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Theta constants associated to coverings of $\boldsymbol{P}^1$ branching at eight points

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Dedicated to Professor Hironori Shiga on his sixtieth birthday

Abstract

In this paper, we construct automorphic forms on the five-dimensional complex ball which give the inverse of the period map for cyclic 4-ple coverings of the complex projective line branching at eight points. We use theta constants associated to the Prym varieties of these coverings.

1. Introduction

Let $\mu_1, \ldots, \mu_n$ be rational numbers such that $0 < \mu_j < 1$, $\sum_{j=1}^{n} \mu_j = 2$, and let d be the common denominator of $\mu_1, \ldots, \mu_n$. For the cyclic d-ple covering C of $\boldsymbol{P}^1$ branching at n points with index $\mu = (\mu_1, \ldots, \mu_n)$ and a homology marking ϕ of C, the period $p(C, \phi)$ of a marked curve (C, ϕ) can be regarded as an element of the $(n-3)$-dimensional complex ball B_μ. The morphism p from the moduli space M_{marked} of marked curves (C, ϕ) to B_μ is called a period map. Since the period map p is equivariant under the monodromy group Γ_μ, it induces the morphism $M_{\text{marked}}/\Gamma_\mu \to B_\mu/\Gamma_\mu$. According to results of [DM86] and [Ter83, Ter85], the period map p is an isomorphism onto a Zariski open set of B_μ/Γ_μ if μ satisfies the condition

$$(1 - \mu_j - \mu_k)^{-1} \in \mathbb{Z} \cup \infty \quad \text{for } j \neq k. \tag{1.1}$$

There are finitely many such μ for $n \geqslant 5$: for $n \geqslant 9$ there is no μ, for $n = 8$ there is one, for $n = 7$ there is one, for $n = 6$ there are seven, and for $n = 5$ there are 27.

It is a natural demand to describe the inverse of the period map in terms of explicit automorphic forms with respect to Γ_μ for such μ. In fact, for several μ for $n = 5, 6$, the inverses of the period maps are studied and some of them are expressed in terms of theta constants; refer to [Koi03], [Mat89], [Mat01], [Pic83], [Shi88] and [Yos97]. In this paper, we construct automorphic forms on B_μ which give the inverse of the period map for the case $n = 8$, $\mu = (\frac{1}{4})^8 = (\frac{1}{4}, \ldots, \frac{1}{4})$. Note that all μ with $d = 4$ satisfying the condition (1.1) can be obtained by confluences of the branching index $(\frac{1}{4})^8$.

Before stating our main theorem, we define a projective embedding of the moduli space $M_{8\text{pts}}$ of eight points on $\boldsymbol{P}^1$, which is isomorphic to $M_{\text{marked}}/\Gamma_\mu$. We divide the set $\{1, \ldots, 8\}$ into a set of four pairs $\{\{j_1, j_2\}, \ldots, \{j_7, j_8\}\}$, which is called a $(2,2,2,2)$-partition of $\{1, \ldots, 8\}$. We denote the set of $(2,2,2,2)$-partitions of $\{1, \ldots, 8\}$ by $P(2^4)$ which has cardinality 105. We associate a polynomial $P_r = \prod_{k=1}^{4} (x_{j_{2k-1}} - x_{j_{2k}})$ for each $(2,2,2,2)$-partition $r = \{\{j_1, j_2\}, \ldots, \{j_7, j_8\}\}$. If we regard the x_j as affine coordinates of eight points on $\boldsymbol{P}^1$, then P_r are relative invariants under projective transformations of $\boldsymbol{P}^1$. Thus the set of polynomials $\{P_r\}_{r \in P(2^4)}$ induces a map $P : M_{8\text{pts}} \to \boldsymbol{P}^{104}$. It is shown in [Koi04] that P is an embedding.

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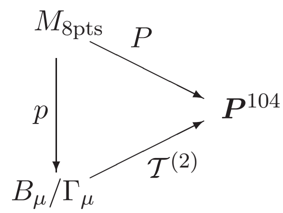
THEOREM 1.1 (Main theorem). *There exist 105 automorphic forms $\mathcal{T}_r^{(2)}$ with respect to Γ_μ such that the following diagram commutes.*



Here the map $\mathcal{T}^{(2)}$ is given by the 105 automorphic forms $\mathcal{T}_r^{(2)}$. Especially, the image of $\mathcal{T}^{(2)}$ coincides with that of P and $\mathcal{T}^{(2)} \circ P^{-1}$ gives the inverse of p.

Let us explain how to construct 105 automorphic forms in terms of theta constants. The curve C is of genus 9 and it can be regarded as a double cover of a hyperelliptic curve of genus 3. The period $p(C, \phi)$ is an element of the five-dimensional complex ball. We consider the Prym variety $\mathrm{Prym}(C)$ of C with respect to ρ^2, which is a six-dimensional sub-abelian variety of the Jacobian $J(C)$ of C obtained by the (-1)-eigenspaces of $H^0(C, \Omega^1)$ and $H_1(C, \mathbb{Z})$ for the action of ρ^2, where ρ is a generator of the group of covering transformations of $C \to \boldsymbol{P}^1$.

Since the polarization of $\mathrm{Prym}(C)$ is not principal, we construct 105 principally polarized abelian varieties isogenous to $\mathrm{Prym}(C)$ as follows. Let $\mathrm{Prym}(C)_{1-\rho}$ be the group of $(1-\rho)$-torsion points of $\mathrm{Prym}(C)$, which is isomorphic to $\boldsymbol{F}_2^6$ with a quadratic form. There are 105 three-dimensional totally singular subspaces Λ_r of $\mathrm{Prym}(C)_{1-\rho}$. For each Λ_r, $A_{L_r} = \mathrm{Prym}(C)/\Lambda_r$ is principally polarized. We show that there is an isomorphism between the automorphism groups $\mathfrak{S}_8$ of the marking of eight points, and the orthogonal group $O_6^+(\boldsymbol{F}_2)$ of $\mathrm{Prym}(C)_{1-\rho}$ yields a one-to-one correspondence between $P(2^4)$ and the set $\{\Lambda_r\}$.

For Λ_1 corresponding to $r_1 = \{\{12\}, \{34\}, \{56\}, \{78\}\} \in P(2^4)$, we study the behavior of the pull back $F_{m_1} = \iota^*(\vartheta_{m_1})$ of a theta function $\vartheta_{m_1}(z)$ associated to A_{L_1} with a characteristic m_1 under the composite map $\iota : C \to A_{L_1}$ of the canonical map $\mathrm{jac}^- : C \to \mathrm{Prym}(C)$ and the natural projection $\mathrm{Prym}(C) \to A_{L_1}$ (see § 4). There are twelve zeros of F_{m_1} in C; eight of them are in the set of fixed points of ρ. There are three theta functions $\vartheta_{m_j}(z)$ $(j = 2, 3, 4)$ on A_{L_1} such that the order of zero of $F_{m_j} = \iota^*(\vartheta_{m_j})$ at every fixed point of ρ is the same as that of F_{m_1}. By considering the four zeros of F_{m_j} not fixed by ρ, we can express the cross-ratio of x_1, x_2, x_5, x_6 in terms of the theta constants $\vartheta_{m_1}, \dots, \vartheta_{m_4}$.

The product $\mathcal{T}_1^{(2)} = \prod_{j=1}^4 \vartheta_{m_j}$ is an automorphic form with respect to the monodromy group Γ_μ. The period map p induces an isomorphism of groups $\mathrm{Aut}(M_{8\mathrm{pts}}) \simeq \mathrm{Aut}(B_\mu/\Gamma_\mu)$. For each $(2,2,2,2)$-partition $r \in P(2^4) = \mathrm{Stab}(r_1)\backslash\mathfrak{S}_8$, we define $\mathcal{T}_r^{(2)}$ by using the action of $\mathfrak{S}_8 \subset \mathrm{Aut}(M_{8\mathrm{pts}}) \simeq \mathrm{Aut}(B_\mu/\Gamma_\mu)$, where $\mathrm{Stab}(r_1)$ is the stabilizer of the partition r_1. In order to prove our main theorem, we investigate the action of $\mathrm{Stab}(r_1)$ on the space generated by the theta constants and relations between theta functions for A_{L_1} and those for A_{L_r}.

Notation 1.2. In this paper, the imaginary unit is denoted by i. For an element $\alpha \in \mathbb{C}$, $\exp(2\pi i\alpha)$ is denoted by $\boldsymbol{e}(\alpha)$. For a square matrix A, the vector consisting of the diagonal elements of A is denoted by A_0. For a vector $v = (v_1, \dots, v_k)$, the diagonal matrix

$$\begin{pmatrix} v_1 & & \\ & \ddots & \\ & & v_k \end{pmatrix}$$

is denoted by $\mathrm{diag}(v)$.

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2. The Prym variety of C

2.1 4-ple covering of $\boldsymbol{P}^1$ branching at eight points

Let C be the projective smooth model of an algebraic curve defined by

$$w^4 = \prod_{j=1}^{8}(z - x_j), \tag{2.1}$$

where $x_1, \ldots, x_8$ are distinct elements of $\mathbb{C}$. The curve C is of genus 9 since it can be regarded as a 4-ple covering of $\boldsymbol{P}^1$ branching at eight points. The automorphism

$$\rho : C \ni (z, w) \mapsto (z, iw) \in C \tag{2.2}$$

induces actions on $H^1(C, \mathbb{Q})$ and $H_1(C, \mathbb{Q})$. We denote the (-1)-eigenspaces of $H^1(C, \mathbb{Q})$ and $H_1(C, \mathbb{Q})$ of ρ^2 by $H^1(C, \mathbb{Q})^-$ and $H_1(C, \mathbb{Q})^-$, respectively. We put

$$H^1(C, \mathbb{Z})^- = H^1(C, \mathbb{Q})^- \cap H^1(C, \mathbb{Z}), \quad H_1(C, \mathbb{Z})^- = H_1(C, \mathbb{Q})^- \cap H_1(C, \mathbb{Z}).$$

Since the action ρ preserves the polarized rational Hodge structure of $H^1(C, \mathbb{Q})$, the (-1) eigensubspace $H^1(C, \mathbb{Q})^-$ of ρ^2 is a polarized rational sub-Hodge structure of $H^1(C, \mathbb{Q})$. The action of ρ induces an action on each factor of the Hodge decomposition

$$H^1(C, \mathbb{Z})^- \otimes \mathbb{C} \simeq H^0(C, \Omega^1)^- \oplus \overline{H^0(C, \Omega^1)^-}.$$

Proposition 2.1. *The multiplicity of the eigenvalue i (respectively $-i$) of ρ on $H^0(C, \Omega^1)^-$ is 5 (respectively 1).*

Proof. Differential 1-forms

$$\varphi_j = \frac{z^j\,dz}{w^3} \quad (j = 0, \ldots, 4), \quad \varphi_5 = \frac{dz}{w}, \quad \varphi_6 = \frac{dz}{w^2}, \quad \varphi_7 = \frac{z\,dz}{w^2}, \quad \varphi_8 = \frac{z^2\,dz}{w^2}$$

span the space $H^0(C, \Omega^1)$. We have $\rho^*(\varphi_j) = i\varphi_j$ for $j = 0, \ldots, 4$, $\rho^*(\varphi_5) = -i\varphi_5$ and $\rho^*(\varphi_k) = -\varphi_k$ for $k = 6, 7, 8$. □

We study the intersection form on $H_1(C, \mathbb{Z})^-$. For the moment, we assume that $x_j \in \mathbb{R}$ $(j = 1, \ldots, 8)$ and $x_1 < x_2 < \cdots < x_8$. Let $\mathcal{U}_0$ be $\boldsymbol{P}^1$ cut along the eight semi-lines

$$l_j = \{x_j + it \in \mathbb{C} \subset \boldsymbol{P}^1 \mid t \leqslant 0\}, \quad j = 1, \ldots, 8.$$

The curve C can be regarded as gluing $\mathcal{U}_0$ and the copies $\mathcal{U}_k = \rho^k(\mathcal{U}_0)$ $(k = 1, 2, 3)$ along $l_1, \ldots, l_8$. The branch of the function w is assigned as its value is in $i^k\mathbb{R}_+$ for $z \in (-\infty, x_1) \subset \mathcal{U}_k$. Let α_j $(1 \leqslant j \leqslant 7)$ be the interval $[x_j, x_{j+1}] \subset \mathbb{R}$ in the sheet $\mathcal{U}_0$ and α_8 be $[-\infty, x_1] \cup [x_8, \infty] \subset \mathbb{R}$ in the sheet $\mathcal{U}_0$. The orientation of α_j is given in Figure 1.

Then the 1-chain $A_j = (1 - \rho^2)\alpha_j$ is a cycle satisfying $\rho^2(A_j) = -A_j$. We put $B_j = \rho A_j$. Figure 1 shows that both of $\sum_{j=1}^{8} \alpha_j$ and $\sum_{j=1}^{8} \rho^j \alpha_j$ are boundaries of 2-chains. Therefore, we have

$$\sum_{j=1}^{8} A_j = 0 \quad \text{and} \quad \sum_{j=1}^{7} \frac{1 - \rho^j}{1 - \rho} A_j = 0$$

in $H_1(C, \mathbb{Z})^-$. The intersection matrix for $\{A_j, B_j\}_{j=1,\ldots,6}$ is given as

$$\begin{pmatrix} P & Q \\ -Q & P \end{pmatrix}, \tag{2.3}$$

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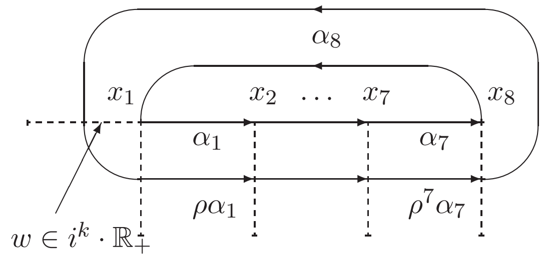

Figure 1. Paths α_j $(j = 1, \dots, 8)$.

where

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Proposition 2.2. *The set $\{A_j, B_j\}_{j=1,\dots,6}$ is a basis of $H_1(C, \mathbb{Z})^-$.*

Proof. The determinant of the matrix (2.3) is 2^6. By Fay's result [Fay73], we have this proposition. □

2.2 Polarization of the Prym variety

The polarized Hodge structure of $H^1(C, \mathbb{Z})$ defines the abelian variety

$$\mathrm{Prym}(C) = \mathrm{Prym}(C, \rho^2) = (H^0(C, \Omega^1)^-)^*/H_1(C, \mathbb{Z})^-,$$

which is called the Prym variety of C. Since the first homology group $H_1(\mathrm{Prym}(C), \mathbb{Z})$ of the Prym variety $\mathrm{Prym}(C)$ is isomorphic to $H_1(C, \mathbb{Z})^-$, the restriction $(\ ,\ ) : H_1(C, \mathbb{Z})^- \times H_1(C, \mathbb{Z})^- \to \mathbb{Z}$ of the intersection form on $H_1(C, \mathbb{Z})$ gives a polarization of $\mathrm{Prym}(C)$. Thus $H_1(C, \mathbb{Z})^- \simeq H_1(\mathrm{Prym}(C), \mathbb{Z})$ naturally has a polarized Hodge structure of weight (-1).

