The Geometry of the Weak Lefschetz Property and Level Sets of Points

Juan C. Migliore

Abstract. In a recent paper, F. Zanello showed that level Artinian algebras in 3 variables can fail to have the Weak Lefschetz Property (WLP), and can even fail to have unimodal Hilbert function. We show that the same is true for the Artinian reduction of reduced, level sets of points in projective 3-space. Our main goal is to begin an understanding of how the geometry of a set of points can prevent its Artinian reduction from having WLP, which in itself is a very algebraic notion. More precisely, we produce level sets of points whose Artinian reductions have socle types 3 and 4 and arbitrary socle degree ≥ 12 (in the worst case), but fail to have WLP. We also produce a level set of points whose Artinian reduction fails to have unimodal Hilbert function; our example is based on Zanello's example. Finally, we show that a level set of points can have Artinian reduction that has WLP but fails to have the Strong Lefschetz Property. While our constructions are all based on basic double G-linkage, the implementations use very different methods.

1 Introduction

Let K be an infinite field, and let $R := K[x_0, x_1, \ldots, x_n]$ and $S := K[x_1, \ldots, x_n]$ (although for most of this paper we will use $n = 3$). For a subscheme $Z \subset \mathbb{P}^n$, we denote by h_Z the Hilbert function $h_Z(t) = \dim(R/I_Z)_t = \dim R_t - \dim(I_Z)_t$. An Artinian graded algebra A over S is *level* if its minimal free resolution ends with a free module $S(-t)^r$ for some integers t and r; equivalently, A is level if its canonical module is generated in one degree. The *socle type* of A is just the integer r. The *socle degree* of A is the degree in which the socle occurs in A, which is the last degree in which A is non-zero. This is equal to $t - n$. (Both the socle type and the socle degree are defined more generally, but we are only interested here in the level case.) An arithmetically Cohen-Macaulay subscheme V of $\mathbb{P}^n$ is said to be *level* if one (hence every) Artinian reduction is level.

An Artinian graded algebra A is said to have the *Weak Lefschetz Property* (henceforth abbreviated WLP) if, for a general linear form L, the induced map

$$\times L \colon A_t \to A_{t+1}$$

has maximal rank, for all t, and A has the *Strong Lefschetz Property* (SLP) if for a general form F of degree d, the induced map $\times F \colon A_t \to A_{t+d}$ has maximal rank, for all t and all d.

In the monograph [11], the authors made an extensive study of level algebras, their properties and some methods to construct such algebras. A study was also made of two important special cases: sets of reduced points whose Artinian reduction is level, and Artinian level algebras satisfying WLP. Important preliminary results were obtained, and an attempt was made to bring some geometry into this very algebraic subject. That monograph left open the following questions *in three variables* (see also Section 6):

- Does every height three Artinian level algebra have the WLP?
- Does every height three Artinian level algebra have unimodal Hilbert function?

Of course, the primary construction to date for Artinian level algebras has been through the use of inverse systems, discovered a century ago by Macaulay [19]. This approach has led to wonderful results and examples, especially in the Gorenstein case, by many authors; these are too numerous to name here, and we refer to Iarrobino and Kanev [15] for a detailed description.

In a very recent paper, Zanello [26] has modified an argument of Iarrobino (communicated privately) to completely answer the two questions listed above. He gives examples of Artinian level algebras with unimodal Hilbert function, but failing to have WLP, and he gives examples of Artinian level algebras with non-unimodal Hilbert function, and even shows that such algebras can have infinitely many "valleys." This is a very exciting development. His main technique is the method of inverse systems (see [15] for a general reference), using at least one form chosen generically. He also gives an example, using monomial ideals, of an Artinian level algebra failing to have WLP, but this latter construction seems somewhat *ad hoc*.

What is missing is a systematic approach to the geometric side of the problem. One of the deep questions, which was also addressed in [11], is to determine which properties that do occur for Artinian algebras can be found for the Artinian reduction of a reduced set of points. Certainly, some interesting properties do not lift. It is known, for instance, that in $k[x_1, \ldots, x_6]$, a general choice of a 17-dimensional vector space of quadrics generates an Artinian ideal with Hilbert function $(1, 6, 4)$ whose minimal free resolution has no ghost (*i.e.,* redundant) terms, while no set of 11 points in $\mathbb{P}^6$ with this h-vector (whose ideal is generated in general by a 17-dimensional vector space of quadrics) has such a resolution [8].

Of course, Zanello's example using monomial ideals lifts to a reduced set of points in $\mathbb{P}^3$ using distractions (*i.e.,* liftings of monomial ideals), so from his work we already have a reduced set of points with h-vector $(1, 3, 5, 5)$ such that a particular Artinian reduction is level but does not have WLP. It is not obvious *a priori* that the *general* Artinian reduction also fails to have WLP (which is an open condition). See [9,14,23] for more on lifting monomial ideals.

This paper represents an attempt to use a variety of geometric methods (*e.g.,* union, hyperplane section) to generalize this, and to address the following questions. The methods are almost disjoint from those of Zanello's, but they are inspired by his beautiful paper.

- Can one recognize from the geometry of a set of points in $\mathbb{P}^3$ that its general Artinian reduction is level but fails to have WLP?
- What socle degrees and types can occur for the general Artinian reduction of a

reduced set of points which is level but fails to have WLP?
- Can the general Artinian reduction of a reduced set of points be level but not have unimodal Hilbert function?
- Can a level, reduced set of points have the property that a particular Artinian reduction fails to have WLP, but the general Artinian reduction *does* have WLP?

The author of the current paper has wondered if WLP, or at least unimodality of the h-vector, was one of the properties guaranteed for the Artinian reduction of a level set of points. Zanello's results, especially the example coming from monomial ideals, began the answer. This paper continues the study. We note that while the underlying construction is very similar in all cases studied here, there are significant differences in the details of these proofs, and some approaches work well in some cases and fail in others. This shows the flexibility of this approach, and leads one to hope that more such results can be obtained following this general method, for instance in higher projective spaces. Our main results are the following.

- For any $d \geq 5$ there is a reduced set of points $Z \subset \mathbb{P}^3$ such that *every* Artinian reduction of Z is level of socle degree d and socle type 4, but fails to have WLP.
- For any odd $d \geq 7$ there is a reduced set of points $Z \subset \mathbb{P}^3$ such that *every* Artinian reduction of Z is level of socle degree d and socle type 3, but fails to have WLP.
- For any even $d \geq 12$ there is a reduced set of points $Z \subset \mathbb{P}^3$ such that *every* Artinian reduction of Z is level of socle degree d and socle type 3, but fails to have WLP.
- There is a reduced, level set of points in $\mathbb{P}^3$ whose h-vector is non-unimodal.
- There exist reduced sets of points $Z \subset \mathbb{P}^3$ such that the general Artinian reduction has WLP but not SLP.
- Let $d \geq 2$ be a positive integer. Then there exists a reduced, level set of points, $Z \subset \mathbb{P}^3$, whose h-vector has a constant value in each of d consecutive degrees, and for which the corresponding $d - 1$ maps in *any* Artinian reduction all fail to be surjective (and hence WLP fails in each of these degrees).

It should be mentioned that Zanello found the first example of level type 3 Artinian algebras without WLP. Our main contribution in this regard is a more detailed description of the possible socle degrees, giving all degrees beyond a certain point, and, of course, the fact that we get reduced sets of points. Again, Zanello's use of inverse systems makes it impossible (with known methods) to use his main methods for points. Another of our contributions is to show how this can nevertheless be mimicked somewhat (see for instance Example 4.2).

At first glance it seems strange to ask that WLP should fail (let us say in the multiplication by a general linear form from degree d to degree $d + 1$) for the Artinian reduction A of the coordinate ring of a set of points Z but the algebra A still be level. The easiest way for WLP to fail is that there is unexpected socle, but this is ruled out by the property of being level. So instead, somehow the kernel must be bigger than expected for more subtle reasons. How can we predict that this will happen, or explain its occurence? In Propositions 2.3 and 2.5 we give necessary and sufficient conditions for surjectivity and injectivity, respectively. Here we wish to give an intuitive explanation.

