ON PERFECT AND EXTREME FORMS

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1. Introduction

Let $f(x) = f(x_1, x_2, \cdots, x_n) = \sum_i \sum_j a_{ij}x_ix_j$ $(a_{ij} = a_{ji})$ be a positive quadratic form with determinant D, and let M be the minimum of f for integral $x \neq 0$. Then f attains the value M for a finite number of integral $x = \pm m_k$ $(k = 1, \cdots, s)$ called its *minimal vectors*.

f is said to be *perfect* if the s equations

$$f(m_k) = \sum_i \sum_j a_{ij}m_{ki}m_{kj} = M \qquad (k = 1, \cdots, s)$$

uniquely determine the $\frac{1}{2}n(n+1)$ coefficients a_{ij} of f; that is, if the equations

$$g(m_k) = \sum_i \sum_j b_{ij}m_{ki}m_{kj} = 0 \qquad (k = 1, \cdots, s; b_{ij} = b_{ji})$$

have only the trivial solution $b_{ij} \equiv 0$.

f is said to be *extreme* if for all infinitesimal variations of the coefficients a_{ij}, M^n/D is a maximum; defining $\Delta = (2/M)^nD$, we see that f is extreme if Δ is a local minimum.

Let $F(y) = \sum_i \sum_j A_{ij}y_iy_j$ be the adjoint of f; we say that f is *eutactic* if $F(y)$ is expressible as

$$(1.1) \qquad F(y) = \sum_{k=1}^{s} \rho_k(m_k'y)^2 = \sum_{k=1}^{s} \rho_k\lambda_k^2 \qquad (\rho_k > 0; k = 1, \cdots, s).$$

Voronoi [9] proved

THEOREM 1.1. *A positive quadratic form is extreme if and only if it is perfect and eutactic.*

For forms with $n \geqq 6$, this is often not a simple criterion to apply; in § 2 I give a useful simplification of the general relation (1.1) in terms of the group of automorphs of the form. A more specialised result of this nature has been obtained by Barnes [1].

All the perfect and extreme forms are now known for $n \leqq 6$. In particular, Korkine and Zolotareff [8] found all the perfect forms for $n \leqq 5$, and recently Barnes [2] has given the complete enumeration of the perfect forms for $n = 6$. Relatively little appears to be known about the forms for

$n > 6$; most of the known perfect forms are listed in Coxeter [6] and Barnes [3,I]. All others are: K_{12} given in [7]; K_{11} of [3,II]; Φ_{10} of [5]; the unclassified forms given in [3,II]; and the sequences of forms of [4] and [10].

In §§ 3 and 5, I define two new classes of forms which considerably extend the list of known perfect forms. Thus for the early values of n, we find that these new forms R_n, S_n contribute:

for $n = 7$, 7 perfect forms;
for $n = 8$, 21 perfect forms;
for $n = 9$, 43 perfect forms.

All except four of these are new, the exceptions being $R_7(3, 2, 2)$, $R_9(5, 3, 1)$, $S_7(6, 2)$ and $S_7(5, 3)$ which appear as extensions in [3, II]. However, these forms are classified here for the first time. Tables of the forms R_n, S_n for $n = 7, 8$ and 9 are given at the end of §§ 3 and 5 respectively.

Suppose the variables of the $(n+1)$-dimensional form $f(x) = f(x_1, \cdots, x_{n+1})$ are made to satisfy the non-trivial linear relation

$$\sum_{1}^{n+1} p_i x_i = 0. \tag{1.2}$$

The form $f(x)$ and the condition (1.2) now define a new form $g(x)$ say; $g(x)$ is said to be the *section* of $f(x)$ by $\sum p_i x_i = 0$. $g(x)$ is in fact an n-dimensional form; in practice, however, because of symmetry considerations, it is often more convenient to leave it expressed in $n+1$ variables. It should be noted that the form $g(x)$ (as an $(n+1)$-variable form) has no unique adjoint form; the adjoint $G(y)$ is in fact found to be dependent on the particular n variables from $x_1, \cdots, x_{n+1}$, remaining after elimination of a variable between $f(x)$ and (1.2). The forms S_n of § 5 are obtained as sections of the forms R_{n+1} defined in § 3. In § 4, I obtain a number of results relating the properties of a form to those of its section. These are: (i) a necessary and sufficient condition that a section of a perfect form be perfect; (ii) formulae giving the adjoint and determinant of the section in terms of the known form. These results are then used to establish the properties of the forms S_n.

The definitions of the forms B_n, L_n^r, P_n and Q_n, referred to in this paper, are given in [3,I].

Finally, I wish to thank Professor E. S. Barnes for his helpful suggestions connected with this work.