Definition 2.3. Let H be a polarized $\mathbb{Z}$-Hodge structure of weight (-1). If H has a basis whose intersection matrix is

$$\begin{pmatrix} 0 & e \\ -e & 0 \end{pmatrix}, \quad e = \mathrm{diag}(\varepsilon_1, \dots, \varepsilon_d), \tag{2.4}$$

then the type of the polarization of H is said to be $(\varepsilon_1, \dots, \varepsilon_d)$. For a polarized abelian variety $(A, (\ ,\ ))$, the type of $(H_1(A, \mathbb{Z}), (\ ,\ ))$ is called the type of the polarized abelian variety. The polarization of type $(1, \dots, 1)$ is called principal.

The curve C can be regarded as a double covering of a hyperelliptic curve of genus 3 branching at eight points. The results in [Fay73] imply that the type of the polarized abelian variety $\mathrm{Prym}(C)$ is $(2, 2, 2, 1, 1, 1)$ (see also [Mum74, § 3, Corollary 1]). We give a symplectic basis Σ of $H_1(C, \mathbb{Z})^-$.

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Proposition 2.4. *Let* $\Sigma = \{\alpha'_1, \dots, \alpha'_6, \beta'_1, \dots, \beta'_6\}$ *be*

$$\begin{aligned}
\alpha'_1 &= 2A_1 + 2A_3 + A_4 + B_4,\\
\alpha'_2 &= 2A_1 + A_2 + 2A_5 + 2A_6 + B_2 - 2B_4 - 2B_5,\\
\alpha'_3 &= A_1 + 2A_2 + A_3 + A_5 + 2A_6 - B_1 + B_3 - B_5,\\
\alpha'_4 &= A_1,\\
\alpha'_5 &= A_1 + A_3,\\
\alpha'_6 &= -A_1 - A_3 + B_5,\\
\beta'_1 &= 2A_1 + 2A_3 + A_5 + 2A_6 - B_5,\\
\beta'_2 &= -A_1 - 2A_2 - 2A_5 - 2A_6 + B_1 + 2B_4 + 2B_5,\\
\beta'_3 &= A_4 - A_6 + B_4 + 2B_5 + B_6,\\
\beta'_4 &= A_2,\\
\beta'_5 &= A_4,\\
\beta'_6 &= A_6.
\end{aligned}$$

Then Σ *is a basis of* $H_1(C,\mathbb{Z})^-$ *whose intersection matrix is given by (2.4) for* $e = \mathrm{diag}(2,2,2,1,1,1)$.

To simplify the notation, the half of the intersection form on $H_1(C,\mathbb{Z})^-$ is denoted by $\langle\, ,\rangle$. Classifying the sub-principally polarized Hodge structures of $(H_1(C,\mathbb{Z}),(\, ,))$ of type $(2,2,2,2,2,2)$ is equivalent to classifying the principally polarized sub-Hodge structures of $(H_1(C,\mathbb{Z}),\langle\, ,\rangle)$.

For a polarized Hodge structure H, the dual Hodge structure $H^\perp$ is defined as

$$H^\perp = \{v \in H \otimes \mathbb{Q} \mid \langle v, w\rangle \in \mathbb{Z} \text{ for all } w \in H\}.$$

It is easy to see that a polarized Hodge structure H is principal if and only if $H = H^\perp$. From now on, we use the polarization $\langle\, ,\rangle$ for sub-Hodge structures of $H_1(C,\mathbb{Z})^-$.

Proposition 2.5.

1) $(H_1(C,\mathbb{Z})^-)^\perp = (1-\rho)H_1(C,\mathbb{Z})^-$.
2) *A principally polarized sub-Hodge structure* L *of* $H_1(C,\mathbb{Z})^-$ *contains* $(1-\rho)H_1(C,\mathbb{Z})^-$ *and is stable under the action of* ρ.

In the next section, we give a combinatorial description of the set of principally polarized sub-Hodge structures of $H_1(C,\mathbb{Z})^-$.

We close this section by giving an example of a principally polarized sub-Hodge structure of $H_1(C,\mathbb{Z})^-$ using the basis A_j, B_j $(j = 1,\dots,6)$ given in Proposition 2.2.

Proposition 2.6. *The sub-Hodge structure* L_1 *of* $H_1(C,\mathbb{Z})^-$ *generated by* $(1-\rho)H_1(C,\mathbb{Z})^-$ *and* A_1, A_3, A_5 *is principal. Actually the set*

$$\Sigma_1 = \{a_1, \dots, a_6, b_1, \dots, b_6\},$$

where

$$\begin{aligned}
a_1 &= A_1,\\
a_2 &= A_1 + A_2 + B_2,\\
a_3 &= A_1 + A_2 + B_2 + B_3,\\
a_4 &= A_1 + A_2 - A_4 + B_2 + B_3 + B_4,\\
a_5 &= A_1 + A_2 + A_5 + B_2 + B_3,
\end{aligned}$$

$$
\begin{aligned}
a_6 &= A_1 + A_2 + A_5 + A_6 + B_2 + B_3 + B_6,\\
b_1 &= -B_1,\\
b_2 &= A_2 - B_1 - B_2,\\
b_3 &= -A_2 - A_3 - A_4 + B_1 + B_2 - B_4,\\
b_4 &= -A_2 - A_3 + B_1 + B_2,\\
b_5 &= A_2 + A_3 - B_1 - B_2 - B_5,\\
b_6 &= A_2 + A_3 + A_6 - B_1 - B_2 - B_5 - B_6,
\end{aligned}
$$

is a symplectic basis of type $(1,1,1,1,1,1)$*. The action* ρ *on this basis is given by*

$$
{}^{\mathrm{t}}(a_1,\dots,a_6,b_1,\dots,b_6) \mapsto \begin{pmatrix} 0 & -U \\ U & 0 \end{pmatrix} {}^{\mathrm{t}}(a_1,\dots,a_6,b_1,\dots,b_6), \tag{2.5}
$$

where

$$
U = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -1 & 0 & 0\\
0 & 0 & -1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
$$

Definition 2.7. A pair (L, Σ_L) of a principal sub-Hodge structure L of $H_1(C,\mathbb{Z})^-$ and a symplectic basis Σ_L of L is called a good basis if the action ρ on Σ_L is given by (2.5).

Remark 2.8. Using a good basis Σ_L, an element of $[1/(1-\rho)]L/L$ can be written as $\frac{1}{2}(\mu, \mu U) \bmod L$ $(\mu \in \mathbb{Z}^6)$.

We will use the lattice L_1 and its symplectic basis Σ_1 in Proposition 2.6 for explicit calculations of theta constants.

3. $O_6^+(2)$-level structure and $\mathfrak{S}_8$-marking

3.1 Total singular subspaces, length 0 elements

We define the standard lattice H_{std} as the free $\mathbb{Z}$-module generated by $\bar{A}_j, \bar{B}_j$ $(j = 1,\dots,6)$. We introduce an alternating form $\langle\,,\rangle$ and an action of ρ on H_{std} by the half of the matrix (2.3) in § 2.1 and $\rho(\bar{A}_j) = \bar{B}_j$, $\rho(\bar{B}_j) = -\bar{A}_j$. Since the action of ρ^2 on H_{std} is equal to multiplication by (-1), H_{std} (respectively $H_{\mathrm{std},\mathbb{R}} = H_{\mathrm{std}} \otimes \mathbb{R}$) becomes a $\boldsymbol{Z}[\rho]/(\rho^2+1)$-module (respectively a vector space over $\boldsymbol{R}(\rho) \simeq \mathbb{C}$). We define a bilinear form $h(x,y)$ on H_{std} by $h(x,y) = \langle x, \rho y\rangle - \langle x, y\rangle i$. Then $h(\,,)$ becomes a hermitian form of the signature $(5,1)$ with respect to the complex structure given by $\boldsymbol{R}(\rho)$. The group of isomorphisms of H_{std} preserving the alternating form $\langle\,,\rangle$ and the action of ρ is denoted by $U(H_{\mathrm{std}})$. The value of associated hermitian metric $\tilde{q}(x) = h(x,x)$ on $H_1(C,\mathbb{Z})^-$ is integral. The class $q(x)$ of $\tilde{q}(x)$ modulo 2 defines a $\mathbb{Z}/2\mathbb{Z}$-valued quadratic form on $H_1(C,\mathbb{Z})^-/(1-\rho)H_1(C,\mathbb{Z})^-$. This quadratic form q has the following simple form.

Let (V,q) be a six-dimensional vector space V over $\boldsymbol{F}_2$ with the quadratic form q of Witt defect 0. Such (V,q) is constructed as follows. The Hamming length of a vector $x \in \boldsymbol{F}_2^8$ is defined by the number of nonzero elements. The subset $\tilde{V}$ consisting of vectors with the even Hamming length is a subspace of $\boldsymbol{F}_2^8$ containing $(1,\dots,1)$. Let V be the quotient space of $\tilde{V}$ by the subspace $(1,\dots,1)\cdot\boldsymbol{F}_2$. The half of the Hamming length modulo 2 becomes the quadratic form q on V of Witt defect 0.

We have an isomorphism

$$
(H_1(C,\mathbb{Z})^-/(1-\rho)H_1(C,\mathbb{Z})^-, q) \to (V,q) \tag{3.1}
$$

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by mapping the class of A_j to that of $e_j - e_{j+1}$, where e_j is the jth unit vector of $\boldsymbol{F}_2^8$. The orthogonal group of (V, q) is denoted by $O_6^+(2)$. The symmetric group $\mathfrak{S}_8$ of degree eight acts on the space V by $e_j \cdot \sigma = e_{j\sigma}$ for $\sigma \in \mathfrak{S}_8$ and $j = 1, \ldots, 8$. This action defines a group homomorphism $\mathfrak{S}_8 \to O_6^+(2)$.

Definition 3.1. For natural numbers $\varepsilon_1, \ldots, \varepsilon_d$ such that $\varepsilon_1 + \cdots + \varepsilon_d = 8$, a (unordered) partition of $\{1, \ldots, 8\}$ into sets of cardinality $\varepsilon_1, \ldots, \varepsilon_d$ is called an $(\varepsilon_1, \ldots, \varepsilon_d)$-partition. The set of $(\varepsilon_1, \ldots, \varepsilon_d)$-partitions of $\{1, \ldots, 8\}$ is denoted by $P(\varepsilon_1, \ldots, \varepsilon_d)$. The sets $P(2, 2, 2, 2)$ and $P(4, 4)$ are denoted by $P(2^4)$ and $P(4^2)$, respectively.

Note that $\#P(2^4) = 105$ and $\#P(4^2) = 35$. Therefore we have the following propositions.

Proposition 3.2.

1) *The map $\mathfrak{S}_8 \to O_6^+(2)$ is an isomorphism.*
2) *By mapping an element $s = \{\{s_1, \ldots, s_4\}, \{s_5, \ldots, s_8\}\} \in P(4^2)$ to a nonzero element $v = e_{s_1} + \cdots + e_{s_4}$ with $q(v) = 0$, we have a one-to-one correspondence*

$$P(4,4) \simeq \{v \in V \mid q(v) = 0, v \neq 0\}.$$

Under this correspondence, the stabilizer of $s \in P(4,4)$ is isomorphic to the stabilizer of v.

For an element $I = \{\{j_1, j_2\}, \ldots, \{j_7, j_8\}\}$ of $P(2^4)$, we define a subspace $V_I = \langle e_{j_1} - e_{j_2}, \ldots, e_{j_7} - e_{j_8} \rangle$ of V.

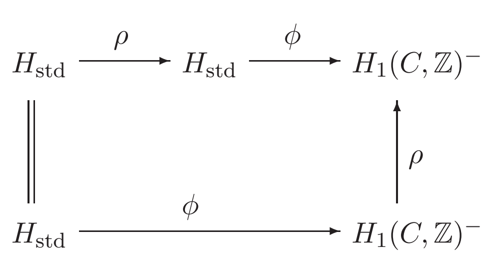
Proposition 3.3. *Let $\psi : H_1(C, \mathbb{Z})^- \to V$ be the natural projection. Then $\psi^{-1}(V_I)$ is a principally polarized sub-Hodge structure in $H_1(C, \mathbb{Z})^-$. This gives a one-to-one correspondence between $P(2^4)$ and the set of sublattices H of $H_1(C, \mathbb{Z})^-$ such that*

1) *$H = H^\perp$,*
2) *$H/(1-\rho)H_1(C, \mathbb{Z})^-$ contains a vector v with $q(v) \neq 0$.*

Under the correspondence of Proposition 3.3, the lattice L_1 given in Proposition 2.6 corresponds to the partition $\{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}\}$.

3.2 Moduli spaces of branched coverings of $\boldsymbol{P}^1$

In this section, we give analytic descriptions of moduli spaces of 4-ple coverings of $\boldsymbol{P}^1$ branching at eight points. Let H_{std} be the module with a symplectic form and the action of ρ as in the last subsection. An isomorphism $\phi : H_{\mathrm{std}} \to H_1(C, \mathbb{Z})^-$ compatible with the intersection pairing and with the action of ρ is called a marking of C. A pair (C, ϕ) is called a marked curve. An isomorphism between two marked curves is an isomorphism between two curves which is compatible with the markings. The set of isomorphic classes of marked curves is denoted by M_{marked}. For an element $\sigma \in U(H_{\mathrm{std}})$, we define $\sigma(C, \phi)$ by $(C, \phi \circ \sigma)$. By the commutative diagram



we have $\rho(C, \sigma) = (C, \sigma)$.