The sets of points that we construct will be unions $Z = X_1 \cup X_2$ of specially

chosen sets of points, put together in a precise way by basic double G-linkage (see Proposition 2.1). These are chosen to have h-vectors of a very particular sort. It turns out that the Artinian reduction A in degrees d and $d + 1$ acts as though these components decomposed into the direct sum of the Artinian reductions of X_1 and X_2, and then it is seen that the multiplication cannot have maximal rank. For example, in Example 2.4 we have $Z = X_1 \cup X_2$ where the h-vectors are given by

deg	0	1	2	3
Δh_{X_1}		1	2	1
Δh_{X_2}	1	2	3	4
Δh_Z	1	3	5	5

We see that even though WLP for Z would require that a general linear form induces an isomorphism from degree 2 to degree 3, the decomposition gives that the rank is only $1 + 3 = 4$, so WLP fails. This is not a completely accurate picture, and Example 5.5 shows that we can set up situations where this philosophy predicts that WLP fails, whereas, in fact, it does occur. The true explanation comes from Propositions 2.3 and 2.5. Nevertheless, this intuitive approach gave us all the examples from which we produced our results.

In what way is the geometry of the set of points related to the failure of WLP? The main tool which we use, Proposition 2.3, says that the multiplication from degree d to degree $d + 1$ should fail to be surjective if and only if a generally chosen line λ fails to impose $d + 2$ independent conditions on $(I_Z)_{d+1}$. We apply this to certain unions of points, where we can see from the geometry that λ has to impose fewer than the expected number of conditions because of fixed two-dimensional components in the base locus of the linear system. Proposition 2.5 gives a condition that is equivalent to the failure of injectivity, and we illustrate its use in Example 5.6.

In this paper we only work in codimension three, studying points in $\mathbb{P}^3$, but the methods generalize, at least in part, to any $\mathbb{P}^n$. Many of our examples were verified with the computer program CoCoA [6].

2 The Machinery

The purpose of this paper is to produce, by very geometric methods (*e.g.*, unions, hyperplane and hypersurface sections), reduced sets of points whose Artinian reductions are level but do not have the Weak Lefschetz Property. In this section we discuss how our sets are constructed, give the reason why they are level, and show how the geometry of the points allows an analysis to help determine if the Weak Lefschetz Property holds or not. These ideas are applied to specific questions in the subsequent sections.

Proposition 2.1 ([17]) *Let $C \subset \mathbb{P}^3$ be an arithmetically Cohen–Macaulay curve and let $Z_1 \subset C$ be any zero-dimensional scheme. Let F be a non-zero divisor on R/I_C of degree d, and let Z_2 be the hypersurface section of C by the surface S defined by F. Let Z be the zero-scheme defined by the ideal $I := F \cdot I_{Z_1} + I_C$. Then the following hold:*

(i) *We have the exact sequence $0 \to I_C(-d) \to I_{Z_1}(-d) \oplus I_C \to I_Z \to 0$.*

(ii) *I is saturated,* i.e., $I = I_Z$.

(iii) *We have the exact sequence*

$$0 \to (R/I_C)(-d) \to (R/I_{Z_1})(-d) \oplus (R/I_C) \to R/I_Z \to 0.$$

(iv) *The Hilbert function of Z is given by the formula*

$$h_Z(t) = h_{Z_1}(t-d) + h_C(t) - h_C(t-d) = h_{Z_1}(t-d) + h_{Z_2}(t)$$

(v) *If Z_1 and C are both level, and if they have the same shift, $-t$, in the last free modules of their minimal free resolutions (which means that the socle degrees of their Artinian reductions differ by 1 since they have different codimensions), then Z is level with minimal free resolution having shift $-t - d$. The socle type is less than or equal to the sum of the socle types of C and of Z_1.*

(vi) *If, in addition, F is a non-zero divisor on R/I_{Z_1} (i.e., Z_1 and Z_2 are disjoint), then $Z = Z_1 \cup Z_2$ as schemes.*

Proof Part (i) can be found in [17, Lemma 4.8], and the rest follows immediately from this. Part (v), in particular, follows from (i) using a mapping cone. ∎

Remark 2.2 Following [17], we will refer to this construction as *basic double G-linkage*. Proposition 2.1 is often used in this paper in the following way. Let $C_1 \subset C_2$ be arithmetically Cohen–Macaulay curves in $\mathbb{P}^3$. Then the general hypersurface section of C_1 of any degree d_1 is also a subscheme of C_2, and so the union of the general degree d_1 hypersurface section of C_1 and the general degree d_2 hypersurface section of C_2 fits into the scheme of Proposition 2.1. These curves and the degrees of the hypersurfaces can be chosen so that the union is level, and then geometric properties can be read off.

In this paper we will use the first difference of the Hilbert function formula above to compute the *h*-vectors.

Proposition 2.3 *Let $Z \subset \mathbb{P}^3$ be a zero-dimensional scheme. Let A be the Artinian reduction of R/I_Z by a linear form L_1 and let L_2 be a second, general linear form. Then the multiplication map $\times L_2 \colon A_{t-1} \to A_t$ is surjective if and only if the line λ defined by (L_1, L_2) imposes $t + 1$ conditions on the linear system $|(I_Z)_t|$.*

Proof Note that L_1 only has to avoid the points of Z. No additional generality is required. We have $A = R/(I_Z + (L_1))$, and the exact sequence

$$[R/(I_Z + (L_1))]_{t-1} \xrightarrow{\times L_2} [R/(I_Z + L_1))]_t \to [R/(I_Z + I_\lambda)]_t \to 0.$$

Hence the multiplication $\times L_2$ is surjective if and only if $(I_Z + I_\lambda)_t = R_t$. Now we also have the exact sequence $0 \to (I_Z \cap I_\lambda)_t \to (I_Z)_t \oplus (I_\lambda)_t \to (I_Z + I_\lambda)_t \to 0$. The statement that λ imposes $t+1$ conditions on $|(I_Z)_t|$ is equivalent to the statement that $\dim(I_Z \cap I_\lambda)_t = \dim(I_Z)_t - (t+1)$. Since $\dim(I_\lambda)_t = \dim R_t - (t+1)$, the statement now follows immediately from exactness of the sequence. ∎

Example 2.4   We illustrate the methods which will be used in this paper by constructing a level reduced set of points with h-vector $(1, 3, 5, 5)$ whose Artinian reduction (arbitrary) does not have WLP. The existence of such a set of points (almost) follows also from Zanello's work [26]: his Example 7 gives an Artinian monomial ideal with this Hilbert function and, as mentioned above, we can lift this example to a reduced set of points. The only thing missing is the question of whether WLP fails for all Artinian reductions or only a special one. In any case, it was an attempt to understand this example from a geometric point of view that motivated our current work.

Let C be a smooth arithmetically Cohen–Macaulay curve with h-vector $(1, 2, 3, 4)$. A general hyperplane section of C consists of 10 points, and it contains a subset consisting of four points in linear general position; we will call this subset X_1. Note that X_1 is a complete intersection of type $(1, 2, 2)$ in $\mathbb{P}^3$. Let X_2 be another general hyperplane section of C, and let $Z = X_1 \cup X_2$. Then Z is clearly reduced. By Proposition 2.1, the h-vector of Z is computed by

deg	0	1	2	3
Δh_{X_1}		1	2	1
Δh_{X_2}	1	2	3	4
Δh_Z	1	3	5	5

The fact that Z is level is computed from an easy mapping cone using the short exact sequence $0 \to I_C(-1) \to I_{X_1}(-1) \oplus I_C \to I_Z \to 0$, which gives the resolution

$$0 \to R(-6)^5 \to \begin{pmatrix} R(-5)^{10} \\ \oplus \\ R(-4)^2 \end{pmatrix} \to \begin{pmatrix} R(-4)^5 \\ \oplus \\ R(-3)^2 \\ \oplus \\ R(-2) \end{pmatrix} \to I_Z \to 0.$$

The failure of WLP to hold comes from Proposition 2.3. Indeed, note that any polynomial of degree 3 containing Z must contain the plane of X_2 as a factor, since the points of X_2 do not lie on any cubic on that plane. Now let λ be a general line, meeting the plane of X_2 in a point, P. The point P is thus a base point in the linear system $|(I_Z)_3|$, so λ imposes at most three conditions on cubics, and hence WLP fails.