2. Simplification of Voronoi's criterion for eutactic forms

As we saw in § 1, the form $f(x) = \sum\sum a_{ij} x_i x_j$, is eutactic if its adjoint $F(x) = \sum\sum A_{ij} x_i x_j$ is expressible as

$$F = \sum_{1}^{s} \rho_k \lambda_k^2, \qquad (\rho_k > 0;\ k = 1, \cdots, s). \tag{2.1}$$

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Let g be the group of automorphs of f. Then under the contragredient group G, the linear forms λ_k fall into the transitive systems

$$(\lambda_1^{(1)}, \cdots, \lambda_{k_1}^{(1)}), \cdots, (\lambda_1^{(r)}, \cdots, \lambda_{k_r}^{(r)}). \tag{2.2}$$

We now rewrite (2.1) as

$$F = \sum_{i=1}^{r} \left(\sum_{k=1}^{k_i} \rho_k^{(i)} (\lambda_k^{(i)})^2 \right), \qquad \rho_k^{(i)} > 0. \tag{2.3}$$

We now prove

LEMMA 2.1. (i) *If F can be expressed in the form* (2.3) *with the $\rho_k^{(i)}$ unrestricted in sign, then there is an expression with*

$$\rho_k^{(i)} = \sigma_i \qquad (k = 1, \cdots, k_i, i = 1, \cdots, r).$$

(ii) *The form f is eutactic if and only if there is now a solution of* (2.3) *with*

$$\sigma_1 > 0, \cdots, \sigma_r > 0.$$

PROOF. If the group G has order h, there are precisely h/k_i $(i = 1, \cdots, r)$ elements of G which transform a form of the ith set of (2.2) into another given form of that set. Applying all the transformations of G to (2.3), and adding, we obtain

$$hF = \sum_{i=1}^{r} \left\{ \frac{h}{k_i} (\rho_1^{(i)} + \rho_2^{(i)} + \cdots + \rho_{k_i}^{(i)}) \sum_{k=1}^{k_i} (\lambda_k^{(i)})^2 \right\}.$$

Thus

$$F = \sigma_1 \sum_{k=1}^{k_1} (\lambda_k^{(1)})^2 + \cdots + \sigma_r \sum_{k=1}^{k_r} (\lambda_k^{(r)})^2, \tag{2.4}$$

where

$$\sigma_i = \frac{1}{k_i} (\rho_1^{(i)} + \rho_2^{(i)} + \cdots + \rho_{k_i}^{(i)}), \qquad (i = 1, \cdots, r). \tag{2.5}$$

This proves (i).

If now there is a solution of (2.4) with

$$\sigma_i > 0 \qquad (i = 1, \cdots, r),$$

clearly this is also a solution of (2.3), and f is eutactic.

If, however, for some i, necessarily $\sigma_i \leqq 0$, then from (2.5) there is at least one value of j $(1 \leqq j \leqq k_i)$, for which

$$\rho_j^{(i)} \leqq 0,$$

and f is not eutactic. This completes the proof.

COROLLARY 1. *If in* (2.4) *there is some value of i for which $\sigma_i < 0$, then from* (2.5), *there is at least one value of j $(1 \leqq j \leqq k_i)$ for which*

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$$\rho_j^{(i)} < 0.$$

In practice, Lemma 2.1 has no great application, as a complete knowledge of the group G is required. However, we can use the lemma to obtain the following more general result.

THEOREM 2.1. *F has a representation of the form*

$$F = \sum_1^s \rho_k \lambda_k^2 \tag{2.6}$$

with either $\rho_k > 0$ $(k = 1, \cdots, s)$ or ρ_k unrestricted in sign $(k = 1, \cdots, s)$, if and only if there is a representation which also satisfies the condition that $\rho_r = \rho_s$ whenever λ_r and λ_s are equivalent under G.

PROOF. The representation provided by Lemma 2.1 satisfies the condition of the theorem, since any two equivalent forms λ_r, λ_s are included in one system of transitivity under G.

3. The form $R_m(r_1, r_2, \cdots, r_k)$

3.1. *Definition, Minimum and Determinant.* We define $R_m = R_m(r_1, r_2, \cdots, r_k)$ to be the form

$$f(x) = \sum_{t=1}^k A_{r_t}(x^{(t)}) \tag{3.1}$$

with lattice the sublattice of the integral lattice

$$\sum_1^m x_i \equiv 0 \ (\mathrm{mod}\ (r_1+1)), \tag{3.2}$$

where

$$r_1 \geqq r_2 \geqq \cdots \geqq r_k \geqq 1; \sum_{t=1}^k r_t = m,$$

and $x = (x^{(1)}, \cdots, x^{(k)})$;
and A_r is the connected, reflexible form of [6], defined by

$$A_r(x) = x_1^2 - x_1x_2 + x_2^2 - \cdots - x_{r-1}x_r + x_r^2.$$

For example $R_7(6, 1)$ is the form

$$f(x) = \{(x_1^{(1)})^2 - x_1^{(1)}x_2^{(1)} + \cdots - x_5^{(1)}x_6^{(1)} + (x_6^{(1)})^2\} + (x_1^{(2)})^2$$

with lattice the sublattice of the integral lattice

$$\sum_1^7 x_i \equiv 0 \ (\mathrm{mod}\ 7),$$

where

$$x = (x^{(1)}, x^{(2)}).$$

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Since A_r has determinant $(r+1)/2^r$, we see that

$$D(R_m) = (r_1+1)^2 \prod_{t=1}^{k} \left(\frac{r_t+1}{2^{r_t}}\right)$$
$$= \frac{1}{2^m} (r_1+1)^2 \prod_{t=1}^{k} (r_t+1).$$