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We define the complex ball $B(H_{\mathrm{std}})$ associated to H_{std} as follows. The complex structure on the vector space $H_{\mathrm{std},\mathbb{R}}$ is given by the action of $\boldsymbol{R}(\rho)$ via the identification $\boldsymbol{R}(\rho) \simeq \mathbb{C}$ given by $\rho = -i$. In the complex projective space $\boldsymbol{P}(H^*_{\mathrm{std},\mathbb{R}})$, the domain

$$B(H_{\mathrm{std}}) = \{h(v,*) \in \boldsymbol{P}(H^*_{\mathrm{std},\mathbb{R}}) \mid h(v,v) < 0\}$$

is isomorphic to a five-dimensional complex ball. Then an element $\sigma \in U(H_{\mathrm{std}})$ induces an isomorphism of $B(H_{\mathrm{std}})$.

For a marked curve (C,ϕ), we construct a point $p(C,\phi)$ in $B(H_{\mathrm{std}})$ as follows. The complex structure arising from ρ is denoted by $\boldsymbol{R}(\rho)$ to distinguish the usual complex structure. By Proposition 2.1, the $(-i)$-eigenspace $H^0(C,\Omega^1)^{-,\rho=-i}$ of ρ in the space $H^0(C,\Omega^1)^-$ is one-dimensional. By the definition of a Hodge structure, we have an $\mathbb{R}$-isomorphism

$$H_1(C,\mathbb{R})^- \to \mathrm{Hom}_{\mathbb{C}}(H^0(C,\Omega^1)^-,\mathbb{C}).$$

By composing the map

$$\mathrm{Hom}_{\mathbb{C}}(H^0(C,\Omega^1)^-,\mathbb{C}) \to \mathrm{Hom}_{\mathbb{C}}(H^0(C,\Omega^1)^{-,\rho=-i},\mathbb{C}),$$

and $H_{\mathrm{std},\mathbb{R}} \overset{\phi}{\simeq} H_1(C,\mathbb{R})^-$, we have an $\mathbb{R}$-linear map

$$H_{\mathrm{std},\mathbb{R}} \to \mathrm{Hom}(H^0(C,\Omega^1)^{-,\rho=-i},\mathbb{C}). \tag{3.2}$$

It is easy to see that if we consider the complex structure on $H_{\mathrm{std},\mathbb{R}}$ by the action of $\boldsymbol{R}(\rho)$, the $\mathbb{R}$-linear map (3.2) is linear for the complex structures via the isomorphism $\boldsymbol{R}(\rho) \simeq \mathbb{C}$ given by $\rho = -i$. This linear form defines a point in $B(H_{\mathrm{std}})$. The corresponding point is denoted by $p(C,\phi)$. This map $p : M_{\mathrm{marked}} \to B(H_{\mathrm{std}})$ is called a period map. More explicitly, this map is given as follows. Let ω be a basis of one-dimensional complex space of $H^0(C,\Omega^1)^{-,\rho=-i}$. Then the map (3.2) $H_{\mathrm{std}} \otimes \mathbb{R} \to \mathbb{C}$ is given by $\gamma \mapsto \int_{\phi(\gamma)} \omega$. By using the dual basis A_j^* of A_j over $\boldsymbol{R}(\rho)$, this map is expressed as $\sum_{j=1}^6 (\int_{\phi(A_j)} \omega) A_j^*$. This period map is holomorphic with respect to the parameters $x_1,\dots,x_8$.

Theorem 3.4 (Terada, Deligne and Mostow). *The map p is an open embedding and the complement of the image is a proper analytic subset. The natural actions of $U(H_{\mathrm{std}})$ on M_{marked} and $B(H_{\mathrm{std}})$ are compatible.*

Let M_{8pts} (respectively M_{unord}) be the set of the isomorphism classes of the ordered (respectively unordered) set of distinct eight points in $\boldsymbol{P}^1$. The set M_{8pts} (respectively M_{unord}) has a natural structure of an algebraic variety which is isomorphic to $((\boldsymbol{P}^1)^8 - \mathrm{Diag})/PGL(2,\mathbb{C})$ (respectively $((\boldsymbol{P}^1)^8 - \mathrm{Diag})/\mathfrak{S}_8 \times PGL(2,\mathbb{C})$), where $\mathrm{Diag} = \{(x_j) \mid x_p = x_q \text{ for some } p < q\}$. By corresponding a marked curve $(C,\phi) \in M_{\mathrm{marked}}$ to the set of branching points $\{x_1,\dots,x_8\}$ of $C \to \boldsymbol{P}^1$, we have a morphism:

$$M_{\mathrm{marked}} \to M_{\mathrm{unord}}.$$

Proposition 3.5 (cf. [MY93]). *Via the open immersion p, M_{unord} is identified with an open set of $B(H_{\mathrm{std}})/U(H_{\mathrm{std}})$. Moreover the covering M_{8pts} of M_{unord} is identified with an open set of $B(H_{\mathrm{std}})/\Gamma(i+1)$, where*

$$\Gamma(i+1) = \{g \in U(H_{\mathrm{std}}) \mid g \equiv 1 \bmod (1+\rho)H_{\mathrm{std}}\}.$$

We define complex reflections $M_{p,p+1} \in U(H_{\mathrm{std}})$ for $p = 1,\dots,7$. We choose an initial point $X = (x_1,\dots,x_8) \in M_{\mathrm{8pts}}$ such that $x_j \in \mathbb{R}$ $(j = 1,\dots,8)$ and $x_1 < \cdots < x_8$ (see Figure 1). The image of

X in M_{unord} is denoted by $\overline{X}$. We consider a path $M_{p,p+1} = (M_{p,p+1,j})_{j=1,\dots,8} : [0,1] \to M_{8\mathrm{pts}}$ by

$$M_{p,p+1,j}(t) = x_j \quad \text{for } j \neq p, p+1,$$
$$M_{p,p+1,p}(t) = \frac{x_{p+1}+x_p}{2} - \frac{x_{p+1}-x_p}{2}\boldsymbol{e}(t/2),$$
$$M_{p,p+1,p+1}(t) = \frac{x_{p+1}+x_p}{2} + \frac{x_{p+1}-x_p}{2}\boldsymbol{e}(t/2),$$

where $\boldsymbol{e}(t) = \exp(2\pi i t)$. Then $M_{p,p+1}$ defines a closed path in M_{unord} with the base point $\overline{X}$. By fixing the point X, we define the monodromy action $M_{p,p+1}$ of H_{std} as follows. Let C be the curve defined by Equation (2.2), where $x_1, \dots, x_8$ are the coordinates of X. A basis $A_1, \dots, B_6$ of $H_1(C,\mathbb{Z})^-$ defined in § 2.1 gives a marking $H_{\mathrm{std}} \to H_1(C,\mathbb{Z}^-)$. We consider the lifting $\tilde{M}_{p,p+1}$ of the path $M_{p,p+1}$ beginning from the point in M_{marked} corresponding to the pair (C,ϕ). Then the end point (C,ϕ') of $\tilde{M}_{p,p+1}$ is a lifting $\overline{X}$ of X in M_{marked}. The composite map $\phi^{-1}\circ\phi'$

$$H_{\mathrm{std}} \xrightarrow{\phi'} H_1(C,\mathbb{Z})^- \xleftarrow{\phi} H_{\mathrm{std}}$$

is denoted by $M_{p,p+1}$. Since the pairing and the action of ρ are preserved in the family $H_1(C_{M_{p,p+1}(t)},\mathbb{Z})^-$, $t \in [0,1]$, we have $M_{p,p+1} \in U(H_{\mathrm{std}})$. Since $\Gamma(i+1)$ is a normal subgroup of $U(H_{\mathrm{std}})$, the covering $B(H_{\mathrm{std}})/\Gamma(i+1) \to B(H_{\mathrm{std}})/U(H_{\mathrm{std}})$ is a Galois covering and

$$\mathrm{Gal}(B(H_{\mathrm{std}})/\Gamma(i+1) \to B(H_{\mathrm{std}})/U(H_{\mathrm{std}})) \simeq U(H_{\mathrm{std}})/\Gamma(i+1) \simeq O_6^+(2).$$

By chasing the action of $M_{p,p+1}$ on $H_1(C,\mathbb{Z})^-$, we have the following lemma.

LEMMA 3.6 (cf. [MY93]). *There exists an isomorphism $O_6^+(2) \simeq \mathfrak{S}_8$ such that the image of $M_{p,p+1} \in U(H_{\mathrm{std}})$ is the transposition $(p,p+1)$ of p and $p+1$. Under the isomorphism*

$$B(H_{\mathrm{std}})/\Gamma(i+1) \simeq M_{\mathrm{marked}},$$

the action of $\mathfrak{S}_8 \subset \mathrm{Aut}(M_{\mathrm{marked}})$ is induced by $\sigma^*(x_j) = x_{j\sigma}$. Here the group $\mathfrak{S}_8$ acts on the set $\{1,\dots,8\}$ from the right. The action of $M_{p,p+1}$ is a complex reflection for the root A_p with the eigenvalue $-\rho$, i.e. $M_{p,p+1}$ is characterized by

$$M_{p,p+1}(v) = \begin{cases} -\rho(A_p) & \text{if } v = A_p, \\ v & \text{if } (v,A_p) = 0. \end{cases}$$

The isomorphism $O_6^+(2) \simeq \mathfrak{S}_8$ in Lemma 3.6 induces a homomorphism $\pi : U(H_{\mathrm{std}}) \to \mathfrak{S}_8$.

We define an inclusion $B(H_{\mathrm{std}})$ to the Siegel upper half space $\mathfrak{H}_6$ of degree 6 by using a good basis (L,Σ_L) of H_{std} as follows. By the Poincaré duality, we have $h(y_1,y_2) = 0$ for elements $y_1 \in \ker(H_{\mathrm{std}}, \mathrm{Hom}(H^0(C,\Omega^1)^{-,\rho=-i},\mathbb{C}))$ and $y_2 \in \ker(H_{\mathrm{std}}, \mathrm{Hom}(H^0(C,\Omega^1)^{-,\rho=i},\mathbb{C}))$, where $H^0(C,\Omega^1)^{-,\rho=i}$ is the i-eigenspace of ρ in the space $H^0(C,\Omega^1)^-$. Let H be the corresponding Hodge structure of $p \in B(H_{\mathrm{std}})$. Let $\Sigma_L = \{a_1,\dots,a_6,b_1,\dots,b_6\}$ and

$$H \otimes \mathbb{C} \simeq H^{(1,0)} \oplus H^{(0,1)}$$

be the Hodge decomposition of H. We choose a basis $\omega_1,\dots,\omega_g$ of $H^{(1,0)}$ such that $\int_{b_j}\omega_k = \delta_{jk}$. We put $\tau_{jk} = \int_{a_j}\omega_k$. Then by the definition of polarized Hodge structure, $\tau = (\tau_{jk})_{jk}$ is an element of $\mathfrak{H}_6$.

3.3 Level 2 structure and exponent 2 covering of configuration space

Let $\mathbb{C}(x_1,\dots,x_8)$ be the rational function field of $x_1,\dots,x_8$ over $\mathbb{C}$. On this field, the groups $PGL(2,\mathbb{C})$ and $\mathfrak{S}_8$ act by

$$g(x_j) = \frac{ax_j+b}{cx_j+d}, \quad \sigma(x_j) = x_{j\sigma}$$

for $g \in PGL(2,\mathbb{C})$ and $\sigma \in \mathfrak{S}_8$, respectively. As in the last section the group $\mathfrak{S}_8$ acts on the set $\{1,\dots,8\}$ from the right. (Note that the action of $\mathfrak{S}_8$ on the space M_{marked} is covariant.) Let K be the fixed subfield of $M = \mathbb{C}(x_1,\dots,x_8)$ under the action of $PGL(2,\mathbb{C})$. Since the action of $\mathfrak{S}_8$ commutes with that of $PGL(2,\mathbb{C})$, $\mathfrak{S}_8$ acts on K. Let L be the fixed subfield of K under the action of $\mathfrak{S}_8$. Then K and L are equal to the function fields of $M_{8\mathrm{pts}}$ and M_{unord}, respectively. The field K is generated by the cross-ratios

$$\lambda_j = \frac{(x_3 - x_1)(x_j - x_2)}{(x_2 - x_1)(x_j - x_3)}$$

of $\{x_1, x_2, x_3, x_j\}$ for $j = 4,\dots,8$ over $\mathbb{C}$.

Let $\tilde{K}$ be the algebraic closure of K in $\tilde{M} = \mathbb{C}(x_j, \sqrt{x_j - x_k})_{j\neq k}$. Since the extension $\tilde{M}/M$ is a Galois extension, so is $\tilde{K}/K$. The extensions M and $\tilde{K}$ are linearly independent over K, therefore the restriction map

$$\mathrm{Gal}(\tilde{M}/M) \to \mathrm{Gal}(\tilde{K}/K)$$

is surjective, and $\tilde{K}$ is generated by $f = \prod_{j<k} \sqrt{x_j - x_k}^{a_{jk}}$ such that f^2 is an element of K. Thus $\tilde{K}$ is generated by $\sqrt{\lambda_j}$, $\sqrt{\lambda_j - 1}$ for $j = 4,\dots,8$, and by $\sqrt{\lambda_j - \lambda_k}$ for $4 \leqslant j < k \leqslant 8$, and $\tilde{K}$ is a Galois extension of L. The inclusions of fields $L \subset K \subset \tilde{K}$ imply the following exact sequence of groups:

$$1 \to N \to \mathrm{Gal}(\tilde{K}/L) \to \mathfrak{S}_8 \to 1,$$

where $N = \mathrm{Gal}(\tilde{K}/K) \simeq (\mathbb{Z}/2\mathbb{Z})^{20}$.

We compare this Galois extension with the analytic description of the corresponding moduli space of 4-ple coverings of $\boldsymbol{P}^1$ branching at eight points.

Proposition 3.7. *Let $\tilde{M}_{8\mathrm{pts}}$ be the normalization of the $M_{8\mathrm{pts}}$ in $\tilde{K}$. This variety $\tilde{M}_{8\mathrm{pts}}$ is identified with an open subset of $B(H_{\mathrm{std}})/\Gamma(2)$ via the period map p, where*

$$\Gamma(2) = \{g \in U(H_{\mathrm{std}}) \mid g \equiv 1 \bmod 2H_{\mathrm{std}}\}.$$

Via this isomorphism, we have

$$\mathrm{Gal}(\tilde{K}/L) \simeq U(H_{\mathrm{std}})/\Gamma(2) \cdot \langle i\rangle,$$

where $\langle i\rangle$ is the cyclic group generated by i.