We remark that the resolution above is also the minimal free resolution of a set of 14 general points on a smooth arithmetically Cohen–Macaulay curve of degree 5 and genus 2, the Artinian reduction of which is also level, and *does* have WLP.

In this paper we are mainly concerned with the failure of WLP occurring by having the multiplication by a general linear form fail to be surjective. However, of course, it can also be injectivity that fails, and there is a criterion for that as well. Although we will not use this criterion for any of our main results, we record it here.

Proposition 2.5   Let $Z \subset \mathbb{P}^3$ be a zero-dimensional scheme. Let A be the Artinian reduction of R/I_Z by a linear form L_1 and let L_2 be a second, general linear form. Let λ be the line defined by the ideal (L_1, L_2). Then the multiplication map $\times L_2\colon A_{t-1} \to A_t$ is injective if and only if $(I_Z \cdot I_\lambda)_t = (I_Z \cap I_\lambda)_t$.

Proof Note that again, L_1 only has to avoid the points of Z. Let

$$J = \frac{I_Z}{(L_1 \cdot I_Z)} \cong \frac{I_Z + (L_1)}{(L_1)}.$$

If $S := R/(L_1)$, then we have $A = S/J$. By [21, Proposition 2.1.5] (see also [20]),

$$\frac{J : L_2}{J}(-1) \cong \frac{I_Z \cap I_\lambda}{I_Z \cdot I_\lambda}.$$

The result then follows from the exact sequence

$$0 \to \frac{J : L_2}{J}(-1) \xrightarrow{\times L_2} S/J(-1) \xrightarrow{\times L_2} S/J \to R/(I_Z + I_\lambda) \to 0. \qquad \blacksquare$$

We will illustrate the use of Proposition 2.5 in Section 5 (see Example 5.6), but we have not found an effective way to apply it for broader results.

3 Small Socle Type

In this section we show how our methods give classes of reduced sets of points with small socle type, and how the geometry of these sets is what determines the fact that the WLP fails for the Artinian reduction.

Theorem 3.1 *For any $d \geq 5$ there is a reduced set of points $Z \subset \mathbb{P}^3$ such that* any *Artinian reduction of Z is level of socle degree d and socle type 4, but fails to have the Weak Lefschetz Property.*

Proof We begin by showing the existence of certain level sets of points Z_d in $\mathbb{P}^2$ of socle type 3 and socle degree d that contain as subsets certain complete intersection sets of points Y_d of socle degree $d - 1$. These can be obtained as liftings (also known as *distractions*) of monomial ideals in two variables, so it is enough to present these monomial ideals. We distinguish the cases d even and d odd. We first give the construction and collect some facts for both of these cases.

Case 1: d odd, $d = 2r - 1$.   **Case 2: d even, $d = 2r$.**
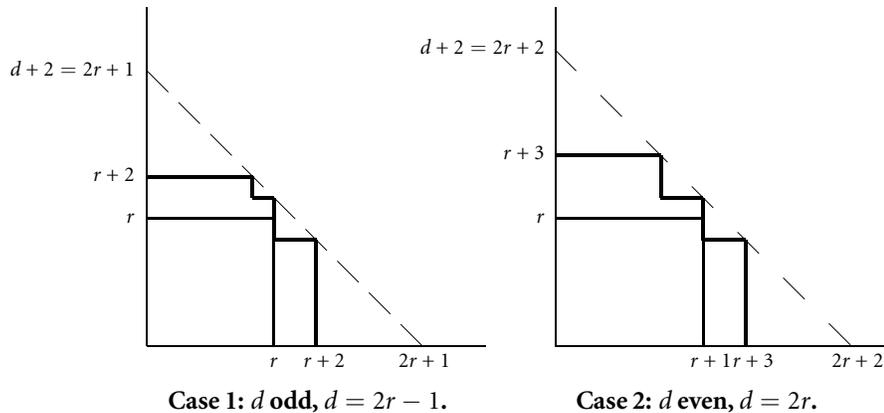
In Case 1, Z_d is the reduced set of points in $\mathbb{P}^2$ obtained by lifting the monomial ideal (in two variables) $(y^{r+2}, x^{r-1}y^{r+1}, x^r y^{r-1}, x^{r+2})$. This ideal is illustrated above by the thickest lines. The complete intersection $Y_d \subset Z_d$ is obtained by lifting the complete intersection (x^r, y^r), illustrated above by the second thickest lines. Then Y_d is a complete intersection of type (r, r). In Case 2, Z_d is obtained by lifting the monomial ideal $(y^{r+3}, x^{r-1}y^{r+1}, x^{r+1}y^{r-1}, x^{r+3})$, and $Y_d \subset Z_d$ is obtained by lifting the complete intersection (x^{r+1}, y^r). Then Y_d is a complete intersection of type $(r, r+1)$.

The following facts are easy to verify. For $r = 3$ some of the higher values of the Hilbert function are not correct below, but in the critical range where we have to show the failure of WLP, they are correct.

Case 1:

(i) Y_d has h-vector

deg	0	1	2	$\cdots$	$r-2$	$r-1$	r	$\cdots$	$2r-4$	$2r-3$	$2r-2$
Δh_{Y_d}	1	2	3	$\cdots$	$r-1$	r	$r-1$	$\cdots$	3	2	1

In particular, the socle degree is $2r - 2 = d - 1$.

(ii) Z_d is level with socle type 3 and socle degree $2r - 1 = d$. It has minimal free resolution (over $S = K[x_1, x_2, x_3]$) which can be read immediately from the diagram above:

$$0 \to S(-2r-1)^3 \to \begin{pmatrix} S(-r-2)^2 \\ \oplus \\ S(-2r+1) \\ \oplus \\ S(-2r) \end{pmatrix} \to I_{Z_d} \to 0.$$

(iii) Z_d has h-vector

deg	0	1	2	$\cdots$	$r+1$	$r+2$	$r+3$	$\cdots$	$2r-3$	$2r-2$	$2r-1$
Δh_{Z_d}	1	2	3	$\cdots$	$r+2$	$r+1$	r	$\cdots$	6	5	3

Case 2:

(i) Y_d has h-vector

deg	0	1	2	$\cdots$	$r-2$	$r-1$	r	$r+1$	$\cdots$	$2r-3$	$2r-2$	$2r-1$
Δh_{Y_d}	1	2	3	$\cdots$	$r-1$	r	r	$r-1$	$\cdots$	3	2	1

In particular, the socle degree is $2r - 1 = d - 1$.

(ii) Z_d is level with socle type 3 and socle degree $2r = d$. It has minimal free resolution (over $S = K[x_1, x_2, x_3]$) which can be read immediately from the diagram above:

$$0 \to S(-2r-2)^3 \to \begin{pmatrix} S(-r-3)^2 \\ \oplus \\ S(-2r)^2 \end{pmatrix} \to I_{Z_d} \to 0.$$

(iii) Z_d has h-vector

deg	0	1	2	$\cdots$	$r+2$	$r+3$	$r+4$	$\cdots$	$2r-2$	$2r-1$	$2r$
Δh_{Z_d}	1	2	3	$\cdots$	$r+3$	$r+2$	$r+1$	$\cdots$	7	6	3

Choose a general point in $\mathbb{P}^3$ and consider the cones $B_d \subset C_d$ in $\mathbb{P}^3$ over $Y_d \subset Z_d \subset \mathbb{P}^2$. Then B_d and C_d are arithmetically Cohen–Macaulay curves, and have the same h-vector and graded Betti numbers (over $R = k[x_0, x_1, x_2, x_3]$) as Y_d and Z_d do, respectively, over S.