We shall also show that

$$M(R_m) = 2, \quad \text{with} \quad \Delta(R_m) = \frac{1}{2^m} (r_1+1)^2 \prod_{t=1}^{k} (r_t+1).$$

We first examine all integral vectors $x \neq 0$ for which

$$f \leqq 2. \tag{3.3}$$

Let $e_i^{(t)}$ denote the unit vector in m-space corresponding to the coordinate $x_i^{(t)}$. Since

$$A_{r_t}(x^{(t)}) \begin{cases} = 0 & \text{if} \quad x^{(t)} = 0, \\ = 1 & \text{if} \quad \pm x^{(t)} = \sum_{i=p+1}^{p+h} e_i^{(t)}, \qquad (0 \leqq p < p+h \leqq r_t), \\ \geqq 2 & \text{otherwise,} \end{cases} \tag{3.4}$$

in order to satisfy (3.3), $A_{r_t}(x^{(t)})$ can be non-zero for at most two values of t.

(i) Suppose a single $A_{r_t}(x^{(t)})$ is non-zero. Since no $x^{(t)}$ for which $A_{r_t}(x^{(t)}) = 1$ satisfies the relation (3.2), we have $A_{r_t}(x^{(t)}) \geqq 2$.

If $r_t \geqq 3$, there are vectors $x^{(t)}$ satisfying (3.2) for which $A_{r_t}(x^{(t)}) = 2$; for example

$$x^{(t)} = e_i^{(t)} - e_j^{(t)} \qquad (j \neq i+1).$$

(ii) Suppose $A_{r_t}(x^{(t)})$ is non-zero for just two values of t, $t = t_1$, and $t = t_2$ say. Then from (3.4), $f \geqq 2$, equality holding when

$$A_{r_{t_1}}(x^{(t_1)}) = A_{r_{t_2}}(x^{(t_2)}) = 1.$$

In this case, we have

$$\pm(x^{(t_1)} \pm x^{(t_2)}) = \sum_{j=p_1+1}^{p_1+h_1} e_j^{(t_1)} \pm \sum_{j=p_2+1}^{p_2+h_2} e_j^{(t_2)} \qquad (0 \leqq p_i < p_i+h_i \leqq r_{t_i},\ i = 1, 2),$$

where $x^{(t_1)}$, $x^{(t_2)}$ are defined with like sign in (3.4_2). Of these, only the following satisfy (3.2), and so are minimal vectors:

$$\pm(x^{(t_1)} + x^{(t_2)}) \quad \text{with} \quad h_1 + h_2 = r_1 + 1,$$
$$\pm(x^{(t_1)} - x^{(t_2)}) \quad \text{with} \quad h_1 = h_2.$$

Hence the form $R_m(r_1, r_2, \cdots, r_k)$ has minimum 2 as required, provided $k = 1, r_1 \geqq 3$; or $k \geqq 2$.

We note that the forms B_m, L_m^r are special cases of R_m with

$$r_1 = r_2 = \cdots = r_m = 1,$$
$$2 = r_1 \geqq r_2 \geqq \cdots \geqq r_k \geqq 1,$$

respectively. To avoid repetition, in what follows we assume $r_1 \geqq 3$.

3.2. *Conditions for Perfection.* We shall need the following minimal vectors of R_m:

$$(3.5) \quad \begin{array}{l} e_i^{(t)} - e_j^{(t)} (1 \leqq i < j \leqq r_t, \quad j \neq i+1; \quad 1 \leqq t \leqq k), \\ e_i^{(t_1)} - e_j^{(t_2)} (1 \leqq i \leqq r_{t_1}, 1 \leqq j \leqq r_{t_2}; \quad 1 \leqq t_1 < t_2 \leqq k), \end{array}$$

$$(3.6) \quad \begin{array}{l} e_i^{(t)} + e_{i+1}^{(t)} - e_j^{(t)} - e_{j+1}^{(t)} (1 \leqq i < j \leqq r_t, j > i+2; 1 \leqq t \leqq k), \\ e_i^{(t_1)} + e_{i+1}^{(t_1)} - e_j^{(t_2)} - e_{j+1}^{(t_2)} (1 \leqq i < i+1 \leqq r_{t_1}, 1 \leqq j < j+1 \leqq r_{t_2}; \\ \hfill 1 \leqq t_1 < t_2 \leqq k), \end{array}$$

$$(3.7) \quad \sum_{i=1}^{r_1} e_i^{(1)} + e_j^{(t)} (1 \leqq j \leqq r_t; 2 \leqq t \leqq k),$$

$$(3.8) \quad \left. \begin{array}{l} \sum_{i=1}^{r_1-1} e_i^{(1)} + e_j^{(t)} + e_{j+1}^{(t)} \\ \sum_{i=2}^{r_1} e_i^{(1)} + e_j^{(t)} + e_{j+1}^{(t)} \end{array} \right\} (1 \leqq j < j+1 \leqq r_t; 2 \leqq t \leqq k).$$

