A Relation between two Ordinary Linear Differential Equations of the Second Order

By D. G. TAYLOR.

1. Between the solutions of the equations

$$\frac{d^2 y}{dx^2} + f'(x)\, y = 0, \tag{1}$$

$$\frac{d^2 \eta}{d\xi^2} + \phi'(\xi)\, \eta = 0, \tag{2}$$

a relation can be established, provided the functional symbols f, ϕ are inverse to one another. For example, let $f(x) = \sin x$, then $\phi(\xi) = \arcsin \xi$, and the two equations are

$$\frac{d^2 y}{dx^2} + (\cos x)\, y = 0, \quad \frac{d^2 \eta}{d\xi^2} + \frac{1}{\sqrt{(1 - \xi^2)}}\, \eta = 0. \tag{3}$$

The process consists in obtaining from (1) the second order equation satisfied by $y_1 \equiv dy/dx$, making a change of independent variable in (2), and comparing the resulting equations.

Writing (1) in the form

$$\frac{1}{f_1} y_2 + y = 0,$$

the suffixes indicating differentiation with respect to x, and differentiating, we have, after multiplying up by f_1,

$$\frac{d^2 y_1}{dx^2} - \frac{f_2}{f_1} \frac{dy_1}{dx} + f_1 y_1 = 0. \tag{4}$$

In (2) write $\xi = \xi(\zeta)$, ζ being a new independent variable. Accents denoting differentiation with respect to ζ, we have

$$\frac{d^2 \eta}{d\zeta^2} - \frac{\xi''}{\xi'} \frac{d\eta}{d\zeta} + \xi'^2 \phi'(\xi)\, \eta = 0. \tag{5}$$

Equations (4) and (5) become identical in *form* if we make

$$f_1 \equiv \xi', \quad \xi' \phi'(\xi) = 1. \tag{6}$$

The first of these conditions affirms an equivalence of functional symbols, which is satisfied if we make the functional symbols f, ξ identical. The second is $d\zeta = \phi'(\xi)\, d\xi$, that is, $\zeta = \phi(\xi)$, the constant of integration being ignored. The functional symbol ϕ is thus the inverse of the functional symbol ξ, that is, the inverse of f.

2. Now, whatever function y_1 be of x in (4), η will be the same function of ζ in (5). Let us assume that we know the solution of (4) in the form

$$y_1 = y_1(x). \tag{7}$$

Then we have also

$$\eta = y_1(\zeta). \tag{8}$$

The solution of (1) is

$$y = \int y_1(x)\, dx, \tag{9}$$

and of (2),

$$\eta = y_1\{\phi(\xi)\}, \tag{10}$$

so that if (9) is known, we immediately have the form of y_1, and therefore (10).

Conversely, let us assume that we know the solution of (2) in the form $\eta = \eta(\xi)$. Since f, ξ are identical functional symbols, we have $\eta = \eta\{f(\zeta)\}$,

whence

$$y_1 = \eta\{f(x)\},$$

or

$$y = \int \eta\{f(x)\}\, dx. \tag{11}$$

Thus η is obtained from y by differentiation, y from η by integration. The integration in (11) introduces a third arbitrary constant, corresponding to the fact that (4) is of the third order as an equation in y; and a relation between the three constants will be obtained by actual substitution in (1).

But, the relations between the two equations being reciprocal, the processes applied to them can be interchanged, and the superfluous constant need not arise.

3. If (1), (2) are replaced by Riccati equations, there will likewise be a relation between the solutions of these. If for example we write

$$b = -\frac{1}{y}\frac{dy}{dx}, \quad \beta = -\frac{1}{\eta}\frac{d\eta}{d\xi}, \tag{12}$$

(1), (2) yield

$$\frac{db}{dx} = b^2 + f'(x), \tag{13}$$

$$\frac{d\beta}{d\xi} = \beta^2 + \phi'(\xi). \tag{14}$$

Thus if we know the solution of (13) in the form $b = b(x)$, we can write

$$y = \exp\left(-\int b\,dx\right), \quad y_1 = -b \exp\left(-\int b\,dx\right),$$

and finally

$$\eta = -b\{\phi(\xi)\}\exp\left[-\int b\{\phi(\xi)\}\phi'(\xi)\,d\xi\right].$$

A Generalisation of a Theorem of Wolstenholme

By Hansraj Gupta.

§ 1. Chowla[1] has generalised Wolstenholme's Theorem as follows:

$$\sum_{a <\cdot\, p^u} \frac{1}{a} \equiv 0 \pmod{p^{2u}}, \quad p > 3, \ u > 0;$$

where p denotes as usual a prime, and $<\cdot$ is used for "less than and prime to."

Denoting the greatest common divisor of two non-zero positive integers n and m by $\{n, m\}$, I here prove that

$$\lambda \,.\, \sum_{a <\cdot\, n} \frac{1}{a} \equiv 0 \pmod{n^2}, \qquad [1]$$

where $\lambda = \left\{n^2, \frac{12}{l}\right\}$, and $l = 1$, 2, or 4, as defined later.

§ 2. Elsewhere[2] I have shown that

$$\sum_{i=1}^{n-1} i^{s-2} = \frac{n^{(s-1)}}{s-1} + G(-s+2, 1)\,.\,\frac{n^{(s-2)}}{s-2} + \ldots + G(-k, s-k-1)\,.\,\frac{n^{(k)}}{k}$$
$$+ \ldots + G(-2, s-3)\,.\,\frac{n^{(2)}}{2}, \qquad [2\cdot1]$$

where $n^{(k)} = n(n-1)(n-2)\ldots(n-k+1)$; $s > 2$;

and $$(k-2)!\,G(-k, s-k-1) = \sum_{j=0}^{k-2}\left\{(-1)^j\binom{k-2}{j}(k-j-1)^{s-3}\right\}, \quad s > k. \qquad [2\cdot2]$$

Evidently $\frac{n^{(k)}}{k} \equiv 0 \pmod{n}$, except when $k \mid n$, and is either 4 or a prime, 2, 3 or > 3.