Let Z be the union of a general hyperplane section X_1 of B_d and a general hyperplane section X_2 of C_d (see Remark 2.2). We will show that Z satisfies all the claims in the statement of the theorem. We will prove it for d odd (Case 1), and leave it to the reader to check the case d even, which is completely analogous.

We first compute the h-vector. Observe that X_1 (resp. X_2) has the same h-vector as Y_d (resp. Z_d), and that in particular $X_1 \subset C_d$. Hence Z is obtained by basic double G-linkage, and if L is the linear form defining the hyperplane of X_2 then we have

$$I_Z = L \cdot I_{X_1} + I_{C_d}.$$

The h-vector of Z is thus obtained by the computation

deg	0	1	2	3	$\cdots$	$r-1$	r	$r+1$	$r+2$	$\cdots$	$2r-3$	$2r-2$	$2r-1$	$2r$
Δh_{X_1}		1	2	3	$\cdots$	$r-1$	r	$r-1$	$r-2$	$\cdots$	3	2	1	
Δh_{X_2}	1	2	3	4	$\cdots$	r	$r+1$	$r+2$	$r+1$	$\cdots$	6	5	3	
Δh_Z	1	3	5	7	$\cdots$	$2r-1$	$2r+1$	$2r+1$	$2r-1$	$\cdots$	9	7	4	

(Notice the shift in Δh_{X_1}.) In particular, note that if T is the ring of an Artinian reduction and J is the corresponding ideal, then $\dim(T/J)_r = \dim(T/J)_{r+1} = 2r+1$. This is where we will focus our attention to show the failure of WLP.

We first check that Z is level, of type 4. This is immediate from the mapping cone associated with the diagram in Figure 1. This gives a minimal (in this case) free resolution of I_Z.

We now verify the failure of WLP to hold for any Artinian reduction of Z. By Proposition 2.3, it is enough to show that a general line λ imposes fewer than the expected $r + 2$ conditions on $|(I_Z)_{r+1}|$.

Let $I_\lambda = (L_1, L_2)$ as in Proposition 2.3. By assumption, L_1 avoids the points of Z and L_2 is general. Hence λ meets the plane of X_2 at a single point P disjoint from X_2. Now, thanks to the h-vector computation of Z_d (which is the same as that of X_2), any form of degree $r + 1$ containing Z must contain the plane of X_2 as a factor, since otherwise it would restrict to a curve of degree $r + 1$ containing X_2. Hence P is a base point of the linear system $|(I_Z)_{r+1}|$. Therefore λ imposes at most $r + 1$ conditions on $|(I_Z)_{r+1}|$, and we have shown that the map induced by L_2 on the Artinian reduction T by L_1 fails to be surjective, by Proposition 2.3. Since $\dim(T/J)_r = \dim(T/J)_{r+1} = 2r + 1$, this means that WLP fails for Z.

One checks that this works for $r \geq 3$ in both the even and odd cases, completing the proof. ∎

Theorem 3.2 *For any* odd $d \geq 7$, *there is a reduced set of points $Z \subset \mathbb{P}^3$ such that any Artinian reduction of Z is level of socle degee d and socle type 3, but fails to have the Weak Lefschetz Property.*

$$0 \to R(-2r-2)^3 \to \begin{matrix} R(-r-3)^2 \oplus R(-2r) \\ \oplus \\ R(-2r) \oplus R(-2r-1) \end{matrix} \to I_{C_d}(-1) \to 0$$

$$\uparrow$$

$$0 \to \begin{matrix} R(-2r-2) \\ \oplus \\ R(-r-2)^2 \oplus R(-2r-1) \end{matrix} \to \begin{matrix} R(-2r-1)^3 \\ \oplus \\ R(-2) \oplus R(-r-1)^2 \end{matrix} \to \begin{matrix} R(-r-2)^2 \oplus R(-2r+1) \oplus R(-2r) \\ \oplus \\ I_{X_1}(-1) \end{matrix} \to I_{C_d} \to 0$$

Figure 1

Proof The proof follows along very similar lines to that of Theorem 3.1, so we will highlight primarily the differences. Let $d = 2r - 1$. We again use basic double G-linkage to produce Z as the union of subsets X_1, X_2, both degenerate but on different planes, so that Z is level of type 3, and the h-vector of Z is computed by

deg	0	1	2	3	$\cdots$	$r-2$	$r-1$	r	$r+1$	$r+2$	$\cdots$	$2r-4$	$2r-3$	$2r-2$	$2r-1$
Δh_{X_1}		1	2	3	$\cdots$	$r-2$	$r-1$	r	$r-1$	$r-2$	$\cdots$	4	3	2	1
Δh_{X_2}	1	2	3	4	$\cdots$	$r-1$	r	$r+1$	$r+2$	$r+1$	$\cdots$	7	6	4	2
Δh_Z	1	3	5	7	$\cdots$	$2r-3$	$2r-1$	$2r+1$	$2r+1$	$2r-1$	$\cdots$	11	9	6	3

Again, WLP will be shown to fail by the failure of the multiplication by a general linear form from degree r to degree $r + 1$ to be surjective. To achieve this, we start by constructing a set of points $Z_d \subset \mathbb{P}^2$ that is level of type 2 and contains a subset Y_d which is a complete intersection of type (r, r). However, this cannot be achieved by monomial ideals, and we instead need a completely different argument.

As a first step, we show that if Y_0 is a general complete intersection in $\mathbb{P}^2$ of type $(3, 3)$ and Y_d is a general complete intersection of type (r, r), then there is a reduced complete intersection of type $(r + 2, r + 2)$ containing $Y_0 \cup Y_d$. To see this, consider the exact sequence

$$0 \to [I_{Y_0} \cap I_{Y_d}]_{r+2} \to [I_{Y_0}]_{r+2} \oplus [I_{Y_d}]_{r+2} \to [I_{Y_0} + I_{Y_d}]_{r+2} \to 0.$$

It is easy to see that $\dim[I_{Y_0}]_{r+2} = \binom{r+4}{2} - 9$ and that $\dim[I_{Y_d}]_{r+2} = 12$. On the other hand, $I_{Y_0} + I_{Y_d}$ is a general choice of forms of degrees $3, 3, r, r$. By a result of Anick [1] we know the Hilbert function of an ideal of general forms of any specified degree in three variables, and we can verify that $\dim[I_{Y_0} + I_{Y_d}]_{r+2} = \binom{r+4}{2}$, *i.e.*, the Hilbert function is zero in degree $r + 2$. This gives

$$\dim[I_{Y_0} \cap I_{Y_d}]_{r+2} = \binom{r+4}{2} - 9 + 12 - \binom{r+4}{2} = 3.$$

We claim that a general choice of two forms in $[I_{Y_0} \cap I_{Y_d}]_{r+2}$ will define a *reduced* complete intersection of type $(r + 2, r + 2)$. Note that to specify a general complete intersection of type $(3, 3)$, it is equivalent to specifying eight general points in $\mathbb{P}^2$; then the ninth point is uniquely determined. It is clear that a general choice of two forms in $[I_{Y_d}]_{r+2}$ (which has dimension 12) defines a reduced complete intersection, and that imposing the further passing through eight general points drops the dimension to 4 and preserves the property that a general choice of two defines a reduced complete intersection. Now the ninth point of Y_0 is determined, but it still imposes a new condition on $[I_{Y_d}]_{r+2}$ by the computation above. Since everything has been chosen generically so far, we preserve the property of reducedness.

Hence by taking two general elements of the linear system $[I_{Y_0} \cap I_{Y_d}]_{r+2}$, we can link Y_0 to a reduced set of points Z_d that is level of type 2, has h-vector given by the line ΔH_{X_2} in the computation at the beginning of this proof, and contains the complete intersection Y_d of type (r, r). By the general choices, everything is reduced.