LEMMA 3.1. *If the form R_m defined by* (3.1) *and* (3.2) *is perfect, then so is the form $R_{m+r_0}(r_0 \leqq r_1)$:*

$$f_0(x, x^{(0)}) = f(x) + A_{r_0}(x^{(0)})$$

with lattice

$$\sum_1^{m+r_0} x_i \equiv 0 \ (\mathrm{mod}\ (r_1+1)).$$

PROOF. The minimal vectors of R_{m+r_0} include

(3.9) (i) the vectors (3.5_1) with $t = 0$; (3.5_2) with $t_2 = 0$;

(3.10) (ii) the vectors (3.6_1) with $t = 0$; (3.6_2) with $t_2 = 0$;

(3.11) (iii) the vectors (3.7) with $t = 0$.

Suppose all the minimal vectors of R_{m+r_0} satisfy the relation

$$(3.12) \qquad \sum_1^{m+r_0} \sum_1^{m+r_0} p_{ij} x_i x_j = 0 \qquad (p_{ij} = p_{ji}).$$

We set

$$q_{ij} = q_{ji} = 2p_{ij} - p_{ii} - p_{jj} \qquad (i \neq j).$$

Since R_m is perfect

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(3.13) $$p_{ij} = 0 \qquad (1 \leqq i \leqq j \leqq m),$$

and so

(3.14) $$q_{ij} = 0 \qquad (1 \leqq i < j \leqq m).$$

From the vectors (3.9),

$$q_{ij} = 0$$

for i, j taken over the ranges given in (3.5).

If $r_0 \geqq 2$, from the vectors (3.10) we obtain

$$q_{i,i+1} + q_{j,j+1} = 0,$$

where i, j take values as in (3.6). Using (3.14),

$$q_{j,j+1} = 0 \qquad (m+1 \leqq j < j+1 \leqq m+r_0),$$

and hence

$$q_{ij} \equiv 0 \qquad (1 \leqq i < j \leqq m+r_0).$$

It follows that (3.12) must be of the form

$$\left(\sum_1^{m+r_0} x_i\right)\left(\sum_1^{m+r_0} p_{jj} x_j\right) = 0.$$

From (3.13), $p_{jj} = 0$ $(1 \leqq j \leqq m)$; now using the vectors (3.11),

$$p_{jj} = 0 \qquad (m+1 \leqq j \leqq m+r_0),$$

and R_{m+r_0} is perfect.

We now examine those forms which cannot be obtained in this way.

I. Forms containing three terms A_{r_i}, $(r_1 \geqq r_2 \geqq r_3 \geqq 2)$.

$$f(x) = A_{r_1}(x^{(1)}) + A_{r_2}(x^{(2)}) + A_{r_3}(x^{(3)}),$$

and

$$\sum_1^m x_i \equiv 0 \pmod{(r_1+1)}.$$

We again consider a quadratic relation

(3.15) $$\sum_1^m \sum_1^m p_{ij} x_i x_j = 0$$

satisfied by all the minimal vectors.

From the vectors (3.5),

$$q_{ij} = 0,$$

where i, j take the values given in (3.5) (with $k = 3$). Similarly, from (3.6) we have

$$q_{i,i+1} + q_{j,j+1} = 0,$$

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again with the ranges of i, j as in (3.6), and since f contains three terms, it follows that

$$q_{i,i+1} = q_{j,j+1} = 0.$$

Hence (3.15) can be written

$$\left(\sum_1^m x_i\right)\left(\sum_1^m p_{ij}x_j\right) = 0.$$

Finally, from the vectors (3.8) it easily follows that

$$p_{jj} = 0 \qquad (1 \leqq j \leqq m)$$

and R_m is perfect.

II. Forms containing just two terms A_{r_1}, $A_{r_2}(r_1 \geqq r_2 \geqq 2)$.

$$f(x) = A_{r_1}(x^{(1)}) + A_{r_2}(x^{(2)})$$

with lattice

$$\sum_1^m x_i \equiv 0 \ (\mathrm{mod}\ (r_1+1)).$$

For $r_1 \geqq 5$, it is easy to show that R_m is perfect, using the same method as in I. However, R_m is not perfect in the following cases:

$R_5(3,2)$: this case is trivial, since now $s < N = \frac{1}{2}m(m+1)$.

$R_6(3,3)$: all minimal vectors satisfy the relation

$$(y_1+y_2+y_3)^2 - (y_4+y_5+y_6)^2 - 4(y_1y_2+y_2y_3-y_4y_5-y_5y_6) = 0.$$

$R_6(4,2)$: we find $s = 20 < N = 21$.