Proof. We already know that

$$U(H_{\mathrm{std}})/\Gamma(i+1) \cdot \langle i\rangle \simeq \mathfrak{S}_8.$$

The group $\Gamma(i+1) \cdot \langle i\rangle/\Gamma(2) \cdot \langle i\rangle \simeq (\mathbb{Z}/2\mathbb{Z})^{20}$ is generated by M_{jk}^2. By restricting the action of $U(H_{\mathrm{std}})$ to $\tilde{K}$, we have a map

$$\Gamma(i+1) \cdot \langle i\rangle/\Gamma(2) \cdot \langle i\rangle \to N. \tag{3.3}$$

We study the action of M_{jk} on $\{\sqrt{x_p - x_q}\}$. We assign each algebraic function $\sqrt{x_j - x_k}$ on $(\boldsymbol{P}^1)^8 -$ Diag a branch as follows. Let X be the initial point in $M_{8\mathrm{pts}}$ as in the last section. For $j < k$, $\sqrt{x_k - x_j}$ denotes the branch of the algebraic function on $(\boldsymbol{P}^1)^8 -$ Diag so that it takes a positive real value at X. The analytic continuation of the function $\sqrt{x_k - x_j}$ along the path $M_{p,p+1}$ is given as

$$M_{p,p+1}(\sqrt{x_k - x_j}) = \begin{cases} i\sqrt{x_k - x_j} & \text{if } j = p \text{ and } p+1 = k,\\ \sqrt{x_{k\sigma} - x_{j\sigma}} & \text{otherwise,}\end{cases}$$

where σ is the transposition $(p, p+1)$. As a consequence, we have

$$M_{p,p+1}^2(\sqrt{x_k - x_j}) = \begin{cases} -\sqrt{x_k - x_j} & \text{if } j = p \text{ and } p+1 = k,\\ \sqrt{x_k - x_j} & \text{otherwise.}\end{cases}$$

By looking at the action of M_{jk}^2 on the set $\{\sqrt{\lambda_p}, \sqrt{1-\lambda_p}, \sqrt{\lambda_p-\lambda_q}\}$, the homomorphism (3.3) is an isomorphism. □

Let $g \in U(H_{\mathrm{std}})$ and $r_1 = \{\{1,2\},\{3,4\},\{5,6\},\{7,8\}\} \in P(2^4)$. We put $r_1\pi(g) = \{\{j_1,j_2\}, \{j_3,j_4\},\{j_5,j_6\},\{j_7,j_8\}\} \in P(2^4)$, where $\pi : U(H_{\mathrm{std}}) \to \mathfrak{S}_8$. We assume that $j_p < j_{p+1}$ for $p = 1,3,5,7$. Then there exists a fourth root of unity $\arg(g)$, called the argument of g, such that

$$g(\sqrt{(x_2-x_1)(x_4-x_3)(x_6-x_5)(x_8-x_7)}) = \arg(g)\sqrt{(x_{j_2}-x_{j_1})(x_{j_4}-x_{j_3})(x_{j_6}-x_{j_5})(x_{j_8}-x_{j_7})}.$$

4. Theta function of standard principal sub-Hodge structure L_1

4.1 Abel–Jacobi map and the order of zero

Let $p_1,\dots,p_8$ be the ramification points of the smooth curve C defined by (2.1) above $x_1,\dots,x_8$, respectively. Let jac be the Abel–Jacobi map

$$\mathrm{jac} : C \longrightarrow J(C) = \mathrm{Hom}(H^0(C,\Omega^1),\mathbb{C})/H_1(C,\mathbb{Z})$$
$$p \mapsto \text{the linear function } \int_{p_1}^{p} \text{ on } H^0(C,\Omega^1) \text{ defined by}$$
$$\int_{p_1}^{p} : H^0(C,\Omega^1) \ni \omega \mapsto \int_{p_1}^{p} \omega \in \mathbb{C}.$$

The endomorphism $(1-\rho^2) : J(C) \to J(C)$ of $J(C)$ factors through the natural inclusion $\kappa : \mathrm{Prym}(C) \to J(C)$, i.e. there is a morphism $\alpha : J(C) \to \mathrm{Prym}(C)$ such that $\kappa \circ \alpha = 1-\rho^2$. Let $J(C)^+$ be connected component of the kernel of α. Then it is easy to see that $J(C)/J(C)^+$ is isomorphic to $\mathrm{Prym}(C)$ and that the morphism

$$J(C)/J(C)^+ \simeq \mathrm{Prym}(C) \to \mathrm{Prym}(C)$$

induced by the morphism α corresponds to the index finite group

$$(1-\rho^2)H_1(C,\mathbb{Z}) = (1-\rho)H_1(C,\mathbb{Z})^- \text{ of } H_1(C,\mathbb{Z})^-.$$

As a consequence, we have the following sequence of morphisms:

$$J(C) \to J(C)/J(C)^+ \simeq \mathrm{Prym}(C) \to \mathrm{Prym}(C) \to J(C).$$

The composite map

$$C \overset{\mathrm{jac}}{\to} J(C) \to J(C)/J(C)^+ \simeq \mathrm{Prym}(C)$$

is denoted by jac^-. Let L_1 be the principal sub-Hodge structure defined in § 2.2. Then $A_{L_1} = \mathbb{C}^6/L_1$ is a principally polarized abelian variety. The inclusions

$$(1-\rho)H_1(C,\mathbb{Z})^- \subset L_1 \subset H_1(C,\mathbb{Z})^-$$

induce homomorphisms of abelian varieties

$$\mathrm{Prym}(C) \xrightarrow{\pi_1} A_{L_1} \xrightarrow{\pi_2} \mathrm{Prym}(C).$$

We define the theta function $\vartheta_m(\Sigma_1, z)$ for the good basis Σ_1 defined in § 2.2 with the characteristic $m = (m', m'') \in \mathbb{Q}^{12}$ by

$$\vartheta_m(\Sigma_1, z) = \sum_{\xi \in \mathbb{Z}^6} \boldsymbol{e}(\tfrac{1}{2}(\xi+m')\tau\,{}^{\mathrm t}(\xi+m') + (z+m'')\,{}^{\mathrm t}(\xi+m')),$$

where $z \in \mathbb{C}^6$ and $\tau = (\tau_{ij})_{ij} \in \mathfrak{H}_6$ is the normalized period matrix for the good basis Σ_1 defined in the last section.

Let $\iota : C \to A_{L_1}$ be the composition $\pi_1 \circ \mathrm{jac}^-$. By Propositions 2.6 and 3.3, we have the following lemma.

Lemma 4.1.

1) *The image of each ramification point* $p_1, \dots, p_8$ *under the map* ι *is contained in the set of* $(1-\rho)$*-torsion points of* $\mathrm{Prym}(C)$.
2) *The image of* π_1 *is identified with the quotient of* $\mathrm{Prym}(C)$ *by the group generated by* $\mathrm{jac}^-(p_2), \mathrm{jac}^-(p_3) - \mathrm{jac}^-(p_4), \mathrm{jac}^-(p_5) - \mathrm{jac}^-(p_6)$.
3) *Using the good basis* Σ_{L_1} *of* L_1, *we have*
$$\iota(p_k) \equiv \tfrac{1}{2}(\xi_k, \xi_k U) \bmod L_1,$$
where $\xi_1 = \xi_2 = 0$, $\xi_3 = \xi_4 = (1,1,0,0,0,0)$, $\xi_5 = \xi_6 = (1,1,1,1,0,0)$ *and* $\xi_7 = \xi_8 = (1,1,1,1,1,1)$.

We study the order of zero of the pull back of $\vartheta_m(\Sigma_1, z)$ by ι. Let $\tilde{C}$ be the universal covering of C and we choose a base point $\tilde{q}_1$ of $\tilde{C}$ as a lifting of p_1. Then we have a lifting $\tilde{\iota} : \tilde{C} \to (H^0(C, \Omega^1)^-)^*$ of ι by sending $\tilde{p}_1$ to 0. Let $\omega_1, \dots, \omega_6$ be the normalized basis of $H^0(C, \Omega^1)^-$ with respect to Σ_1. Via the isomorphism
$$(H^0(C, \Omega^1)^-)^* \ni \gamma \mapsto \left(\int_\gamma \omega_1, \dots, \int_\gamma \omega_6 \right) \in \mathbb{C}^6,$$
$\tilde{\iota}$ is identified with $\tilde{C} \to \mathbb{C}^6$. We define a map $F_m : \tilde{C} \to \mathbb{C}$ by
$$\tilde{C} \ni \tilde{p} \mapsto F_m(\tilde{p}) = \vartheta_m(\Sigma_1, \tilde{\iota}(\tilde{p})) \in C.$$
Since $\vartheta_m(\tau, z)$ is a nonzero section of a line bundle $\mathcal{L}_m$, the order of zero at the lifting $\tilde{p}$ of p depends only on the point p in C. It is called the order of zero at p and denoted by $\mathrm{ord}_p(F_m)$.

Proposition 4.2. *The total sum of the order* $\mathrm{ord}_p(F_m)$ *of* p *on* C *is 12.*

Proof. Use a similar argument in [MT03] just after Proposition 4.9. □

The next proposition is fundamental for determining the distribution of zeros of F_m.

Proposition 4.3. *Let* $m = \frac{1}{2}(\mu, \mu U)$ $(\mu \in \mathbb{Z}^6)$ *be an element of* $[1/(1-\rho)]L_1$ *(see Remark 2.8). Let* ξ_j *be an element of* $\mathbb{Z}^6$ *such that* $\iota(\tilde{p}_j) \equiv \frac{1}{2}(\xi_j, \xi_j U) \pmod{\mathbb{Z}^{12}}$, *where* $\tilde{p}_j$ *is a lifting of* p_j *to* $\tilde{C}$. *We put* $q = \mu + \xi_j$. *Then* $\mathrm{ord}_{p_j}(F_m)$ *is equal to* $-qU\,{}^{\mathrm{t}}q$ *modulo* 4.

Proof. Let z be the coordinate for the universal covering of A_{L_1}. Since the point p_j is fixed under the action of ρ, $\rho z = z + l$ $(l \in L_1)$. By the transformation formula in [Igu72, p. 85], we have $F(\rho z) = u(z)F(z)$, where
$$\lim_{z \to \tilde{p}_j} u(z) = \boldsymbol{e}\left(\frac{-qUq}{4} \right).$$
Therefore the order of F_m at p_j is congruent to $-qUq \bmod 4$. □

Let $m_k = \frac{1}{2}(\mu_k, \mu_k U)$ $(k = 1, \dots, 4)$, where
$$\begin{aligned} \mu_1 &= (0,0,0,0,0,0), \quad \mu_2 = (0,0,1,1,1,1), \\ \mu_3 &= (1,1,0,0,1,1), \quad \mu_4 = (1,1,1,1,0,0). \end{aligned} \tag{4.1}$$
Then by Proposition 4.3, the table of $\mathrm{ord}_{p_j}(F_{m_k})$ (mod 4) is given by
$$\begin{array}{cccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 \\ 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 \end{array} \tag{4.2}$$
for $k = 1, \dots, 4$.

4.2 Determination of extra zeros of theta functions

Let $\mu_1, \ldots, \mu_4$ and $m_1, \ldots, m_4$ be as in § 4.1. Then the sum of the known zeros of $F_{m_k}(\tilde{p})$ is eight for $k = 1, \ldots, 4$. Since the action of ρ on the curve $\tilde{C}$ and $H^0(C, \Omega^1)$ is compatible, the remaining $(12 - 8 =)$ four zeros are stable under the action of ρ, and if its support contains one of p_j, then its multiplicity should be 4 by the modulo 4 condition.

Proposition 4.4. *The function $R_{jk} = F_{m_j}(\tilde{p})/F_{m_k}(\tilde{p})$ is a rational function of $p = (z, w) \in C$ for $1 \leqslant j, k \leqslant 4$. Moreover $R_{jk}(\tilde{p}) = c \cdot (z - s)/(z - t)$, with some constants s, t and $c \neq 0$.*

Proof. Since the image of the fundamental group of C in L_1 is equal to $(1 - \rho)H_1(C, \mathbb{Z})^-$, if the theta functions $\vartheta_{m_j}(\tau, z)$ and $\vartheta_{m_k}(\tau, z)$ have the same quasi-periodicity, the quotient F_{m_j}/F_{m_k} is a rational function on C. By comparing the zeros of the numerator and the denominator of R_{jk} (see (4.2)), we have the proposition. □

Proposition 4.5.