We then proceed as in Theorem 3.1, taking cones over $Y_d \subset Z_d$ to produce arithmetically Cohen–Macaulay curves curves $B_d \subset C_d \subset \mathbb{P}^3$ over $Y_d \subset Z_d \subset \mathbb{P}^2$, and form the union $Z = X_1 \cup X_2$, where X_1 is a general hyperplane section of B_d and X_2 is a general hyperplane section of C_d. The verification of the Hilbert function, the level property and the WLP is identical to that in Theorem 3.1 and is left to the reader.

We remark that for even socle degree this method does not yield a complete intersection containing the analogous $Y_0 \cup Y_d$, and so it fails to work. In the next result, we will use a different construction for the even case.   ∎

Theorem 3.3   *For any* even $d \geq 12$, *there is a reduced set of points $Z \subset \mathbb{P}^3$ such that any Artinian reduction of Z is level of socle degree d and socle type 3, but fails to have the Weak Lefschetz Property.*

Proof   The methods are rather different, even if the underlying construction is based on basic double G-linkage. We will use basic double G-linkage to produce Z as the union of subsets X_1, X_2. This time, however, neither is degenerate. One, X_1, will be arithmetically Gorenstein (not a complete intersection), and the other, X_2, will be a quadric hypersurface section of a suitable arithmetically Cohen–Macaulay curve in $\mathbb{P}^3$. Another difference is that this time it will not be enough to take a cone over a set of points in $\mathbb{P}^2$. Rather, we will apply a direct geometric construction, and from that use liaison methods to produce the reduced, arithmetically Gorenstein set of points that we need.

Let $d = 2r$. We must construct an arithmetically Cohen–Macaulay curve with the following properties:

(i) C is a stick figure, *i.e.,* it is a union of lines with at most two passing through any given point.

(ii) C contains as a subset an arithmetically Cohen–Macaulay curve Y with maximal h-vector $(1, 2, 3, \ldots, r-1, r)$. We will construct on Y (and hence on C) a reduced, arithmetically Gorenstein set of points, X_1, with properties described below.

(iii) Any Artinian reduction of R/I_C is a compressed level algebra with socle type 2 and socle degree $2r - 1$. Any Artinian reduction of R/I_{X_1} is a compressed Gorenstein algebra with socle degree $2r - 2 = d - 2$. (But notice that C has codimension two and X_1 has codimension three.)

Once we have these basic objects, we will construct Z with basic double G-linkage as follows: $Z = X_1 \cup X_2$, where X_2 is a general *quadric* hypersurface section of C.

First let us construct C. We have three cases: $r \equiv 0 \pmod 3$, $r \equiv 1 \pmod 3$, and $r \equiv 2 \pmod 3$. In all cases, C will be residual to a complete intersection D of type (t, t), where

$$t = \begin{cases} \frac{2}{3}r & \text{if } r \equiv 0 \pmod 3, \\ \frac{2}{3}(r-1) + 1 & \text{if } r \equiv 1 \pmod 3, \\ \frac{2}{3}(r-2) + 1 & \text{if } r \equiv 2 \pmod 3, \end{cases}$$

inside a complete intersection E of type

$$
\begin{cases}
\left(\frac{4r}{3}, \frac{4r}{3} + 1\right) & \text{if } r \equiv 0 \ (\mathrm{mod}\ 3), \\
\left(\frac{4(r-1)}{3} + 2, \frac{4(r-1)}{3} + 2\right) & \text{if } r \equiv 1 \ (\mathrm{mod}\ 3), \\
\left(\frac{4(r-2)}{3} + 3, \frac{4(r-2)}{3} + 3\right) & \text{if } r \equiv 2 \ (\mathrm{mod}\ 3).
\end{cases}
$$

From a standard mapping cone argument it is easy to see that such a C is level of type 2, since it is linked to a complete intersection of balanced type (t, t). We will need information about the Hilbert function of such a C. We will work it out for $r \equiv 0 \ (\mathrm{mod}\ 3)$, and leave the remaining cases to the reader. We now compute the h-vector, using the formula for Hilbert functions under linkage [7, 21]. (We read the h-vector of D from right to left, noting that it is symmetric, and that of C from left to right.)

deg	0	1	2	$\cdots$	$\frac{4r}{3} - 2$	$\frac{4r}{3} - 1$	$\frac{4r}{3}$	$\frac{4r}{3} + 1$	$\frac{4r}{3} + 2$	$\cdots$
E	1	2	3	$\cdots$	$\frac{4r}{3} - 1$	$\frac{4r}{3}$	$\frac{4r}{3}$	$\frac{4r}{3} - 1$	$\frac{4r}{3} - 2$	$\cdots$
D							0	1	2	$\cdots$
C	1	2	3	$\cdots$	$\frac{4r}{3} - 1$	$\frac{4r}{3}$	$\frac{4r}{3}$	$\frac{4r}{3} - 2$	$\frac{4r}{3} - 4$	$\cdots$

deg	$\cdots$	$2r - 2$	$2r - 1$	$2r$	$2r + 1$	$\cdots$	$\frac{8r}{3} - 3$	$\frac{8r}{3} - 2$	$\frac{8r}{3} - 1$
E		$\frac{2r}{3} + 2$	$\frac{2r}{3} + 1$	$\frac{2r}{3}$	$\frac{2r}{3} - 1$	$\cdots$	3	2	1
D		$\frac{2r}{3} - 2$	$\frac{2r}{3} - 1$	$\frac{2r}{3}$	$\frac{2r}{3} - 1$	$\cdots$	3	2	1
C	$\cdots$	4	2						

Now we want C to contain as a subcurve an arithmetically Cohen–Macaulay curve with maximal h-vector $(1, 2, 3, \ldots, r)$. In order to be able to construct C with the above numerical properties and with a subcurve having this h-vector (and satisfying the other requirements listed above), we use a trick similar to that used in [24], and related to liftings of monomial ideals. That is, consider two families of planes in $\mathbb{P}^3$, $L_1, \ldots, L_{4r/3}$ and $M_1, \ldots, M_{(4r/3)+1}$, where the L_i and the M_i are chosen generically. In Figure 2, each horizontal line represents an L_i and each vertical line represents an M_j, and the intersection point represents the line of intersection of L_i and M_j. It is clear that if we denote by $L := \prod L_i$ and $M := \prod M_j$ then (L, M) is a complete intersection of type $(\frac{4r}{3}, \frac{4r}{3} + 1)$ which is a stick figure.

There are r solid dots along the top line, decreasing down to 1 on the r-th line; these dots represent the lines that form the subcurve Y required in (ii) above. It is not hard, using methods of [23, 24], to check that Y is arithmetically Cohen–Macaulay with maximal h-vector $(1, 2, \ldots, r)$. The open dots represent the residual complete intersection of type (t, t) (in our case, with $r \equiv 0 \ (\mathrm{mod}\ 3)$, we have $t = \frac{2r}{3}$). Hence the dots that are not open represent our curve C, and clearly it has the required subcurve.

Now, observe that the initial degree of I_Y is r, and that it contains $r + 1$ minimal generators of degree r (this can be read immediately from the h-vector). By looking at suitable products of L_i and of M_j, we can link Y to a residual curve Y' inside a
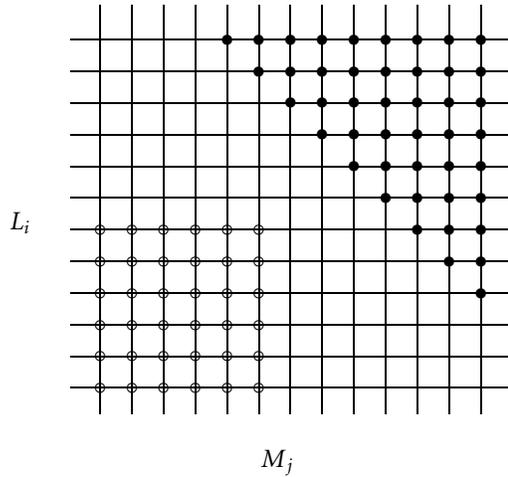
L_i

M_j

Figure 2

complete intersection of type $(r, r+1)$ (it does not matter if Y' overlaps with the open dots). Note that Y' has the same h-vector as Y, and that the complete intersection is a stick figure. Hence the sum $I_Y + I_{Y'}$ is the saturated ideal of a reduced, arithmetically Gorenstein set of points X_1 (as was done in [24]) with h-vector

$$\left(1, 3, 6, \ldots, \binom{r}{2}, \binom{r+1}{2}, \binom{r}{2}, \ldots, 6, 3, 1\right)$$

that lies on C (in particular).