$R_7(4,3)$: all minimal vectors satisfy

$$-\left(\sum_1^4 y_i\right)^2 + \left(\sum_5^7 y_i\right)^2 + 5(y_1y_2+y_2y_3+y_3y_4-y_5y_6-y_6y_7) = 0.$$

$R_8(4,4)$: all minimal vectors satisfy

$$g(y) = -\left(\sum_1^4 y_i\right)^2 + \left(\sum_5^8 y_i\right)^2 + 5(y_1y_2+y_2y_3+y_3y_4-y_5y_6-y_6y_7-y_7y_8) = 0.$$

We note here that $R_9(4,4,1)$ is perfect. For, consider the relation

$$kg(y) + 2\sum_{i<9} p_{i9}y_iy_9 + p_{99}y_9^2 = 0.$$

From the minimal vectors $e_i - e_9$, we have

$$2p_{i9} = p_{99} - k \qquad (1 \leqq i \leqq 4),$$
$$2p_{i9} = p_{99} + k \qquad (5 \leqq i \leqq 8).$$

Now using the vectors

$$e_1+e_2+e_3+e_4+e_9, \qquad e_5+e_6+e_7+e_8+e_9,$$

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we obtain

$$p_{99}+k = 0, \qquad p_{99}-k = 0;$$

hence f is perfect.

Similarly $R_7(3,3,1)$ is perfect.

III. Forms containing a single term A_m.

$$f(x) = A_m(x) = x_1^2-x_1x_2+x_2^2-\cdots-x_{m-1}x_m+x_m^2$$

with lattice

$$\sum_1^m x_i \equiv 0 \ (\mathrm{mod}\ (m+1)).$$

If we apply the unimodular transformation

$$x = Ty = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ \cdot & 1 & 1 & \cdots & 1 \\ \cdot & \cdot & 1 & \cdots & 1 \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdots & 1 \end{pmatrix} y,$$

we obtain the form

$$2f(x) = \sum_1^m y_i^2 + \left(\sum_1^m y_i\right)^2$$

with lattice

$$\sum_1^m iy_i \equiv 0 \ (\mathrm{mod}\ (m+1)).$$

This is the form P_m, known to be perfect and extreme for $m \geqq 6$. (For $m \geqq 8$, perfection can be established as in I).

3.3. *Equivalences to Known Forms.* We have the following equivalences:

(i) $R_7(3,3,1) \sim P_7$ under the transformation

$$y = T_1x = \tfrac{1}{4}\begin{pmatrix} -3 & -2 & -1 & \cdot & 1 & -2 & -1 \\ -2 & \cdot & 2 & \cdot & 2 & \cdot & -2 \\ -1 & -2 & 1 & \cdot & -1 & -2 & -3 \\ 1 & 2 & -1 & \cdot & 1 & -2 & -1 \\ 2 & 4 & 2 & 4 & 2 & \cdot & 2 \\ 3 & 2 & 1 & 4 & 3 & 2 & 1 \\ 2 & \cdot & 2 & \cdot & 2 & \cdot & 2 \end{pmatrix} x$$

(ii) $R_8(3,3,1,1) \sim Q_8$ under the transformation

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$$y = T_2 x = \frac{1}{4}\begin{pmatrix} 2 & -1 & \cdot & 1 & -2 & -1 & \cdot & 1 \\ 2 & \cdot & 2 & \cdot & -2 & \cdot & -2 & \cdot \\ \cdot & -1 & 2 & 1 & \cdot & -1 & -2 & 1 \\ -1 & \cdot & 1 & 2 & -1 & \cdot & 1 & -2 \\ \cdot & 2 & \cdot & 2 & \cdot & -2 & \cdot & -2 \\ -1 & 2 & 1 & \cdot & -1 & -2 & 1 & \cdot \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} x.$$

3.4 *The Eutaxy of* $R_m(r_1, \cdots, r_k)$. The adjoint $F(x)$ of the form (3.1) is a multiple of

$$f^*(x) = \sum_{t=1}^{k} A^*_{r_t}(x^{(t)}) \tag{3.16}$$

where

$$\tfrac{1}{2}(r_t+1)A^*_{r_t}(x^{(t)}) = \sum_{i=1}^{r_t} (x_i^{(t)})^2 + \sum_{i=1}^{r_t-1} (x_i^{(t)}+x_{i+1}^{(t)})^2 + \cdots + \left(\sum_{i=1}^{r_t} x_i^{(t)}\right)^2.$$

We next consider the problem of deciding when R_m is eutactic, i.e. when its adjoint $F(x)$ is expressible as

$$F(x) = \sum_1^s \rho_k \lambda_k^2, \qquad \rho_k > 0, \tag{3.17}$$

where $\lambda_k (k = 1, \cdots, s)$ are the associated linear forms.

If for some $i, j,$ $(1 < i < j \leqq k)$ we have

$$r_i + r_j < r_t + 1,$$

then $R_m(r_1, \cdots, r_k)$ is not eutactic.