1) *In the expression of the rational function $R_{13} = c \cdot (z - s)/(z - t)$, s and t are determined by the equations:*

$$\frac{x_1 - s}{x_1 - t} + \frac{x_2 - s}{x_2 - t} = 0, \quad \frac{x_5 - s}{x_5 - t} + \frac{x_6 - s}{x_6 - t} = 0. \tag{4.3}$$

2) *The rational function R_{12} on C is a constant.*

Proof. By Proposition 4.4, we have

$$c \cdot \frac{x_j - s}{x_j - t} = R_{13}(\tilde{p}_j) = \frac{F_{m_1}(\tilde{p}_j)}{F_{m_3}(\tilde{p}_j)}.$$

On the other hand, we have

$$\frac{F_{m_1}(\tilde{p}_1)}{F_{m_3}(\tilde{p}_1)} = -\frac{F_{m_1}(\tilde{p}_2)}{F_{m_3}(\tilde{p}_2)}$$

by the quasi-periodicity of theta functions; thus we have the statement 1. We can prove the statement 2 similarly. □

Proposition 4.6. *Let $\vartheta_k = \vartheta_{m_k}(\tau)$. Then we have*

$$\frac{(\vartheta_2 + i\vartheta_3)^2(\vartheta_1 - i\vartheta_4)^2}{4\vartheta_1^2\vartheta_3^2} = \frac{(x_1 - x_5)(x_2 - x_6)}{(x_1 - x_2)(x_5 - x_6)}.$$

Proof. By the definition of theta constants and $R_{jk}(\tilde{p})$, we have

$$R_{13}(p_1) = \frac{\vartheta_1}{\vartheta_3} = c \cdot \frac{x_1 - s}{x_1 - t}, \quad R_{13}(p_5) = -\frac{\vartheta_4}{\vartheta_2} = c \cdot \frac{x_5 - s}{x_5 - t}.$$

The equality $R_{12}(p_1) = R_{12}(p_5)$ implies

$$\frac{\vartheta_1}{\vartheta_2} = \frac{\vartheta_4}{\vartheta_3}.$$

By computing $R_{13}(p_1)/R_{13}(p_5)$, we have

$$\frac{(x_1 - s)(x_5 - t)}{(x_1 - t)(x_5 - s)} = -\frac{\vartheta_1\vartheta_2}{\vartheta_3\vartheta_4} = -\frac{\vartheta_1^2}{\vartheta_4^2} = -\frac{\vartheta_2^2}{\vartheta_3^2} \tag{4.4}$$

and

$$\left(1+\frac{\vartheta_4^2}{\vartheta_1^2}\right)\left(1+\frac{\vartheta_1^2}{\vartheta_4^2}\right) = \frac{(\vartheta_1^2+\vartheta_4^2)^2}{\vartheta_4^2\vartheta_1^2} = \frac{\vartheta_2^2+\vartheta_3^2}{\vartheta_2\vartheta_3}\cdot\frac{\vartheta_1^2+\vartheta_4^2}{\vartheta_4\vartheta_1} = \frac{(\vartheta_2+i\vartheta_3)(\vartheta_2-i\vartheta_3)(\vartheta_1+i\vartheta_4)(\vartheta_1-i\vartheta_4)}{\vartheta_1\vartheta_2\vartheta_3\vartheta_4} = \frac{(\vartheta_2+i\vartheta_3)^2(\vartheta_1-i\vartheta_4)^2}{\vartheta_1^2\vartheta_3^2}.$$

Here we used the relation $(\vartheta_2-i\vartheta_3)(\vartheta_1+i\vartheta_4)=(\vartheta_2+i\vartheta_3)(\vartheta_1-i\vartheta_4)$. On the other hand, by (4.3) and (4.4), we have

$$\frac{1}{4}\left(1+\frac{\vartheta_4^2}{\vartheta_1^2}\right)\left(1+\frac{\vartheta_1^2}{\vartheta_4^2}\right) = \frac{(x_1-x_5)(x_2-x_6)}{(x_1-x_2)(x_5-x_6)}.$$ □

4.3 An application of the quadratic theta relation

In this section, we fix a principally polarized sub-Hodge structure $L=L_1$ and study quadratic relations between theta constants. We recall the quadratic relation between theta functions in [Igu72]. For the next proposition, see [Igu72, p. 139].

Proposition 4.7. *Put*

$$n_1=(n_1',n_1'')=\tfrac{1}{2}(m_1+m_2),\quad n_2=(n_2',n_2'')=\tfrac{1}{2}(m_1-m_2),$$

for $m_1=(m_1',m_1'')$, $m_2=(m_2',m_2'')\in\mathbb{Q}^{12}$. Let S be a complete set of representatives of $(\frac{1}{2}\mathbb{Z})^6/\mathbb{Z}^6$. We have

$$\vartheta_{m_1}(\tau)\vartheta_{m_2}(\tau)=\frac{1}{2^6}\sum_{a''\in S}\boldsymbol{e}(-2m_1'\,{}^{\mathrm t}a'')\,\vartheta_{2n_1',n_1''+a''}\left(\frac{\tau}{2}\right)\,\vartheta_{2n_2',n_2''+a''}\left(\frac{\tau}{2}\right).$$

We apply this formula to

$$n_1=(\tfrac{1}{2}v_1-\tfrac{1}{4}U_0,\tfrac{1}{2}U_0),\quad n_2=(-\tfrac{1}{4}U_0,0),$$

where $v_1\in\mathbb{Z}^6$, and we use Notation 1.2 for $U_0=(1,1,0,0,1,1)$. Replace τ by $\tau+U$, then we have

$$\vartheta_{\frac{1}{2}v_1-\frac{1}{2}U_0,\frac{1}{2}U_0}(\tau+U)\,\vartheta_{\frac{1}{2}v_1,\frac{1}{2}U_0}(\tau+U) = \frac{1}{2^6}\sum_{a''\in S}\boldsymbol{e}(-(v_1-U_0)\,{}^{\mathrm t}a'')\,\vartheta_{v_1-\frac{1}{2}U_0,\frac{1}{2}U_0+a''}(\tfrac{1}{2}(\tau+U))\,\vartheta_{-\frac{1}{2}U_0,a''}(\tfrac{1}{2}(\tau+U)). \tag{4.5}$$

We assume that τ is the normalized period matrix of the principally polarized Hodge structure L with respect to the symplectic basis Σ_1. Then we have

$$(\tau U)^2=-I. \tag{4.6}$$

By applying the transformation formula

$$\vartheta_{m',m''}(\tau+U)=\boldsymbol{e}(-\tfrac{1}{2}m'U\,{}^{\mathrm t}m'+\tfrac{1}{2}m'\,{}^{\mathrm t}U_0)\,\vartheta_{m',m''+m'U+\frac{1}{2}U_0}(\tau)$$

to the left hand side of (4.5), we have

$$\vartheta_{\frac{1}{2}v_1-\frac{1}{2}U_0,\frac{1}{2}U_0}(\tau+U)\,\vartheta_{\frac{1}{2}v_1,\frac{1}{2}U_0}(\tau+U) = \boldsymbol{e}(-\tfrac{1}{4}v_1U\,{}^{\mathrm t}v_1+\tfrac{3}{4}v_1\,{}^{\mathrm t}U_0-\tfrac{3}{8}U_0\,{}^{\mathrm t}U_0)\,\vartheta_{\frac{1}{2}(v_1-U_0),\frac{1}{2}(v_1-U_0)U}(\tau)\,\vartheta_{\frac{1}{2}v_1,\frac{1}{2}v_1U}(\tau). \tag{4.7}$$

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To compute the right-hand side, we apply the transformation formula in [Igu72, p. 85],

$$\vartheta_{m^\#}(\tau^\#) = \det(C\tau + D)^{1/2} \cdot u \cdot \vartheta_m(\tau), \tag{4.8}$$

for

$$\sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & U \\ -U & I \end{pmatrix}$$

and $m = (-(a'' + \frac{1}{2}U_0)U, v_1U + a'' + \frac{1}{2}U_0)$ (respectively $m = (-a''U, a'')$). Here $\tau^\# = (A\tau + B)(C\tau + D)^{-1}$ is equal to $\frac{1}{2}(\tau + U)$ and $\det(C\tau + D) = -8$ by the relation (4.6). The theta characteristic

$$m^\# = m \cdot \sigma^{-1} + \tfrac{1}{2}((C\,{}^{\mathrm{t}}D)_0, (A\,{}^{\mathrm{t}}B)_0)$$

is equal to $(v_1 - \frac{1}{2}U_0, \frac{1}{2}U_0 + a'')$ (respectively $(-\frac{1}{2}U_0, a'')$). We fix a branch of $\det(C\tau + D)^{1/2} = \sqrt{8}i$ once and for all. We compute the constant u in the formula (4.8) which depends only on m.

Definition 4.8. We define a nonzero complex number $c(a, b)$ by

$$\vartheta_{a,b}(U(-U\tau + I)^{-1}, z^\#) = c(a, b)(-8)^{1/2}\,\vartheta_{c,d+\frac{1}{2}U_0}(\tau, z), \tag{4.9}$$

where $z^\# = z(-U\tau + I)^{-1}$ and $c = -bU, d = aU + b$.

Proposition 4.9.

1) $c(a, b)/c(0, 0) = \boldsymbol{e}(\frac{1}{2}bU\,{}^{\mathrm{t}}b + a\,{}^{\mathrm{t}}b + \frac{1}{2}b\,{}^{\mathrm{t}}U_0)$.
2) $c(0, 0)^2 = 1$.

Proof. 1) This is the direct consequence of the formula in [Igu72, p. 85].

2) Since $c(0, 0)$ is independent of τ, we evaluate both sides of (4.9) at $\tau = iI + U$. Then we have

$$\begin{aligned}\vartheta_0(iI) &= \det(-U\tau + I)^{1/2}c(0, 0)\vartheta_{0,\frac{1}{2}U_0}(iI + U) \\ &= (-1) \cdot \det(-iU)^{1/2}c(0, 0)\vartheta_{0,0}(iI).\end{aligned}$$

Since $\vartheta_0(iI) \neq 0$, we have $c(0, 0)^2 = 1$. □

We can compute the right-hand side of (4.5) by Proposition 4.9. As a consequence we have the following theorem. For an element $m \in \frac{1}{2}\mathbb{Z}^6$, a representative of the class of m in $\frac{1}{2}\mathbb{Z}^6/\mathbb{Z}^6$ in $\{0, \frac{1}{2}\}^6$ is denoted by $\langle m\rangle$.

Theorem 4.10.

1) *Let v_1 be an element of $\mathbb{Z}^6$. Then we have*

$$\begin{aligned}&8\boldsymbol{e}(-\tfrac{1}{4}v_1U\,{}^{\mathrm{t}}v_1 + \tfrac{3}{4}v_1\,{}^{\mathrm{t}}U_0 - \tfrac{3}{8}U_0\,{}^{\mathrm{t}}U_0) \cdot \vartheta_{\frac{1}{2}(v_1-U_0),\frac{1}{2}(v_1-U_0)U}(\tau)\,\vartheta_{\frac{1}{2}v_1,\frac{1}{2}v_1U}(\tau) \\ &\quad= \sum_{a''} \boldsymbol{e}(a''U\,{}^{\mathrm{t}}a'' + \tfrac{3}{2}a''\,{}^{\mathrm{t}}U_0 - v_1\,{}^{\mathrm{t}}a'')\,\vartheta_{a''U+\frac{1}{2}U_0,a''+\frac{1}{2}U_0}(\tau)\,\vartheta_{a''U,a''}(\tau),\end{aligned}$$

where a'' runs over the complete set S of representatives in $\frac{1}{2}\mathbb{Z}^6/\mathbb{Z}^6$.

2) *Let $v_1 \in \{0, 1\}^6$ and choose b'' as*

$$b'' \equiv \tfrac{1}{2}v_1 - \tfrac{1}{2}U_0, \quad b'' \in \{0, \tfrac{1}{2}\}^6.$$

Then we have

$$\begin{aligned}&8\boldsymbol{e}(-\tfrac{1}{4}v_1U\,{}^{\mathrm{t}}v_1 + \tfrac{3}{4}v_1\,{}^{\mathrm{t}}U_0 - \tfrac{3}{8}U_0\,{}^{\mathrm{t}}U_0) \cdot \boldsymbol{e}(\tfrac{1}{2}v_1\,{}^{\mathrm{t}}U_0)\,\vartheta_{b'',b''U}(\tau)\,\vartheta_{\frac{1}{2}v_1,\frac{1}{2}v_1U}(\tau) \\ &\quad= \sum_{a''} \boldsymbol{e}(\tfrac{1}{2}a''\,{}^{\mathrm{t}}U_0 + v_1\,{}^{\mathrm{t}}a'')\,\vartheta_{\langle a''+\frac{1}{2}U_0\rangle U,\langle a''+\frac{1}{2}U_0\rangle}(\tau)\,\vartheta_{a''U,a''}(\tau).\end{aligned}$$

Moreover if $\frac{1}{4}v_1U\,{}^{\mathrm t}v_1 \in \mathbb{Z}$, *then we have*

$$-8\boldsymbol{e}(\tfrac{1}{4}v_1\,{}^{\mathrm t}U_0)\cdot\vartheta_{b'',b''U}(\tau)\,\vartheta_{\frac{1}{2}v_1,\frac{1}{2}v_1U}(\tau) = \sum_{a''}\boldsymbol{e}(\tfrac{1}{2}a''\,{}^{\mathrm t}U_0+v_1\,{}^{\mathrm t}a'')\,\vartheta_{\langle a''+\frac{1}{2}U_0\rangle U,\langle a''+\frac{1}{2}U_0\rangle}(\tau)\,\vartheta_{a''U,a''}(\tau).$$

We have the following corollary of Theorem 4.10.

Corollary 4.11. *Let* $v_1 \in \{0,1\}^6$ *such that* $v_1U\,{}^{\mathrm t}v_1 \in 4\mathbb{Z}$, $v_1 \neq 0$, *and let* b'' *be the element in* $\{0,\frac{1}{2}\}^6$ *defined in Theorem 4.10. Then we have*

$$\boldsymbol{e}(\tfrac{1}{4}v_1\,{}^{\mathrm t}U_0)\cdot\,\vartheta_{b'',b''U}(\tau)\,\vartheta_{\frac{1}{2}v_1,\frac{1}{2}v_1U}(\tau)+\vartheta_{0,0}(\tau)\,\vartheta_{\frac{1}{2}U_0,\frac{1}{2}U_0}(\tau)=0.$$

5. Comparison for theta constants of different lattices

5.1 Translation vector arising from changing lattices

Let L be a principally polarized sub-Hodge structure of $H_1(C,\mathbb{Z})^-$ and let $\Sigma=\Sigma_L=\{a_1,\dots,a_6,b_1,\dots,b_6\}$ be a good symplectic basis of L. Let $\{\alpha'_1,\dots,\alpha'_6,\beta'_1,\dots,\beta'_6\}$ be the basis of B defined in Proposition 2.4. Put

$$\alpha_j=\alpha'_j \text{ (for } j=1,\dots,6),$$
$$\beta_j=2\beta'_j \text{ (for } j=1,2,3), \quad \beta_j=\beta'_j \text{ (for } j=4,5,6),$$

and $\Sigma_B=\{\alpha_1,\dots,\alpha_6,\beta_1,\dots,\beta_6\}$. Then the lattice B generated by Σ_B admits a principally polarized sub-Hodge structure of $H_1(C,\mathbb{Z})^-$. In this section we compare theta functions of (B,Σ_B) and those of (L,Σ_L). The elements a_j,b_j in Σ_L are linear combinations of Σ_B as

$$a_j=\sum_{j=1}^{6}a_{jk}\alpha_k+\sum_{j=1}^{6}b_{jk}\beta_k,$$
$$b_j=\sum_{j=1}^{6}c_{jk}\alpha_k+\sum_{j=1}^{6}d_{jk}\beta_k,$$

respectively. The column vector consisting of α_j (respectively β_j, a_j and b_j) for $j=1,\dots,6$ is denoted by α (respectively β, $\boldsymbol{a}$ and $\boldsymbol{b}$). We put $A=(a_{jk})$, $B=(b_{jk})$, $C=(c_{jk})$, $D=(d_{jk})$. Then we see that

$$\sigma=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in \mathrm{Sp}(6,\mathbb{Q}).$$

Let $p=(p_1,\dots,p_6)$, $q=(q_1,\dots,q_6)$ be elements in $\mathbb{Q}^6$. We define two vectors $r=(r_1,\dots,r_6)$ and $s=(s_1,\dots,s_6)$ in $\mathbb{Q}^6$ by

$$(r,s)=(p,q)\begin{pmatrix}A & B\\ C & D\end{pmatrix}.$$

We have $(r,s)\,{}^{\mathrm t}(\alpha,\beta)=(p,q)\,{}^{\mathrm t}(\boldsymbol{a},\boldsymbol{b})$. Then $(r,s)\,{}^{\mathrm t}(\alpha,\beta)$ is an element in $H_1(C,\mathbb{Z})^-$ (respectively $(1-\rho)H_1(C,\mathbb{Z})^-$) if and only if $r\in\mathbb{Z}^6$ and $s\in\frac{1}{2}\mathbb{Z}^3\oplus\mathbb{Z}^3$ (respectively $r\in 2\mathbb{Z}^3\oplus\mathbb{Z}^3$ and $s\in\mathbb{Z}^6$).