To compute the h-vector of X_2 (the quadric hypersurface section of C), we "integrate" the h-vector $\Delta^2 h_C$ of C and then take suitable differences. We obtain (focusing on the important part of the computation)

deg	0	1	2	3	$\cdots$	$\frac{4r}{3}-2$	$\frac{4r}{3}-1$	$\frac{4r}{3}$	$\cdots$	$2r-3$	$2r-2$	$2r-1$	$2r$
$\Delta^2 h_C$	1	2	3	4	$\cdots$	$\frac{4r}{3}-1$	$\frac{4r}{3}$	$\frac{4r}{3}$	$\cdots$	6	4	2	
Δh_C	1	3	6	10	$\cdots$	$\binom{4r/3}{2}$	$\binom{(4r/3)+1}{2}$	$\binom{(4r/3)+2}{2}-1$	$\cdots$	$e-6$	$e-2$	e	e
Δh_{X_2}	1	3	5	7	$\cdots$	$\frac{8r}{3}-3$	$\frac{8r}{3}-1$	$\frac{8r}{3}$	$\cdots$		10	6	2

(where e is the degree of C, whose precise value is not important here). Note that we are assuming that $d \geq 12$, so $r \geq 6$ and $r+1 \leq \frac{4r}{3}-1$. We now have all the ingredients, and we again use basic double G-linkage, with the modifications mentioned already. We obtain the computation

deg	0	1	2	3	4	$\cdots$	r	$r+1$	$r+2$	$\cdots$	$2r-2$	$2r-1$	$2r$
Δh_{X_1}		1	3	6		$\cdots$	$\binom{r}{2}$	$\binom{r+1}{2}$	$\binom{r}{2}$	$\cdots$	6	3	1
Δh_{X_2}	1	3	5	7	9	$\cdots$	$2r+1$	$2r+3$	$2r+5-\delta$	$\cdots$	10	6	2
Δh_Z	1	3	6	10	15	$\cdots$	$\binom{r+2}{2}$	$\binom{r+3}{2}$	$\binom{r+2}{2}+4-\delta$	$\cdots$	16	9	3

where $\delta = 1$ if $r = 6$ and $\delta = 0$ otherwise. The important thing to notice is that from degree $r + 1$ to $r + 2$, Δh_{X_1} is decreasing, Δh_{X_2} is increasing, and Δh_Z is decreasing. Hence, because of the latter, we need surjectivity if WLP is to hold. To show that we in fact do not have surjectivity, by Proposition 2.3 we must verify that a sufficiently general line λ does not impose $r + 3$ independent conditions on $(I_Z)_{r+2}$. (By "sufficiently general" we mean that the first linear form only has to miss the points of Z, and the second is general.) We focus on the quadric cutting out X_2, and more precisely on the two points of intersection of λ with this quadric. It is not hard to see that when $r = 6$, one of the two points imposes a new condition but the second does not; when $r > 6$, neither point imposes a new condition.

Cases $r \equiv 1 \pmod 3$ and $r \equiv 2 \pmod 3$ are similar and are left to the reader. ■

Remark 3.4 The reader will note from the last example that we can have consecutive values of the h-vector of Z such that the second is strictly smaller than the first, and still have multiplication by a general linear form fail to be surjective.

4 Non-Unimodality

Zanello [26] proved the following theorem.

Theorem 4.1 ([26, Theorem 3]) *There exist non-unimodal level h-vectors of codimension 3.*

He did this by constructing an explicit example [26, Example 2]. In this section we show how, despite the very algebraic nature of Zanello's non-unimodal example, we can mimic his essential numerical data geometrically to produce a level set of points with non-unimodal h-vector. The h-vector of our example is exactly the same as the Hilbert function of Zanello's example.

Example 4.2 Let $C \subset \mathbb{P}^3$ be an arithmetically Cohen–Macaulay curve with h-vector $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$. That is, C has degree 55 and arithmetic genus 276. We further assume that C contains as a subset an arithmetically Gorenstein set of points X_1 with h-vector $(1, 3, 6, 10, 6, 3, 1)$. This can be constructed, for example, as follows. We start with an arithmetically Cohen–Macaulay curve C_1 with h-vector $(1, 2, 3, 4)$, and link to a similar curve C_2 via a complete intersection of type $(4, 5)$. The ideal $I_G := I_{C_1} + I_{C_2}$ is the saturated ideal of an arithmetically Gorenstein zero-dimensional scheme X_1 with the stated h-vector, which, of course, then lies on both C_1 and C_2. By making general choices, we can arrange that X_1, C_1 and C_2 be reduced [12]. By using liaison addition, we can add plane curves to C_1 to obtain an arithmetically Cohen–Macaulay curve C with the stated h-vector, and since $C \supset C_1 \supset X_1$, we have the desired inclusion. We do not need that C be irreducible, but we do need that it be reduced.

Now we let X_2 be a general cubic hypersurface section of C. The h-vector of X_2 can be computed from that of C, and is given below. Let $Z_1 = X_1 \cup X_2$. We make the

computation

deg	0	1	2	3	4	5	6	7	8	9	10	11
Δh_{X_1}				1	3	6	10	6	3	1		
Δh_{X_2}	1	3	6	9	12	15	18	21	24	27	19	10
Δh_{Z_1}	1	3	6	10	15	21	28	27	27	28	19	10

Now, both X_1 and X_2 are level, but Z_1 is not. Its Artinian reduction has one-dimensional socle in degree 9 and 10-dimensional socle in degree 11. However, we can find a subset Z of Z_1 with any truncated Hilbert function [10], and in particular, a subset with h-vector $(1, 3, 6, 10, 15, 21, 28, 27, 27, 28)$. Since I_Z agrees with I_{Z_1} in all degrees ≤ 8, the only socle elements for the Artinian reduction of R/I_Z occur in degree 9, *i.e.*, Z is level.

Hence we have shown the following.

Theorem 4.3 *There exists a reduced, level set of points in $\mathbb{P}^3$ whose Artinian reduction has non-unimodal Hilbert function.*

Remark 4.4 Zanello also gives a more general result, namely that Artinian level algebras can be found which not only fail to be unimodal, but in fact have arbitrarily many "valleys." We have tried to reproduce this result in the context of points in $\mathbb{P}^3$. Unfortunately, our methods have thus far been unable to extend the result beyond one "valley."

Remark 4.5 Zanello mentions that Iarrobino found an example of an Artinian, codimension 3 level algebra with socle type 6 and non-unimodal Hilbert function. More recently, Iarrobino informed us that he has found an example of an Artinian, codimension 3 level algebra with socle type 5 and socle degree 63. Both examples have been verified on the computer by Art Weiss, a graduate student at Tufts University.

We have not been able to find any example other than the one given above, in the context of reduced sets of points in $\mathbb{P}^3$. We have not seen Iarrobino's example, but perhaps seeing his approach would suggest a way to apply our methods to find reduced sets of points with non-unimodal h-vector and smaller socle type.

5   Other Unusual Behavior

In this section we exhibit other unusual behavior that can be obtained for a level reduced set of points, answering some natural questions. We also show some limitations of our method.

We begin by showing that even for reduced sets of points, it is not true that WLP implies SLP.