For the coefficient of $x_1^{(i)} x_1^{(j)}$ in $F(x)$ is zero, and now the only linear forms λ_k for which λ_k^2 involves a term in $x_1^{(i)} x_1^{(j)}$ are

$$\begin{aligned} \lambda_a &\equiv x_1^{(i)} - x_1^{(j)}, \\ \lambda_b &\equiv x_1^{(i)} + x_2^{(i)} - x_1^{(j)} - x_2^{(j)}, \\ &\vdots \\ \lambda_d &\equiv \sum_{k=1}^{r_j} (x_k^{(i)} - x_k^{(j)}). \end{aligned}$$

Equating coefficients of $2x_1^{(i)} x_1^{(j)}$ in (3.17), we obtain

$$-\rho_a - \rho_b - \cdots - \rho_d = 0,$$

and so R_m cannot be eutactic.

There appears to be no completely general result for the remaining forms R_m. However, the calculations required for any particular form are greatly simplified by the use of Theorem 2.1.

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For completeness, we note the following elements of the group G of automorphs of $F(\boldsymbol{x})$:

$$U_i = (x_j^{(i)} \to x_{r_i+1-j}^{(i)},\ (j = 1, \cdots, r_i));\ (i = 1, \cdots, k);$$
$$V_{ij} = (x_k^{(i)} \to x_k^{(j)},\ (k = 1, \cdots, r_i));\ \text{provided } r_i = r_j;$$
$$W = (x_i^{(1)} \to x_{i+1}^{(1)},\ (i = 1, \cdots, r_1-1);\ x_{r_1}^{(1)} \to -\textstyle\sum_{i=1}^{r_1} x_i^{(1)}).$$

Finally, in view of the equivalence $R_8(3,3,1,1) \sim Q_8$ we note that the form Q_8 is not extreme, contrary to the statement made in [3, I], p. 79.

In Table 1 are listed the new forms $R_m(r_1, \cdots, r_k)$ for $m = 7, 8, 9$. The columns give respectively the value of m; the values of the parameters $r_1, \cdots, r_k$ as a partition of m; the quantity $\Delta = (2/M)^m D$; the number s of pairs of opposite minimal vectors; and whether the form is extreme (E), or perfect and not extreme (P).

TABLE 1

The forms $R_m(r_1, r_2, \ldots, r_k)$ for $m = 7, 8, 9$.

m	Partition of m	Δ	s	P or E
7	6+1	$7^3/2^6$	28	E
	5+2	$3^4/2^4$	30	E
	3+2+2	$3^3/2$	32	E
8	7+1	4	44	P
	6+2	$3 \cdot 7^3/2^8$	42	E
	5+3	$3^3/2^3$	49	P
	6+1+1	$7^3/2^6$	36	P
	5+2+1	$3^4/2^4$	38	P
	4+2+2	$3^2 \cdot 5^3/2^8$	40	P
	3+3+2	3	52	E
	3+2+2+1	$3^3/2$	40	P
9	8+1	$3^6/2^8$	63	E
	7+2	3	60	E
	6+3	$7^3/2^7$	64	E
	5+4	$3^3 \cdot 5/2^6$	76	E
	7+1+1	4	53	P
	6+2+1	$3 \cdot 7^3/2^8$	51	P
	5+3+1	$3^3/2^3$	58	P
	5+2+2	$3^5/2^6$	53	P
	4+4+1	$5^4/2^8$	70	E
	4+3+2	$3 \cdot 5^3/2^7$	60	E
	3+3+3	2	78	E
	6+1+1+1	$7^3/2^6$	45	P
	5+2+1+1	$3^4/2^4$	47	P
	4+2+2+1	$3^2 \cdot 5^3/2^8$	49	P
	3+3+2+1	3	62	P
	3+2+2+2	$3^3/2^3$	56	E
	3+3+1+1+1	4	55	P
	3+2+2+1+1	$3^3/2$	49	P

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4. Theorems on sections of positive quadratic forms

4.1. *The Perfection of a Section.* Let $g(x_1, \cdots, x_{n+1})$ be an arbitrary positive definite form with minimum M for integral $\mathbf{x} \neq \mathbf{0}$, and let $f(x_1, \cdots, x_n)$ be the section obtained by setting

$$\sum_1^{n+1} \alpha_i x_i = 0 \qquad (\alpha_{n+1} \neq 0,\ \alpha_i \text{ integral}). \tag{4.1}$$

THEOREM 4.1. *The section f is perfect if and only if any quadratic relation*

$$\sum_1^{n+1} \sum_1^{n+1} p_{ij} x_i x_j = 0 \qquad (p_{ij} = p_{ji}) \tag{4.2}$$

satisfied by all the minimal vectors common to f and g, is necessarily of the form

$$\left(\sum_1^{n+1} p_i x_i\right)\left(\sum_1^{n+1} \alpha_i x_i\right) = 0. \tag{4.3}$$

PROOF. After applying a suitable integral unimodular transformation, we may take (4.1) to be

$$x_{n+1} = 0$$

in which case

$$f(x_1, \cdots, x_n) = g(x_1, \cdots, x_n, 0),$$

and (4.3) becomes

$$\left(\sum_1^{n+1} p_i x_i\right) x_{n+1} = 0. \tag{4.4}$$

(i) If any quadratic relation satisfied by the minimal vectors common to f and g is of the form (4.4), then f is perfect, since for all such vectors, x_{n+1} is identically zero.