Proposition 5.1. *Let* $e=\mathrm{diag}(2,2,2,1,1,1)$. *Then all entries of* $e\,{}^{\mathrm t}B$, $e\,{}^{\mathrm t}D$ *are integers.*

Let τ and $\tau^{\#}$ be the normalized period matrix of B and L with respect to the symplectic bases Σ_B and Σ_L, respectively. For an element $z\in\mathbb{C}^6$, we define $z^{\#}=z(C\tau+D)^{-1}$.

For a rational vector $n = (n', n'') \in \mathbb{Q}^{12}$, we consider the following functional equations for a function F of $z^{\#} \in \mathbb{C}^6$:

$$(\mathrm{Eq}_n^{\#})\colon F(z^{\#} + p\tau^{\#} + q) = \boldsymbol{e}(-\tfrac{1}{2}p\tau^{\#}\,{}^{\mathrm{t}}p - p\,{}^{\mathrm{t}}z^{\#})\boldsymbol{e}(n'\,{}^{\mathrm{t}}q - n''\,{}^{\mathrm{t}}p)F(z^{\#}),$$
$$(\mathrm{Eq}_m)\colon F(z + r\tau + s) = \boldsymbol{e}(-\tfrac{1}{2}r\tau r - r\,{}^{\mathrm{t}}z)\boldsymbol{e}(m'\,{}^{\mathrm{t}}s - m''\,{}^{\mathrm{t}}r)F(z).$$

Note that the theta function $\vartheta_n(\Sigma_L, z^{\#})$ satisfies the functional equations $(\mathrm{Eq}_n^{\#})$ for $p, q \in \mathbb{Z}^6$.

Before studying the relation between $\vartheta_n(\tau^{\#}, z^{\#})$ and $\vartheta_m(\tau, z)$, we define a translation vector δ relative to the matrix σ. We define $\delta' \in \mathbb{Z}^6$ and $\delta'' \in \frac{1}{2}\mathbb{Z}^3 \oplus \mathbb{Z}^3$ as

$$\delta' = ({}^{\mathrm{t}}CA)_0, \quad \delta'' = (e\,{}^{\mathrm{t}}DBe)_0 e^{-1}.$$

The vector (δ', δ'') is denoted by δ_Σ. To describe properties of δ_Σ, it is convenient to consider a quadratic form q on $[1/(1-\rho)]H_1(C,\mathbb{Z})^-/H_1(C,\mathbb{Z})^-$ induced by the quadratic form q on $H_1(C,\mathbb{Z})^-/(1-\rho)H_1(C,\mathbb{Z})^-$ defined in § 3.1 via the isomorphism

$$\frac{1}{1-\rho}H_1(C,\mathbb{Z})^-/H_1(C,\mathbb{Z})^- \to H_1(C,\mathbb{Z})^-/(1-\rho)H_1(C,\mathbb{Z})^-$$
$$x \mapsto (1-\rho)x.$$

Let (L_1, Σ_1) be the good basis defined in § 2.2 and $g \in U(H_1(C,\mathbb{Z})^-)$. We put $L_g = g(L_1)$ and $\Sigma_g = g(\Sigma_1)$. Then it easy to see that (L_g, Σ_g) is a good symplectic basis of L_g. The translation vector δ_{Σ_g} is denoted by δ_g.

Proposition 5.2.

1) *The vector $\delta_g = (\delta', \delta'')$ is contained in $2\mathbb{Z}^3 \oplus \mathbb{Z}^3 \oplus \mathbb{Z}^6$.*
2) *Let $\overline{\Delta}$ and $\overline{\frac{1}{2}\delta_g}$ be $e_1 + e_2 + e_5 + e_6$ under the mapping (3.1) and the class of $\frac{1}{2}\delta_g\,{}^{\mathrm{t}}(\alpha, \beta)$ in $[1/(1-\rho)]H_1(C,\mathbb{Z})^-/H_1(C,\mathbb{Z})^-$. Then*

$$c_0 = \overline{\tfrac{1}{2}\delta_g} - \overline{\Delta}g$$

is independent of g. Moreover we have $q(\overline{\Delta}) = 0$ and $\overline{\Delta} \neq 0$.

Proof. For any (L_g, Σ_g), we compute vectors δ_g by definition. As a consequence, we obtain this proposition. □

For $m = (m', m'') \in \mathbb{Q}^{12}$, we set $\tilde{m} = m + \frac{1}{2}\delta_\Sigma$, $n = \tilde{m}\sigma^{-1}$, i.e. $n\,{}^{\mathrm{t}}(\boldsymbol{a}, \boldsymbol{b}) = \tilde{m}\,{}^{\mathrm{t}}(\alpha, \beta)$. The next proposition is fundamental for comparing theta functions for different lattices. We define $\theta_m(z^{\#})$ by

$$\theta_m(z^{\#}) = \boldsymbol{e}(\tfrac{1}{2}z(C\tau + D)^{-1}C\,{}^{\mathrm{t}}z)\vartheta_m(\Sigma_B, z).$$

In this definition, z denotes the function of $z^{\#}$ by the relation $z^{\#} = z(C\tau + D)^{-1}$.

Proposition 5.3.

1) *The function $\theta_m(z^{\#})$ satisfies the functional equation $(\mathrm{Eq}_n^{\#})$ for $(p,q)\,{}^{\mathrm{t}}(\boldsymbol{a}, \boldsymbol{b}) \in (1-\rho)H_1(C,\mathbb{Z})^-$.*
2) *The function $\theta_{m'}(z^{\#})$ satisfies the functional equation $(\mathrm{Eq}_n^{\#})$ for $m' \in \mathbb{Q}^{12}$, $(p,q)\,{}^{\mathrm{t}}(\boldsymbol{a}, \boldsymbol{b}) \in (1-\rho)H_1(C,\mathbb{Z})^-$ if and only if $m - m' \in \mathbb{Z}^6 \oplus \frac{1}{2}\mathbb{Z}^3 \oplus \mathbb{Z}^3$, i.e. $(m - m')\,{}^{\mathrm{t}}(\alpha, \beta) \in H_1(C,\mathbb{Z})^-$.*

Let $\Theta(\Sigma_B, m)$ and $\Theta(\Sigma_L, n)$ be the spaces of functions of τ and $\tau^{\#}$ satisfying the functional equations (Eq_m) and $(\mathrm{Eq}_n^{\#})$ for $(r,s)\,{}^{\mathrm{t}}(\alpha, \beta) \in (1-\rho)H_1(C,\mathbb{Z})^-$ and $(p,q)\,{}^{\mathrm{t}}(\boldsymbol{a}, \boldsymbol{b}) \in (1-\rho)H_1(C,\mathbb{Z})^-$, respectively. Since the spaces $\Theta(\Sigma_B, m)$ and $\Theta(\Sigma_L, n)$ are eight-dimensional by the Riemann–Roch theorem, Proposition 5.3 implies the following proposition.

Proposition 5.4. *Let $n = (m + \frac{1}{2}\delta_\Sigma)\sigma^{-1}$. By mapping a function $f(z)$ of z to a function $f^{\#}(z^{\#}) = \boldsymbol{e}(\frac{1}{2}z(C\tau + D)^{-1}C\,{}^{\mathrm{t}}z)f(z)$, of $z^{\#} = z(C\tau + D)^{-1}$, we have an isomorphism*

$$\Theta(\Sigma_B, m) \to \Theta(\Sigma_L, n).$$

5.2 Σ-trace

In order to express $\vartheta_0(\Sigma_L, z^\#)$ as a linear combination of translations of $\theta_m(z^\#)$ in the last section with simple exponential coefficients, we introduce the Σ-trace.

Definition 5.5 (Σ-trace). Let m be an element such that $m + \frac{1}{2}\delta_\Sigma \in \mathbb{Z}^6 \oplus \frac{1}{2}\mathbb{Z}^3 \oplus \mathbb{Z}^3$. Let S be a representative of (p, q) for

$$\mathbb{Z}^{12}/\{(c, d) \mid (c, d)\,{}^{\mathrm{t}}(\boldsymbol{a}, \boldsymbol{b}) \in (1 - \rho)H_1(C, \mathbb{Z})^-\}.$$

The Σ-trace $\mathrm{tr}_\Sigma(f)(z^\#)$ is defined by

$$\mathrm{tr}_\Sigma(f)(z^\#) = \sum_{(p,q)\in S} f(z^\# + p\tau^\# + q)\boldsymbol{e}(\tfrac{1}{2}p\tau^{\#}\,{}^{\mathrm{t}}p + p\,{}^{\mathrm{t}}z^\#).$$

Proposition 5.6. *The Σ-trace $\mathrm{tr}_\Sigma(\theta_m)$ is independent of the choice of the representative S, and it is a constant multiple of $\vartheta_0(\Sigma_L, z^\#)$. Moreover there exists $m \in \mathbb{Z}^6 \oplus \frac{1}{2}\mathbb{Z}^3 \oplus \mathbb{Z}^3 - \frac{1}{2}\delta_\Sigma$ such that $\mathrm{tr}_\Sigma(\theta_m)$ is nonzero.*

Proof. Note that the Σ-trace $\mathrm{tr}_{\Sigma_L}(\theta_m)$ satisfies $(\mathrm{Eq}_n^\#)$ for (p, q) in a sufficiently small lattice in L. By Proposition 5.3 and the characterization of the space generated by theta functions for principally polarized abelian varieties, we have this proposition. ☐

Definition 5.7 (Φ_g,$\Phi_{g,n}$).

1) For each $g \in U(H_1(C, \mathbb{Z})^-)$, we choose $m_g \in \mathbb{Z}^6 \oplus \frac{1}{2}\mathbb{Z}^3 \oplus \mathbb{Z}^3 - \frac{1}{2}\delta_g$ such that $\mathrm{tr}_{\Sigma_g}(\theta_{m_g})$ is nonzero. We define Φ_g by $\mathrm{tr}_{\Sigma_g}(\theta_{m_g})$ and

$$c_g = \frac{\vartheta_0(\Sigma_g, z^\#)}{\Phi_g(z^\#)}.$$

2) For $n = (p_0, q_0) \in \mathbb{Q}^{12}$, we define

$$\Phi_{g,n}(z^\#) = \boldsymbol{e}(\tfrac{1}{2}p_0\tau\,{}^{\mathrm{t}}p_0 + p_0\,{}^{\mathrm{t}}(z^\# + q_0))\,\Phi_g(z^\# + p_0\tau^\# + q_0)$$

and $\Phi_{g,n} = \Phi_{g,n}(0)$.

Recall that $\Phi_{g,n}(z^\#)$ is a linear combination of translations of $\vartheta(\Sigma_B, z^\#)$. In the rest of this section, we compute its coefficients.

Proposition 5.8. *For $n_0 = (m_0+\frac{1}{2}\delta_g)\sigma^{-1}$, we choose representatives $S_g(n_0)$ and $S_B(m_0)$ of $n_0+\{\tilde{n} \mid \tilde{n}\,{}^{\mathrm{t}}(\boldsymbol{a}, \boldsymbol{b}) \in H_1(C, \mathbb{Z})^-\}/\mathbb{Z}^{12}$ and $m_0 + \mathbb{Z}^6 \oplus \frac{1}{2}\mathbb{Z}^3 \oplus \mathbb{Z}^3/\mathbb{Z}^{12}$, respectively. Then $\{\Phi_n(z^\#)\}_{n\in S_g(n_0)}$ and $\{\vartheta_m(\Sigma_B, z)\}_{m\in S_B(m_0)}$ are bases of the eight-dimensional vector spaces $\Theta(\Sigma_g, n_0)$ and $\Theta(\Sigma_B, m)$, respectively.*

Let $(p_0, q_0) \in \mathbb{Q}^{12}$. By the definition of $\Phi_{g,(p_0,q_0)}(z^\#)$, we have the following proposition by simple calculation.

Proposition 5.9. *For $(p_0, q_0) \in \mathbb{Q}^{12}$, we have*

$$\Phi_{g,(p_0,q_0)}(z^\#) = \sum_{(p,q)\in S} c^{(g)}_{(p_0,q_0),(p,q)}\vartheta_{m_g+(p+p_0,q+q_0)\sigma_g}(\Sigma_B, z) \cdot \boldsymbol{e}(\tfrac{1}{2}z(C\tau + D)^{-1}C\,{}^{\mathrm{t}}z),$$

and

$$\Phi_{g,(p_0,q_0)} = \sum_{(p,q)\in S} c^{(g)}_{(p_0,q_0),(p,q)}\vartheta_{m_g+(p+p_0,q+q_0)\sigma_g}(\Sigma_B),$$

where

$$c^{(g)}_{(p_0,q_0),(p,q)} = \boldsymbol{e}(-\tfrac{1}{2}q\,{}^{\mathrm{t}}p - \tfrac{1}{2}s\,{}^{\mathrm{t}}r)\,\boldsymbol{e}(-r\,{}^{\mathrm{t}}m_i'')\,\boldsymbol{e}(\tfrac{1}{2}p_0\,{}^{\mathrm{t}}q_0 - \tfrac{1}{2}r_0\,{}^{\mathrm{t}}s_0 - r_0\,{}^{\mathrm{t}}m_i'' - r_0\,{}^{\mathrm{t}}s).$$

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Here, m_g and S are in the definition of Σ_g-trace and $\Phi_g(z^\#)$, $(r_0, s_0) = (p_0, q_0)\sigma_g$ and $(r, s) = (p, q)\sigma_g$.