Example 5.1 Let $C_1 \subset C_2$ be arithmetically Cohen–Macaulay curves with h-vectors $(1, 2, 3, 3, 2, 1)$ and $(1, 2, 3, 4, 5, 5, 5)$, respectively. Then C_1 will be a complete intersection of type $(3, 4)$. To produce these curves, we will again use our trick of beginning with points in $\mathbb{P}^2$ and taking a cone. So let Z_1 be a general complete intersection of type $(3, 4)$ in $\mathbb{P}^2$. Let X be a general set of eight points in $\mathbb{P}^2$. Then $Z_1 \cup X$ has h-vector $(1, 2, 3, 4, 5, 5)$. Let F be a plane curve of degree 5 containing $Z_1 \cup X$. Let Y be the complete intersection set of points cut out by F and a general quadric. Then $Z_2 := (Z_1 \cup X) \cup Y$ is a basic double link, and has the desired h-vector given above. Choosing a general point in $\mathbb{P}^3$ and taking cones as before, we obtain the desired curves C_1, C_2.

Let X_1, X_2 be general hyperplane sections (with different hyperplanes!) of C_1, C_2 and let $Z := X_1 \cup X_2$. Then as before, Z is obtained by basic double G-linkage, and its h-vector is computed by

deg	0	1	2	3	4	5	6
Δh_{X_1}		1	2	3	3	2	1
Δh_{X_2}	1	2	3	4	5	5	5
Δh_Z	1	3	5	7	8	7	6

We claim (and have verified on CoCoA) that Z has WLP, but the multiplication from degree 3 to degree 5 by a general quadric is not surjective, and hence Z does not have SLP. We must check that (i) WLP holds (rather than fails, as has been the case until now in this paper), and (ii) SLP fails.

For (i) we defer to CoCoA, since our methods for checking injectivity are not as effective, but it could also be checked by hand. Once CoCoA verifies that a specific linear form gives the predicted multiplication, then it is true for the general linear form. For (ii) it is not enough to check that a "random" form produced by CoCoA fails to give the expected multiplication. Instead, we take λ to be a general complete intersection of type $(1, 2)$ rather than $(1, 1)$, reflecting that first we have to reduce by a linear form to get the Artinian reduction, and then we have to multiply by a general quadric. Note that again it is a question of the failure to impose independent conditions. To check that SLP does not hold, we note that the first point of intersection of the conic λ with the plane of X_2 does impose a new condition, but then the second point does not. So we do not get the expected number of conditions and SLP fails, exactly as WLP failed in our other situations. We leave the details to the reader, although we point out that the computation above suggests that multiplication by a general linear form should be of maximal rank (by imagining a decomposition of the form suggested by the diagram), but multiplication by a general quadric should have rank at most $2 + 4 = 6$, rather than 7, from degree 3 to degree 5.

As a result of this example, we have shown the following.

Proposition 5.2 *There exist reduced, level zero-dimensional schemes whose general Artinian reductions have WLP but not SLP.*

The following result shows that the value of the h-vector of our constructed set of points can be constant for arbitrarily many consecutive values (but the value gets

correspondingly bigger), and WLP can correspondingly fail for arbitrarily many consecutive degrees.

Proposition 5.3 *Let $d \geq 2$ be a positive integer. Then there exists a reduced, level set of points $Z \subset \mathbb{P}^3$ whose h-vector has a constant value in each of d consecutive degrees, and for which the corresponding $d-1$ maps in* any *Artinian reduction all fail to be surjective (and hence WLP fails in each of these degrees).*

Proof We choose a reduced complete intersection $Y_1 \subset \mathbb{P}^2$ of type (d, d), and extend it to a set of points $Y_2 \subset \mathbb{P}^2$ by adding $\binom{2d+1}{2} - d^2 = d(d+1)$ *general* points. Choose a general point $P \in \mathbb{P}^3$ and let C_1, C_2 be the cones over Y_1, Y_2 with vertex P. Let X_1, X_2 be general hyperplane sections (different hyperplanes) of C_1 and C_2, respectively, and let $Z = X_1 \cup X_2$. The h-vector of Z is computed by

deg	0	1	2	3	$\cdots$	$d-1$	d	$d+1$	$\cdots$	$2d-3$	$2d-2$	$2d-1$
Δh_{X_1}		1	2	3	$\cdots$	$d-1$	d	$d-1$	$\cdots$	3	2	1
Δh_{X_2}	1	2	3	4	$\cdots$	d	$d+1$	$d+2$	$\cdots$	$2d-2$	$2d-1$	$2d$
Δh_Z	1	3	5	7	$\cdots$	$2d-1$	$2d+1$	$2d+1$	$\cdots$	$2d+1$	$2d+1$	$2d+1$

We see that there are d consecutive degrees with value $2d+1$. The fact that each of the corresponding $d-1$ maps (multiplication by a general linear form) in the Artinian reduction all fail to be surjective comes as before, using Proposition 2.3 and the fact that the point of intersection of λ with the plane of X_2 is forced to be part of the base locus in the corresponding degree, and hence fails to impose an additional condition. ∎

Remark 5.4 The first case of Proposition 5.3, namely the case $d = 2$, which produces the h-vector $(1, 3, 5, 5)$, should be isomorphic to the distraction (lifting) of Zanello's example coming from a monomial ideal. We have not verified this, but numerically the examples are the same. See also Example 2.4.

Example 5.5 One of the cases left open in this paper is whether there exists a level reduced set of points whose Artinian reduction has type 2 but does not have WLP. We now give a construction that, according to the "philosophy" described in the introduction, *should* fail to have WLP, but does not. Consider a reduced arithmetically Cohen–Macaulay curve C_1 in $\mathbb{P}^3$ with h-vector $(1, 2, 3)$; it has degree 6 and arithmetic genus 3. Then C_1 can be linked by a sufficiently general complete intersection C of type $(3, 4)$ to another curve, C_2, of the same degree and genus. The intersection of C_1 and C_2 is a reduced, arithmetically Gorenstein set of points X_1 [12] with h-vector $(1, 3, 6, 3, 1)$. On the other hand, X_1 lies on the complete intersection curve C. Taking a cubic hypersurface section of C, we obtain a complete intersection set of points X_2. The union $Z = X_1 \cup X_2$ is level of type 2, and has Hilbert function computed by

deg	0	1	2	3	4	5	6	7
Δh_{X_1}				1	3	6	3	1
Δh_{X_2}	1	3	6	8	8	6	3	1
Δh_Z	1	3	6	9	11	12	6	2

In order for the Artinian reduction of this set of points to have WLP, in particular the homomorphism from degree 4 to degree 5 induced by multiplication by a general linear form must be injective, *i.e.,* have rank 11. From the successful applications of this method in the previous sections, we would hope that the contribution to the rank from X_1 would be 3 and from X_2 to be 6, combining for a rank of 9 rather than 11. However, CoCoA confirms that indeed the rank is 11, and, in fact, that the Artinian reduction of Z does have WLP. The problem comes from the fact that the three points of intersection of the general line λ must fail to impose independent conditions on forms of degree 7, and this degree is too large for the geometric type of obstructions that helped us before. We have found many such examples.

Example 5.6 We now illustrate that Proposition 2.5 can also be used in a geometric way, this time to prove the failure of *injectivity* for a level algebra. However, we have not found as many useful applications as we did for Proposition 2.3.

Let $C_1 \subset \mathbb{P}^3$ be a reduced complete intersection curve of type $(2, 2)$, and let C_2 be a reduced complete intersection curve of type $(5, 5)$ containing C_1. Let X_1 be a general hyperplane section of C_1, cut out by a general linear form L, and let X_2 be a general quadric section of C_2, cut out by a general quadric Q. Let $Z' = X_1 \cup X_2$. The h-vector of Z' is computed by

deg	0	1	2	3	4	5	6	7	8	9
Δh_{X_1}			1	2	1					
Δh_{X_2}	1	3	5	7	9	9	7	5	3	1
$\Delta h_{Z'}$	1	3	6	9	10	9	7	5	3	1

Using the machinery described above, we note first that Z' is not level: it has socle in degree 4 and in degree 9. However, as before (Example 4.2), we can choose a subset Z of Z' by truncation, with h-vector $(1, 3, 6, 9, 10)$, and this will be level of type 10. We will show that for any Artinian reduction of R/I_Z, the multiplication by a general linear form from degree 3 to degree 4 is not injective, thereby showing that WLP does not hold. Since R/I_Z agrees with $R/I_{Z'}$ in degrees ≤ 4, it is enough to show this for $R/I_{Z'}$.