(ii) Assume f is perfect. Now in (4.2) we have

$$p_{ij} \equiv 0 \qquad (1 \leqq i \leqq j \leqq n)$$

and the relation becomes

$$\left(2\sum_{i=1}^{n} p_{i,n+1} x_i + p_{n+1,n+1} x_{n+1}\right) x_{n+1} = 0$$

which is essentially the same as (4.4).

4.2. *The Adjoint of a Section.* Let $f(x_1, \cdots, x_n, x_{n+1}) = \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} a_{ij} x_i x_j$ be a positive quadratic form with inverse $F(y_1, \cdots, y_n, y_{n+1}) = \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} A_{ij} y_i y_j$. We define $g(x_1, \cdots, x_n)$ to be the n-dimensional section of f obtained by the elimination of x_{n+1} using the relation

$$\sum_1^{n+1} p_i x_i = 0. \tag{4.5}$$

THEOREM 4.2. *The adjoint of* $g(x_1, \cdots, x_n)$ *is a multiple of*

$$\omega(y_1, \cdots, y_n) = \kappa F(y_1, \cdots, y_n, y_{n+1}) - \left(\sum_1^{n+1} q_i y_i\right)^2 \tag{4.6}$$

where $y_{n+1} = 0$, *and*

$$\kappa = F(p_1, \cdots, p_n, p_{n+1}), \tag{4.7}$$

$$q_i = \sum_j A_{ij} p_j \qquad (1 \leqq i \leqq n+1). \tag{4.8}$$

PROOF. In this proof and in § 4.3 it is convenient to obtain the section of $f(x_1, \cdots, x_n, x_{n+1})$ by eliminating the first variable. We therefore cyclically permute the variables to bring x_{n+1} into the first position and rename it x_0.

Since f is a positive definite form, there now exists a transformation $(x_0, \cdots, x_n) = T(z_0, \cdots, z_n)$, where T is a regular $(n+1) \times (n+1)$ triangular matrix with elements t_{ij} $(0 \leqq i \leqq j \leqq n)$, such that

$$f(x_0, \cdots, x_n) = \sum_0^n z_i^2. \tag{4.9}$$

Under this transformation (4.5) becomes

$$\sum_0^n \alpha_i z_i = 0 \tag{4.10}$$

for some coefficients α_i.

We now need the following result:

LEMMA 4.1. *In the variables* z_i, $g(x_1, \cdots, x_n)$ *is given by*

$$g(x_1, \cdots, x_n) = \sum_1^n z_i^2 + \left(\sum_1^n \frac{\alpha_i}{\alpha_0} z_i\right)^2, \tag{4.11}$$

obtained by eliminating z_0 *between* (4.9) *and* (4.10).

PROOF. Under the transformation T we have

$$f(x_0, x_1, \cdots, x_n) = \sum_0^n z_i^2. \tag{4.12}$$

Let $U = (u_{ij})$ $(0 \leqq i \leqq j \leqq n)$ be the inverse of T. Now if A is the matrix of the form f,

$$U'U = A, \tag{4.13}$$

and $z_0, \cdots, z_n$, and $x_0, \cdots, x_n$ are related by

$$(4.14)\qquad \begin{aligned} z_0 &= u_{00}x_0+u_{01}x_1+\cdots+u_{0n}x_n\\ z_1 &= \qquad\quad u_{11}x_1+\cdots+u_{1n}x_n\\ &\vdots \qquad\qquad\qquad\qquad \vdots\\ z_n &= \qquad\qquad\qquad\quad u_{nn}x_n. \end{aligned}$$

Eliminating x_0 from both sides of (4.12), using (4.14) and the relation

$$\sum_0^n p_i x_i = 0,$$

we obtain

$$g(x_1, \cdots, x_n) = \left\{\frac{u_{00}}{p_0}\left(\sum_1^n p_i x_i\right) - (u_{01}x_1 + \cdots + u_{0n}x_n)\right\}^2 + \sum_1^n z_i^2.$$

Eliminating z_0 between (4.9) and (4.10) we obtain a form h say, where

$$h(z_1, \cdots, z_n) = \left(\sum_1^n \frac{\alpha_i}{\alpha_0} z_i\right)^2 + \sum_1^n z_i^2.$$

We shall now prove that the forms $g(x_1, \cdots, x_n)$, $h(z_1, \cdots, z_n)$ are identical. Clearly it will suffice to show that

$$(4.15)\qquad \left\{\frac{u_{00}}{p_0}\left(\sum_1^n p_i x_i\right) - (u_{01}x_1 + \cdots + u_{0n}x_n)\right\} = \sum_1^n \frac{\alpha_i}{\alpha_0} z_i.$$