Let $n = (p_0, q_0)$ and (p, q) be elements of $S_g(n_0)$ and $\mathbb{Z}^{12}$, respectively. For $(r_0, s_0) = (p_0, q_0)\sigma_g$ and $(r, s) = (p, q)\sigma_g$, we have

$$\begin{aligned} m_g + (p + p_0, q + q_0)\sigma_g &\in m_g + n_0\sigma_g + \mathbb{Z}^6 \oplus \tfrac{1}{2}\mathbb{Z}^3 \oplus \mathbb{Z}^3 \\ &= m_g + m_0 + \tfrac{1}{2}\delta_g + \mathbb{Z}^6 \oplus \tfrac{1}{2}\mathbb{Z}^3 \oplus \mathbb{Z}^3 \\ &= m_0 + \mathbb{Z}^6 \oplus \tfrac{1}{2}\mathbb{Z}^3 \oplus \mathbb{Z}^3. \end{aligned}$$

For an element $m \in S_B(m_0)$, we put

$$I(m) = \{(p, q) \in S \mid m_g + (p + p_0, q + q_0)\sigma - m \in \mathbb{Z}^{12}\},$$

$$d^{(g,B)}_{n,m} = \sum_{(p,q)\in I(m)} c^{(g)}_{(p_0,q_0),(p,q)} \frac{\vartheta_{m_g+(p+q_0,q+q_0)\sigma}(\Sigma_B, z)}{\vartheta_m(\Sigma_B, z)}.$$

Then by Proposition 5.9, for $n \in S_g(n_0)$, $\Phi_{g,n}(z^\#)$ can be written as

$$\Phi_{g,n}(z^\#) = \sum_{m\in S_B(m_0)} d^{(g,B)}_{n,m} \vartheta_m(\Sigma_B, z) \cdot \boldsymbol{e}(\tfrac{1}{2}z(C\tau + D)^{-1}C\,{}^{\mathrm{t}}z). \tag{5.1}$$

We put

$$D^{(g,B)} = (d^{(g,B)}_{n,m})_{n\in S_g(n_0), m\in S_B(m_0)}.$$

This is the base change matrix for $\{\vartheta_m(\Sigma_B, z)\}_{m\in S_B(m_0)}$ and $\{\Phi_{g,n}(z^\#)\}_{n\in S_g(n_0)}$ up to a constant exponential multiple.

6. Main theorem

6.1 Action of the stabilizer of length 0 element on theta functions

By choosing a principal lattice L, we get an injective homomorphism from $U(H_{\mathrm{std}})$ to $\mathrm{Sp}(6, \mathbb{R}) = \mathrm{Aut}(\mathfrak{H}_6)$ and an inclusion $\jmath : B(H_{\mathrm{std}}) \to \mathfrak{H}_6$. By this inclusion, we identify $U(H_{\mathrm{std}})$ as a subgroup of $\mathrm{Sp}(6, \mathbb{R})$. We consider the function $\det(C\tau + D)^{1/2}$ on $\mathrm{Sp}(6, \mathbb{R}) \times \mathfrak{H}_6$ defined in § 4.3. Let

$$\sigma_g = \begin{pmatrix} A_g & B_g \\ C_g & D_g \end{pmatrix}$$

be a matrix such that $\Sigma_{L_g} = \sigma_g(\Sigma_B)$. We put $n_0 = (m_0 + \frac{1}{2}\delta_g)\sigma_g^{-1}$ for $m_0 \in \mathbb{Q}^{12}$. We choose $S_g(n_0)$ and $S_B(m_0)$ as in the last section. For any $n \in S_L(n_0)$, there exist complex numbers $u^{(g,B)}_{n,m}$ $(m \in S_B(m_0))$ independent of τ and z such that

$$\begin{aligned} \vartheta_n(\Sigma_g, z^\#_g) = \det(C_g\tau + D_g)^{-1/2}&\boldsymbol{e}(z(C_g\tau + D_g)C_g^{-1}\,{}^{\mathrm{t}}z) \\ &\times \sum_{m\in S_B(m_0)} u^{(g,B)}_{n,m} \vartheta_m(\Sigma_B, z), \end{aligned} \tag{6.1}$$

where $\tau^\#_g = (A_g\tau + B_g)(C_g\tau + D_g)^{-1}$ and $z^\#_g = z \cdot (C_g\tau + D_g)^{-1}$ by the transformation formula in [Igu72, p. 84]. Moreover this expression is unique. By comparing the right-hand sides of (5.1) and (6.1), we have the following proposition.

Proposition 6.1. *The matrix $U^{(g,B)} = (u^{(g,B)}_{n,m})_{n\in S_L(n_0), m\in S_B(m_0)}$ is a nonzero constant multiple c_g of $D^{(g,B)}$. Moreover the nonzero constant c_g does not depend on τ. Especially, for $n \in S_g(0)$, we* *have*

$$\begin{aligned}\vartheta_n(\Sigma_g, z_g^{\#}) &= \sum_{m\in S_1(0)} u_{n,m}^{(g)}\vartheta_m(\Sigma_1, z_1^{\#})\\ &= c_g \sum_{m\in S_1(0)} d_{n,m}^{(g)}\vartheta_m(\Sigma_1, z_1^{\#}).\end{aligned} \tag{6.2}$$

We define

$$U(H_{\mathrm{std}})_{\overline{\Delta}} = \{g \in U(H_{\mathrm{std}}) \mid \overline{\Delta}g = \overline{\Delta}\}.$$

We fix a representative $S_1(0)$. For example we choose $S_1(0) = \{\frac{1}{2}(\mu_j, \mu_j U)\}$ with

$$\mu_1 = (0,0,0,0,0,0),\quad \mu_2 = (0,0,1,1,1,1),\quad \mu_3 = (1,1,0,0,1,1),\quad \mu_4 = (1,1,1,1,0,0),$$
$$\mu_5 = (1,1,1,1,1,1),\quad \mu_6 = (1,1,0,0,0,0),\quad \mu_7 = (0,0,1,1,0,0),\quad \mu_8 = (0,0,0,0,1,1).$$

If $n\,{}^{\mathrm{t}}(\boldsymbol{a},\boldsymbol{b}) \in H_1(C,\mathbb{Z})^-$, then $n\,{}^{\mathrm{t}}(g(\boldsymbol{a}), g(\boldsymbol{b})) = g(n\,{}^{\mathrm{t}}(\boldsymbol{a},\boldsymbol{b})) \in g(H_1(C,\mathbb{Z})^-)$ for $g \in U(H_{\mathrm{std}})_{\overline{\Delta}}$. Therefore we can take a representative $S_g(0)$ as $S_1(0)$. Then the vector spaces generated by $\Phi_n(z_g^{\#})$ ($n \in S_g(0)$) and $\Phi_n(z^{\#})$ ($n \in S_1(0)$) are isomorphic via the map defined in Proposition 5.4:

$$\Theta(\Sigma_g, 0) \xleftarrow{\simeq} \Theta(\Sigma_B, -\overline{\Delta}) \xrightarrow{\simeq} \Theta(\Sigma_1, 0),$$

where $z_g^{\#}$ and $z^{\#}$ are related by $z^{\#} = z(C\tau + D)^{-1}$ and $z_g^{\#} = z(C_g\tau + D_g)^{-1}$. We put $D^{(g)} = D^{(g,B)}(D^{(\mathrm{id},B)})^{-1}$ and $U^{(g)} = U^{(g,B)}(U^{(\mathrm{id},B)})^{-1}$. By the definition of $U^{(g,B)}$, the map $g \in U(H_{\mathrm{std}})_{\overline{\Delta}} \mapsto U^{(g)}$ defines a projective representation of $U(H_{\mathrm{std}})_{\overline{\Delta}}$, which is denoted by χ.

Since $U^{(g)} = c_g c_1^{-1} D^{(g)}$, we have the following corollary of Proposition 6.1.

Corollary 6.2. *The map*

$$U(H_{\mathrm{std}})_{\overline{\Delta}} \ni g \mapsto D^{(g,B)}(D^{(1,B)})^{-1} \in \mathrm{Aut}(\mathbb{C}^{S_1(0)})$$

becomes a projective representation of $U(H_{\mathrm{std}})_{\overline{\Delta}}$*, which is isomorphic to* χ.

Let $S_1(0)_{\mathrm{ev}}$ be the subset of $S_1(0)$ consisting of $v\,{}^{\mathrm{t}}Uv \in 4\mathbb{Z}$. In the example given as above, we have $S_1(0)_{\mathrm{ev}} = \{\frac{1}{2}(\mu_j, \mu_j U)\}$ with $j = 1,4,6,7$. By evaluating (6.2) at $z_1^{\#} = z_g^{\#} = 0$, for $n_2 \in S_1(0)_{\mathrm{ev}}$, we have

$$\begin{aligned}\vartheta_n(\Sigma_g) &= \sum_{m\in S_1(0)_{\mathrm{ev}}} u_{n,m}^{(g)}\vartheta_m(\Sigma_1)\\ &= c_g c_1^{-1} \sum_{m\in S_1(0)_{\mathrm{ev}}} d_{n,m}^{(g)}\vartheta_m(\Sigma_1).\end{aligned} \tag{6.3}$$

We put $D_{\mathrm{ev}}^{(g)} = (d_{n,m}^{(g,B)})_{n,m\in S_1(0)_{\mathrm{ev}}}$. We define a projective representation χ_{const} as

$$\chi_{\mathrm{const}}(g) = D_{\mathrm{ev}}^{(g)} \in PGL(4,\mathbb{C})$$

on the space of theta constants. Note that an element g in $U(H_{\mathrm{std}})$ is in $U(H_{\mathrm{std}})_{\overline{\Delta}}$ if and only if $\pi(g) \in (\mathfrak{S}_4(1,2,5,6) \times \mathfrak{S}_4(3,4,7,8)) \rtimes \mathfrak{S}_2$ under the homomorphism $\pi : U(H_{\mathrm{std}}) \to \mathfrak{S}_8$ defined just after Lemma 3.6. Here $\mathfrak{S}_4(1,2,5,6)$ is the symmetric group of permutations of index $\{1,2,5,6\}$. Let $M_{2,5}$ be the (complex) reflection corresponding to the transposition of the points p_2 and p_5. Then we have $M_{2,5} \in U(H_{\mathrm{std}})_{\overline{\Delta}}$,

$$D_{\mathrm{ev}}^{(M_{2,5})} = \begin{pmatrix} \frac{1}{2}-\frac{1}{2}i & -\frac{1}{2}-\frac{1}{2}i & 0 & 0\\ -\frac{1}{2}-\frac{1}{2}i & \frac{1}{2}-\frac{1}{2}i & 0 & 0\\ 0 & 0 & \frac{1}{2}-\frac{1}{2}i & -\frac{1}{2}-\frac{1}{2}i\\ 0 & 0 & -\frac{1}{2}-\frac{1}{2}i & \frac{1}{2}-\frac{1}{2}i \end{pmatrix}$$

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and

$$ {}^{\mathrm{t}}(\vartheta_{m_j}(\tau_g^{\#}))_{j=1,4,3,2} = c_{M_{2,5}} c_1^{-1} \cdot \det(\gamma_g \tau^{\#} + \delta_g)^{1/2} \cdot D_{\mathrm{ev}}^{(M_{2,5})\mathrm{t}} (\vartheta_{m_j}(\tau_1^{\#}))_{j=1,4,3,2}, $$

where

$$ \sigma_g \cdot \sigma_1^{-1} = \begin{pmatrix} \alpha_g & \beta_g \\ \gamma_g & \delta_g \end{pmatrix} \in \mathrm{Sp}(6, \mathbb{Q}). $$

6.2 Theta constants and cross-ratios of coordinates

In this section, we combine the results in §§ 4.2 and 6.1 to get an $\mathfrak{S}_8$-equivariant presentation of a projective map from the moduli space $M_{8\mathrm{pts}}$ to $\boldsymbol{P}^{104}$.

We define a function $\mathcal{T}_g(\tau)$ of $\tau \in M_{\mathrm{marked}}$ by

$$ \mathcal{T}_g(\tau) = \det(\gamma_g \tau + \delta_g)^{-1} \vartheta_{m_1}(\tau_g^{\#}) \vartheta_{m_3}(\tau_g^{\#}), \tag{6.4} $$

where

$$ \sigma_g = \begin{pmatrix} \alpha_g & \beta_g \\ \gamma_g & \delta_g \end{pmatrix} $$

and $\tau_g^{\#}$ are defined as before for $g \in U(H_{\mathrm{std}})$. Then we have

$$ \mathcal{T}_g(h \cdot \tau) = \mathcal{T}_{gh}(\tau). \tag{6.5} $$

Since $\mathcal{T}_g^2$ depends only on the image $\pi(g) \in \mathfrak{S}_8$, it is also denoted by $\mathcal{T}_{\pi(g)}^2$.

By the result of the last section, if $g \in U(H_{\mathrm{std}})_{\overline{\Delta}}$, then $\mathcal{T}_g(\tau)$ is a homogeneous polynomial of $\vartheta_{m_j}(\tau)$ with constant coefficients. For example, if $g = M_{25}$, we have

$$ \mathcal{T}_g = \frac{-i}{2} \cdot (c_{M_{2,5}} c_1^{-1})^2 (\vartheta_{m_1}(\tau_1^{\#}) - i\vartheta_{m_4}(\tau_1^{\#})) \cdot (\vartheta_{m_3}(\tau_1^{\#}) - i\vartheta_{m_2}(\tau_1^{\#})). $$

Therefore we have

$$ \begin{aligned} \frac{\mathcal{T}_g^2}{\mathcal{T}_1^2} &= \frac{-1}{4} \cdot (c_{M_{2,5}} c_1^{-1})^4 \frac{(\vartheta_{m_1}(\tau_1^{\#}) - i\vartheta_{m_4}(\tau_1^{\#}))^2}{\vartheta_{m_1}(\tau_1^{\#})^2} \cdot \frac{(\vartheta_{m_3}(\tau_1^{\#}) - i\vartheta_{m_2}(\tau_1^{\#}))^2}{\vartheta_{m_3}(\tau_1^{\#})^2} \\ &= c \frac{(x_1 - x_5)(x_2 - x_6)}{(x_1 - x_2)(x_5 - x_6)}, \end{aligned} \tag{6.6} $$

where $c = (c_{M_{2,5}} c_1^{-1})^4$.