By Proposition 2.5, we must show that for a sufficiently general line λ, $(I_{Z'} \cdot I_\lambda)_4 \neq (I_{Z'} \cap I_\lambda)_4$ (where $I_\lambda = (L_1, L_2)$ and L_1 is chosen simply to avoid the points of Z', while L_2 is chosen generically). Since $\dim(I_{Z'})_3 = 1$ (with basis LQ), we note that $(I_{Z'} \cdot I_\lambda)_4$ is generated by $(LQ \cdot L_1, LQ \cdot L_2)$. So it is enough to show that $\dim(I_{Z'} \cap I_\lambda)_4 > 2$. By considering multiplication by Q, we see that it is enough to show that $\lambda \cup X_1$ lies on at least three independent quadrics. But λ alone lies on 7 quadrics, and X_1 imposes at most 4 additional conditions, so we are finished.

6 Open questions

We remark that our methods become more and more difficult to apply as the socle type decreases. In this section we list some questions that remain open, having to do with low socle type.

Zanello asks [26, Question 6] "What is the maximum type t_0 such that all the codimension 3 level h-vectors of type $t \leq t_0$ are unimodal? In particular, is there always unimodality for $t = 2$?" The following questions may be viewed as a continuation of Zanello's line of questioning.

In codimension 3, it is not known whether every Artinian Gorenstein algebra has the WLP, although it is known that they all have unimodal Hilbert function [25]. The only result to date in this direction comes from [13], where it is shown that every height 3 complete intersection has WLP. It is also not known if there is an Artinian level algebra of type 2 that fails to have WLP.[1] Of course, as a result it is not known if there is a reduced set of points whose Artinian reduction has these properties, although it is known [12] that for any set of graded Betti numbers that occurs for an Artinian Gorenstein algebra of codimension 3, there is a reduced set of points in $\mathbb{P}^3$ with these graded Betti numbers. One can also ask, for any t_0, what is the smallest socle degree for which there is an Artinian level algebra (resp. a level set of points) with socle type t_0 and which fails to have WLP or fails to have unimodal h-vector.

In codimension 4 it is was shown by Ikeda [16] that not all graded Artinian Gorenstein algebras have WLP. It is not known if they all have unimodal Hilbert function. It is also not known if Ikeda's example lifts to points, or, more generally, if there is a reduced set of points in $\mathbb{P}^4$ whose Artinian reduction fails to have WLP, even if the Hilbert function is unimodal.

In codimension 5 and higher it is known that not all graded Artinian Gorenstein algebras have unimodal Hilbert function (see [2–4]). But again, it is not known if this extends to points. It would be extremely interesting to know if all arithmetically Gorenstein, reduced sets of points have (at least) the general Artinian reduction with WLP, because then, thanks to [24], we would have a complete description of the Hilbert functions of reduced, arithmetically Gorenstein sets of points.

It would be very interesting to know if we can use these (or related) methods to show that the postulation Hilbert scheme is reducible in some cases, for level sets of points. Basic double linkage was used in [22] to show the reducibility of the postulation Hilbert scheme in a different setting: there a fundamental role was, in fact, played by *forcing* the existence of socle elements in degrees other than the last one, so those results do not apply at all to our situation.

References

[1]   D. Anick, *Thin algebras of embedding dimension three.* J. Algebra **100**(1986), no. 1, 235–259.

[2]   D. Bernstein and A. Iarrobino, *A nonunimodal graded Gorenstein Artin algebra in codimension five.* Comm. Alg. **20**(1992), no. 8, 2323–2336.

---

[1]Recently an affirmative answer was given by Boij and Zanello [5].

[3]   M. Boij, *Graded Gorenstein Artin algebras whose Hilbert functions have a large number of valleys.* Comm. Algebra **23**(1995), no. 1, 97–103.

[4]   M. Boij and D. Laksov, *Nonunimodality of graded Gorenstein Artin algebras.* Proc. Amer. Math. Soc. **120**(1994), no. 4, 1083–1092.

[5]   M. Boij and F. Zanello, *Level algebras with bad properties.* Proc. Amer. Math. Soc. **135**(2007), no. 9, 2713–2722.

[6]   CoCoA: *A System for Doing Computations in Commutative Algebra.* Available at http://cocoa.dima.unige.it.

[7]   E. Davis, A. V. Geramita, and F. Orecchia, *Gorenstein algebras and the Cayley-Bacharach theorem.* Proc. Amer. Math. Soc. **93**(1985), no. 4, 593–597.

[8]   D. Eisenbud and S. Popescu, *Gale duality and free resolutions of ideals of points.* Invent. Math. **136**(1999), no. 2, 419–449.

[9]   A. V. Geramita, D. Gregory, and L. Roberts, *Monomial ideals and points in projective space.* J. Pure Appl. Algebra **40**(1986), no. 1, 33–62.

[10]   A. V. Geramita, P. Maroscia, and L. Roberts, *The Hilbert function of a reduced k-algebra.* J. London Math. Soc. **28**(1983), no. 3, 443–452.

[11]   A. V. Geramita, T. Harima, J. Migliore, and Y. S. Shin, *The Hilbert function of a level algebra*, Mem. Amer. Math. Soc. **186**(2007), no. 872.

[12]   A. V. Geramita and J. Migliore, *Reduced Gorenstein codimension three subschemes of projective space.* Proc. Amer. Math. Soc. **125**(1997), no. 4, 943–950.

[13]   T. Harima, J. Migliore, U. Nagel, and J. Watanabe, *The Weak and Strong Lefschetz Properties for Artinian K-algebras.* J. Algebra **262**(2003), no. 1, 99–126.

[14]   R. Hartshorne, *Connectedness of the Hilbert scheme.* Inst. Hautes Études Sci. Publ. Math. **29**(1966), 5–48.

[15]   A. Iarrobino and V. Kanev, *Power Sums, Gorenstein Algebras, and Determinantal Loci.* Lecture Notes in Mathematics 1721, Springer-Verlag, Berlin, 1999.

[16]   H. Ikeda, *Results on Dilworth and Rees numbes of Artinian local rings.* Japan. J. Math. **22**(1996), no. 1, 147–158.

[17]   J. Kleppe, J. Migliore, R. M. Miró-Roig, U. Nagel, and C. Peterson, *Gorenstein liaison, complete intersection liaison invariants and unobstructedness.* Mem. Amer. Math. Soc. **154**(2001), no. 732.

[18]   A. Lorenzini, *The minimal resolution conjecture.* J. Algebra **156**(1993), no. 1, 5–35.

[19]   F. S. Macaulay, *Some properties of enumeration in the theory of modular systems.* Proc. Lond. Math. Soc. **26**(1927) 531–555.

[20]   J. Migliore, *Submodules of the deficiency module.* J. Lond. Math. Soc. **48**(1993), no. 3, 396–414.

[21]   _____, *Introduction to Liaison Theory and Deficiency Modules.* Progress in Mathematics 165, Birkhäuser Boston, Boston, MA, 1998.

[22]   _____, *Families of reduced, zero-dimensional schemes.* Collect. Math **57**(2006), no. 2, 173–192.

[23]   J. Migliore and U. Nagel, *Lifting monomial ideals.* Comm. Algebra **28**(2000), no. 12, 5679-5701.

[24]   _____, *Reduced arithmetically Gorenstein schemes and simplicial polytopes with maximal Betti numbers.* Adv. Math. **180**(2003), no. 1, 1–63.

[25]   R. Stanley, *Hilbert Functions of graded algebras.* Adv. Math. **28**(1978), no. 1, 57–83.

[26]   F. Zanello, *A non-unimodal codimension 3 level h-vector.* J. Algebra **305**(2006), no. 2, 949–956.

Department of Mathematics
University of Notre Dame
Notre Dame, IN 46556
U.S.A.
e-mail: juan.c.migliore.1@nd.edu