| , \\ \mathbb{Q}^t , & i = |{}_k\Pi-J-1| , \quad t \neq 0 . \end{cases}$$

Putting together the results of (3.11), (4.1) and (4.3) we obtain

(4.4) **THEOREM.** *Let* $J \subset {}_k\Pi$, *with* $D = |{}_k\Pi-J| + d - 1 > 0$ *where* $d = k$-rank *of* $Z(G)$. *Then the rational homology groups of* $\mathcal{B}_J(G)$ *and* $\mathcal{B}_J(S)$ *are given by*

$$H_i(\mathcal{B}_J(G)) = \begin{cases} \mathbb{Q} , & i \neq 0 , \\ 0 , & 0 < i < D , \\ \mathbb{Q}^t , & t \neq 0 , \quad i = D , \end{cases}$$

$$H_i(\mathcal{B}_J(S)) = \begin{cases} \mathbb{Q} & , \quad i \neq 0 , \\ 0 & , \quad 0 < i < D , \\ \mathbb{Q}^{|Y_J|} & , \quad i = D , \end{cases}$$

where $Y_J = \{w \in {}_kW \mid wJ \subset \Phi^+ , w({}_k\Pi-J) \subset \Phi^-\}$.

5. Representations associated with the spaces $\mathcal{B}_J(G)$

From Theorem (4.4) it is apparent that we have a representation M_J of $G(k)$ on $H_D(\mathcal{B}_J(G))$. Moreover when $|{}_k\Pi-J| > 1$, it follows from (3.10) *(i)* that M_J is the same representation as that of $G(k)$ on $H_{|{}_k\Pi-J|-1}(\Delta_J(G, k))$. Applying the Hopf trace formula to this latter complex we see that (when $|{}_k\Pi-J| > 1$)

$$(5.1) \quad M_J = \sum_{L,\ {}_k\Pi \supset L \supset J} (-1)^{|L-J|} \operatorname{Ind}_{{}_kP_L(k)}^{G(k)}(1) .$$

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Moreover, we may always assume that the k-rank d of $Z(G)^0$ is non-zero (by suspending $\mathcal{B}(G)$ if necessary); thus the formula (5.1) holds without restriction on J. This is equivalent to stipulating that $\mathcal{B}_J(G)$ is connected for all J.

The representations M_J were introduced by Solomon [11] and studied by Surowski [16] and Stanley [14]. Our object in reintroducing them here is to show how the trace of M_J at regular semisimple elements of $G(k)$ may be evaluated directly from our main result (Theorem (2.5)).

(5.2) **DEFINITION.** For $L \subset {}_k\Pi$ define $\alpha_L : W \to \mathbb{N}$ by

$$\alpha_L(x) = \#\{W_{\bar{L}}w \mid w \in {}_kW,\ W_{\bar{L}}wx = W_{\bar{L}}w\} .$$

Note that in the split case, $\alpha_L = \mathrm{Ind}_{W_L}^{W}(1)$ $(W = {}_kW,\ L = \bar{L})$.

(5.3) **THEOREM.** *Let s be a regular semisimple element of $G(k)$, and let w_s be the corresponding element of W, chosen as in the statement of Theorem (2.5). If μ_J is the character of the representation M_J above, then*

$$\mu_J(s) = \sum_{{}_k\Pi \supseteq L \supseteq J} (-1)^{|L-J|}\alpha_L(w_s) .$$

Proof. We have by the Lefschetz principle that

$$\sum_{i=0}^{D} (-1)^i \mathrm{tr}\big(s, H_i(\mathcal{B}_J(G))\big) = \chi_E\big(\mathcal{B}_J(G)\big)^s ,$$

where χ_E is the Euler characteristic.

But by (4.4) the left side reduces to $1 + (-1)^D\mu_J(s)$. Now by Theorem (2.5), the right hand side is

$$\chi_E\big[\mathcal{B}_J(G) \cap \mathcal{B}(Z_S(w_s))\big] = \chi_E\big[\mathcal{B}_J(Z_S(w_s))\big] .$$

To identify the topological nature of the space $\mathcal{B}_J(Z_S(w_s))$ we wish to relate it to the complex Γ_J (*cf.* (3.10) *(ii)*). For this we use (2.7) to

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prove

(5.4) **LEMMA.** *Let* $\Gamma_J^{w_s}$ *be the subcomplex of* Γ_J *defined by the condition*

$$W_L w \in \Gamma_J^{w_s} \Leftrightarrow W_{\bar{L}} w w_s = W_{\bar{L}} w \ .$$

Then under the identification of (3.10) *(ii),* $\mathcal{B}_J(Z_S(w_s))$ *is identified with the* *d-fold suspension of* $\left|\Gamma_J^{w_s}\right|$.