Let R be a set of representatives of the composite surjection

$$ U(H_{\mathrm{std}}) \xrightarrow{\pi} \mathfrak{S}_8 \to \mathrm{Stab}\{\{1,2\},\{5,6\},\{3,4\},\{7,8\}\} \backslash \mathfrak{S}_8, $$

where π is the natural surjection and $\mathrm{Stab}\{\{1,2\},\{5,6\},\{3,4\},\{7,8\}\}$ is the stabilizer of $\{\{1,2\}, \{5,6\},\{3,4\},\{7,8\}\}$. We fix this set R once and for all.

Definition 6.3 (Polynomial map). Set

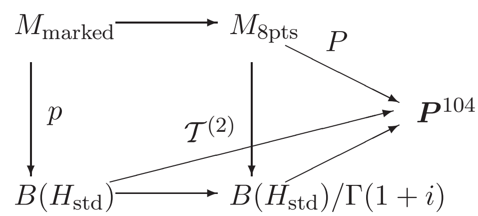
$$ P_1 = (x_1 - x_2)(x_3 - x_4)(x_5 - x_6)(x_7 - x_8) $$

and $P_r = \pi(r)^*(P_1)$ for $r \in R$. Since each P_r is relative invariant under the action of $PGL(2, \mathbb{C})$, the map $P : (\boldsymbol{P}^1)^8 - \mathrm{Diag} \to \boldsymbol{P}^{104}$ defined by the ratio of $(P_r)_{r \in R}$ descends to a morphism $M_{8\mathrm{pts}} \to \boldsymbol{P}^{104}$, which is also denoted by P. The composite $M_{\mathrm{marked}} \to M_{8\mathrm{pts}} \to \boldsymbol{P}^{104}$ is also denoted by P.

By Definition 6.3, the last term of (6.6) is equal to $c \cdot P_g / P_1$.

We have the following theorem.

Theorem 6.4. *Let $\mathcal{T}^{(2)}$ be the map from M_{marked} to $\boldsymbol{P}^{104}$ defined by $(\mathcal{T}_r^2)_{r \in R}$. Then the following diagram is commutative.*



In order to prove Theorem 6.4, we first give some lemmas. Let $[r]$ denote the class of $r \in \mathfrak{S}_8$ in $\mathrm{Stab}\{\{1,2\},\{5,6\},\{3,4\},\{7,8\}\}\backslash\mathfrak{S}_8$.

LEMMA 6.5.

1) *If $[r]=[r']$, then $\mathcal{T}_r^2$ is a constant multiple of $\mathcal{T}_{r'}^2$.*
2) *The map from $\mathrm{Stab}\{\{1,2\},\{5,6\},\{3,4\},\{7,8\}\}$ to $\{\pm 1\}$ defined by $g \mapsto \mathcal{T}_g^2/\mathcal{T}_1^2$ is a character and this coincides with the restriction of the signature on $\mathfrak{S}_8$.*

Proof. 1) By the equality (6.5), we have the following equation of rational functions of $M_{8\mathrm{pts}}$,

$$\frac{\mathcal{T}_g^2}{\mathcal{T}_1^2}(h(\tau)) = \frac{\mathcal{T}_{gh}^2}{\mathcal{T}_h^2}(\tau) \tag{6.7}$$

for $h \in \mathfrak{S}_8$. Thus we have only to prove the lemma for the case $r'=1$. If $[r]=[1]$, $\mathcal{T}_r^2$ is a constant multiple of $\mathcal{T}_1^2$ by Corollary 4.11 and the expression of the projective representation χ_{const}.

2) The first statement is a consequence of part 1. Using the transformation formula of [Igu72, p. 85], we have

$$\mathcal{T}_{M_{12}} = \mathcal{T}_{M_{56}} = i\mathcal{T}_1. \tag{6.8}$$

By applying $M_{12}M_{56}$ to the equality (6.6), we have

$$\mathcal{T}_{gM_{12}M_{56}}^2 = \mathcal{T}_g^2. \tag{6.9}$$

Equalities (6.8) and (6.9) characterize the character of $\mathrm{Stab}\{\{1,2\},\{5,6\},\{3,4\},\{7,8\}\}$ and the character $\mathcal{T}_g^2/\mathcal{T}_1^2$ coincides with the restriction of the signature. □

LEMMA 6.6. *Let $[r]$ be an element of $\mathrm{Stab}\{\{1,2\},\{5,6\},\{3,4\},\{7,8\}\}\backslash\mathfrak{S}_8$ and $(2,6) \in \mathfrak{S}_8$ be the transposition of 2 and 6. Then there exist sequences of $g_1,\dots,g_{k+1}$ and $h_1,\dots,h_k$ of $\mathfrak{S}_8$ such that*

1) $[r] = [g_1h_1\cdots g_kh_kg_{k+1}]$,
2) $[g_1h_1\cdots g_l] = [g_1h_1\cdots g_lh_l]$ *for* $l=1,\dots,k$,
3) $[(2,6)] = [(2,6)g_1]$, $[(2,6)g_1h_1\cdots g_lh_l] = [(2,6)g_1h_1\cdots g_lh_lg_{l+1}]$ *for* $l=1,\dots,k$.

We are now in a position to prove the theorem.

Proof of Theorem 6.4. Using (6.7), we have

$$\frac{\mathcal{T}_{M_{2,5}}^2}{\mathcal{T}_1^2}(M_{2,6}(\tau)) = \frac{\mathcal{T}_{M_{2,5}M_{2,6}}^2}{\mathcal{T}_{M_{2,6}}^2}(\tau) = -\frac{\mathcal{T}_{M_{2,5}}^2}{\mathcal{T}_{M_{2,6}}^2}(\tau). \tag{6.10}$$

We put $c = (c_{M_{2,5}}c_1^{-1})^4$. Since the map p is equivariant under the action of $\mathfrak{S}_8$, we have

$$\frac{\mathcal{T}_{M_{2,5}}^2}{\mathcal{T}_1^2}(h(\tau)) = c\cdot\frac{(x_{1h}-x_{5h})(x_{2h}-x_{6h})}{(x_{1h}-x_{2h})(x_{5h}-x_{6h})} \tag{6.11}$$

for $h \in \mathfrak{S}_8$. Equations (6.10) and (6.11) yield

$$\frac{\mathcal{T}_{(2,6)}^2}{\mathcal{T}_1^2}(\tau) = \frac{\mathcal{T}_{M_{2,6}}^2}{\mathcal{T}_1^2}(\tau) = \frac{P_{(2,6)}}{P_1}. \tag{6.12}$$

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For any $r \in \mathfrak{S}_8$, there exist sequences $g_1, \ldots, g_{k+1}$ and $h_1, \ldots, h_k$ such that

$$\begin{aligned} [(2,6)] &= [(2,6)g_1], \qquad [(2,6)g_1h_1] = [(2,6)g_1h_1g_2], \\ [g_1] &= [g_1h_1], \qquad [g_1h_1g_2] = [g_1h_1g_2h_2] \\ \ldots \quad [(2,6)g_1\cdots h_k] &= [(2,6)g_1\cdots h_kg_{k+1}] \\ \ldots \quad [g_1h_1\cdots h_kg_{k+1}] &= [r], \end{aligned}$$

by Lemma 6.6. By applying g_1 and g_1h_1 to the equality (6.12), we have

$$\frac{\mathcal{T}^2_{(2,6)}}{\mathcal{T}^2_{g_1}}(\tau) = \frac{\mathcal{T}^2_{(2,6)g_1}}{\mathcal{T}^2_{g_1}}(\tau) = \frac{P_{(2,6)g_1}}{P_{g_1}} = \frac{P_{(2,6)}}{P_{g_1}}, \tag{6.13}$$

$$\frac{\mathcal{T}^2_{(2,6)g_1h_1}}{\mathcal{T}^2_{g_1}}(\tau) = \frac{\mathcal{T}^2_{(2,6)g_1h_1}}{\mathcal{T}^2_{g_1h_1}}(\tau) = \frac{P_{(2,6)g_1h_1}}{P_{g_1h_1}} = \frac{P_{(2,6)g_1h_1}}{P_{g_1}}. \tag{6.14}$$

From the equalities (6.12) and (6.13), we have

$$\frac{\mathcal{T}^2_{g_1}}{\mathcal{T}^2_{1}}(\tau) = \frac{P_{g_1}}{P_1} \tag{6.15}$$

and from the equalities (6.14) and (6.15), we have

$$\frac{\mathcal{T}^2_{(2,6)g_1h_1}}{\mathcal{T}^2_{1}}(\tau) = \frac{P_{(2,6)g_1h_1}}{P_1}.$$

We continue this procedure. We get an identity

$$\frac{\mathcal{T}^2_{g}}{\mathcal{T}^2_{1}}(\tau) = \frac{P_{g}}{P_1} \quad \text{for all } g \in \mathfrak{S}_8,$$

which completes the proof. □

6.3 Branched covering of $M_{8\mathrm{pts}}$ corresponding to $\Gamma(2)$

In this section, we study the map from M_{marked} to $\tilde{M}_{8\mathrm{pts}}$ defined by the theta constants on $B(H_{\mathrm{std}})$. As in § 3.3, we choose an initial point $X = (x_1, \ldots, x_8)$ and specify the branch of the function $\sqrt{x_k - x_j}$.

Let r_1 be 2^4-partition $\{\{1,2,\},\{3,4\},\{5,6\},\{7,8\}\}$ and $\pi : U(H_{\mathrm{std}}) \to \mathfrak{S}_8$ be the natural projection. Using the argument $\arg(g)$ defined in § 3.3, we define a multivalued function Q_r ($r \in R$), on $\mathbb{C}^8 - \mathrm{Diag}$ by

$$Q_r = \arg(g)\sqrt{(x_{j_2} - x_{j_1})(x_{j_4} - x_{j_3})(x_{j_6} - x_{j_5})(x_{j_8} - x_{j_7})},$$

where $r_1\pi(r) = \{\{j_1,j_2\},\{j_3,j_4\},\{j_5,j_6\},\{j_7,j_8\}\}$ and $j_p < j_{p+1}$ for $j = 1,3,5,7$. Here we chose the branch of the square root as in § 3.3. Let $\tilde{N}$ be the covering of $\mathbb{C}^8 - \mathrm{Diag}$ defined by $\sqrt{x_j - x_k}$ ($1 \leqslant k < j \leqslant 8$). Then the functions Q_r on $\tilde{N}$ define a morphism $Q = (Q_r)_{r\in R} : \tilde{N} \to \boldsymbol{P}^{104}$. Let $pr_1 : \boldsymbol{P}^{104} \to \boldsymbol{P}^{104}$ be the morphism defined by $(y_r)_{r\in R} \mapsto (y_r^2)_{r\in R}$. Since the coordinates of the inverse image of $P(\lambda_4, \ldots, \lambda_8)$ under pr_1 can be expressed by polynomials of $\sqrt{\lambda_j}$, $\sqrt{1-\lambda_j}$, $\sqrt{\lambda_j - \lambda_k}$, the morphism $\tilde{M}_{8\mathrm{pts}} \to M_{8\mathrm{pts}} \to \boldsymbol{P}^{104}$ factors through pr_1 (see the following diagram).

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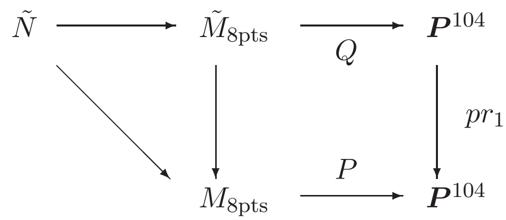
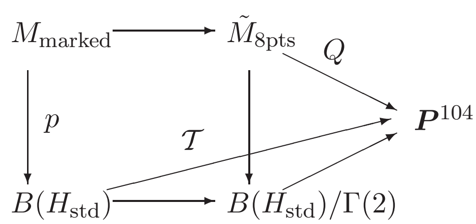


We define the morphism Q by the above diagram.

Theorem 6.7. *Let $\mathcal{T} = (\mathcal{T}_r)_{r\in R}$ be the morphism from $B(H_{\text{std}})$ to $\boldsymbol{P}^{104}$ defined by $\mathcal{T}_r$ $(r \in R)$. Then the following diagram is commutative.*



Proof. We determine the branch of the square root of the last term in (6.6). Let $h = M_{1,5}, g = M_{2,5}$ be elements of $U(H_{\text{std}})$. Then we have $h(L_1) = gh(L_1)$. By the transformation formula [Igu72, p. 85], we have

$$\mathcal{T}_{gh} = i\mathcal{T}_h.$$

By Theorem 6.4, we have

$$\frac{\mathcal{T}_h}{\mathcal{T}_1} = i \cdot \epsilon \cdot \sqrt{\frac{(x_5 - x_2)(x_6 - x_1)}{(x_2 - x_1)(x_6 - x_5)}} \tag{6.16}$$

with $\epsilon = \pm 1$. By applying g^* to both sides of (6.16), we get

$$i \cdot \frac{\mathcal{T}_h}{\mathcal{T}_g} = \frac{\mathcal{T}_{hg}}{\mathcal{T}_g} = \frac{-\epsilon}{\arg(g)} \cdot \sqrt{\frac{(x_5 - x_2)(x_6 - x_1)}{(x_5 - x_1)(x_6 - x_2)}}. \tag{6.17}$$

By (6.16) and (6.17), we have

$$\frac{\mathcal{T}_g}{\mathcal{T}_1} = \frac{\mathcal{T}_g}{\mathcal{T}_h} \cdot \frac{\mathcal{T}_h}{\mathcal{T}_1} = \arg(g) \cdot \sqrt{\frac{(x_5 - x_1)(x_6 - x_2)}{(x_2 - x_1)(x_6 - x_5)}}.$$

Using Lemma 6.6 and the same argument as in the proof of Theorem 6.4, we have the theorem. □

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