Proof of lemma. From (2.7) we have that $b \in \mathcal{B}(S)$ is in $\mathcal{B}(Z_S(w_s)) \Leftrightarrow \dot{w}_s \in P(b)$. Now for $b \in \mathcal{B}_J(S)$, $P(b) = w^{-1}P_{\bar{L}}w$ for some $L \in {}_k\Pi$, with $L \supset J$ (recall ${}_kP_L = P_{\bar{L}}$), and some $w \in {}_kW$. Thus the condition $\dot{w}_s \in P(b)$ is equivalent to $w_s \in w^{-1}W_{\bar{L}}w$, or that $W_{\bar{L}}ww_s = W_{\bar{L}}w$. Thus under the identification of (3.10) *(ii)*, $\mathcal{B}_J(Z_S(w_s))$ is precisely the image of the suspensions of the simplexes in $\Gamma_J^{w_s}$. This proves the lemma. □

Completion of the proof of Theorem (5.3). The Euler characteristic of $\mathcal{B}_J(Z_S(w_s))$ is now easily computed in terms of that of $\Gamma_J^{w_s}$, which is obtained by using the Hopf trace formula, and recalling that $\chi_E(SX) = 2 - \chi_E(X)$, where SX is the suspension of the topological space X .

(5.5) **COROLLARY.** *We have, for each subset* $J \subset {}_k\Pi$,

$$\mathrm{Ind}_{{}_kP_J(k)}^{G(k)}(1)(s) = \alpha_J(w_s) \ .$$

This is obtained by applying Möbius inversion with respect to the partially ordered set of subsets of ${}_k\Pi$ to the formula of (5.3).

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6. Concluding remarks

(6.1) It is not immediately apparent that our definition of $\alpha_L(x)$ is F-conjugacy invariant. This is because the element w_s is somewhat special. However, let us define

$$\alpha'_L(x) = \#\{W_L w \mid W_L wx = W_L F(x)\} .$$

It is then not difficult to show that for the elements w_s of (5.5), one has $\alpha_L(w_s) = \alpha'_L(w_s)$, and α'_L is patently F-conjugacy invariant: $\alpha'_L(wxF(w)^{-1}) = \alpha'_L(x)$ for each $w \in W$.

(6.2) Next, we remark that in the split case (when F acts trivially on W and $S = \bar{S}$ in §2) the results (and proofs) simplify. In particular, in the split case, one has a W-action on $\mathcal{B}(S)$ and Theorem (2.5) has the simple formulation given in (2.6).

The result (5) is then deduced from the simple geometric observation that

(5.5)' we have $\Lambda(s, \mathcal{B}_J(G)) = \Lambda(w_s, \mathcal{B}_J(S))$ (where $\Lambda(-, X)$ denotes the Lefschetz number: $\sum_{i=0}^{\infty} (-1)^i \operatorname{tr}(-, H_i(X))$.)

(6.3) As mentioned in the introduction, the formula (5.5) is not new. One of the by-products of the present geometric setting for it is that it may be generalized to the case where we have an equivariant $G(k)$-sheaf $\mathcal{F}$ of complex vector spaces on $\mathcal{B}(G)$ (in the sense of Grothendieck [6]). Suppose for simplicity that G is split; we assume always that the cohomology modules $H^i_c(\mathcal{B}(G), \mathcal{F})$ (cohomology with compact supports) are of finite type (that is, have finite complex dimension).

One then has a virtual $G(k)$ module $\Lambda_G = \sum_{i=0}^{\infty} (-1)^i H^i_c(\mathcal{B}(G), \mathcal{F})$ whose associated trace function will be written $\lambda_G(\mathcal{B}(G), \mathcal{F})(x)$. Now the methods of Verdier [17] may be generalized (see Donovan and Lehrer [5]) to prove that

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$$\lambda_G\big(B(G),\ F\big)(x) = \lambda_{\langle x\rangle}\big(B(G)^x,\ F|B(G)^x\big)(x) \ ,$$

where $\langle x\rangle$ denotes the cyclic group generated by x and $F|Y$ denotes the sheaf F restricted to the subspace Y of $B(G)$. Hence for s a regular semisimple element of $G(k)$ (and G split), we have

$$(6.4)\quad \lambda_G\big(B(G),\ F\big)(s) = \lambda_{\langle s'\rangle}\big(B(S)^{{}^{w}s},\ F|B(S)^{{}^{w}s}\big)(s')$$

where s' is an appropriate $G(k)$-conjugate of s.

We note finally that since $B_J(G)$ is a closed subspace of $B(G)$, the discussion in §5 may be thought of as a special case of (6.4), where F is taken as constant on $B_J(G)$ and zero outside $B_J(G)$.

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