From (4.5) and (4.10) we have

$$\begin{aligned} p_0x_0 + \left(\sum_1^n p_i x_i\right) &= \alpha_0 z_0 + \left(\sum_1^n \alpha_i z_i\right)\\ &= \alpha_0(u_{00}x_0+u_{01}x_1+\cdots+u_{0n}x_n) + \sum_1^n \alpha_i z_i. \end{aligned}$$

Since $z_1, \cdots, z_n$ do not involve x_0, we have

$$(4.16)\qquad p_0 = \alpha_0 u_{00},$$

$$(4.17)\qquad \sum_1^n p_i x_i = \alpha_0(u_{01}x_1 + \cdots + u_{0n}x_n) + \sum_1^n \alpha_i z_i.$$

Equation (4.15) now follows immediately from (4.16) and (4.17), and this completes the proof of the lemma.

The adjoint of the form (4.11), in variables contragredient to those in (4.11), is easily found to be

$$(4.18)\qquad \begin{aligned} G(y_1, \cdots, y_n) &= \sum_{i=1}^n \left\{1 + \sum_{\substack{k=1\\k\neq i}}^n \left(\frac{\alpha_k}{\alpha_0}\right)^2 w_i^2\right\} - 2\sum\sum_{1\leq i<j\leq n} \left(\frac{\alpha_i}{\alpha_0}\right)\left(\frac{\alpha_j}{\alpha_0}\right) w_i w_j\\ &= \left(1 + \frac{1}{\alpha_0^2}\sum_1^n \alpha_k^2\right)\sum_1^n w_i^2 - \frac{1}{\alpha_0^2}\left(\sum_1^n \alpha_i w_i\right)^2. \end{aligned}$$

Clearly $\sum_0^n w_i^2$ is the inverse of the form (4.9), and (4.18) can be written

$$\alpha_0^2 G(y_1, \cdots, y_n) = \left(\sum_0^n \alpha_k^2\right) \sum_0^n w_i^2 - \left(\sum_0^n \alpha_i w_i\right)^2, \tag{4.19}$$

subject to the condition

$$w_0 = 0. \tag{4.20}$$

Finally, applying the transformation $(w_0, \cdots, w_n) = T'(y_0, \cdots, y_n)$ to (4.19) and (4.20), and writing $\omega(y_1, \cdots, y_n) = \alpha_0^2 G(y_1, \cdots, y_n)$, we obtain

$$\omega(y_1, \cdots, y_n) = \kappa F(y_0, y_1, \cdots, y_n) - \left(\sum_0^n q_i y_i\right)^2 \tag{4.21}$$

where

$$y_0 = 0,$$
$$\kappa = \sum_0^n \alpha_k^2,$$

and $q_1, \cdots, q_n$ are coefficients to be determined.

It now only remains to prove (4.7) and (4.8).

From (4.5) and (4.10), using (4.14) we now obtain

$$p_i = \sum_{j=0}^n \alpha_j u_{ji}. \tag{4.22}$$

Similarly, $(y_0, \cdots, y_n) = U'(w_0, \cdots, w_n)$, and from (4.19) and (4.21) we have

$$\alpha_i = \sum_{j=0}^n q_j u_{ij}. \tag{4.23}$$

Substituting (4.23) in (4.22) now gives

$$\begin{aligned} p_i &= \sum_j \sum_k q_k u_{jk} u_{ji} \\ &= \sum_k \left(\sum_j u_{ji} u_{jk}\right) q_k \\ &= \sum_k a_{ik} q_k \end{aligned}$$

using (4.13).

Hence

$$q_i = \sum_j A_{ij} p_j \tag{4.24}$$

as required.

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Now

$$\kappa = \sum_{i=0}^{n} \alpha_i^2$$
$$= \sum_i (\sum_j \sum_k q_j q_k u_{ij} u_{ik})$$

from (4.23). Changing the order of summation,

$$\kappa = \sum_j \sum_k q_j q_k (\sum_i u_{ij} u_{jk})$$
$$= \sum_j \sum_k a_{jk} q_j q_k \qquad \text{(by (4.13))},$$
$$= \sum_j \sum_k A_{jk} p_j p_k$$

using the result (4.24).

4.3. *The Determinant of a Section.* In the terminology of § 4.2, the form

$$\sum_1^n z_i^2 + \left(\sum_1^n \frac{\alpha_i}{\alpha_0} z_i\right)^2 \tag{4.25}$$

is easily found to have determinant κ/α_0^2. The form $g(x_1, \cdots, x_n)$ is transformed into (4.25) under the transformation $(x_0, x_1, \cdots, x_n) = T(z_0, z_1, \cdots, z_n)$. Since the transforming matrix consists of only the last n rows and columns of T, we have

$$D(g) = \frac{t_{00}^2}{|T|^2} \cdot \frac{\kappa}{\alpha_0^2}.$$

Substituting $D(f) = 1/|T|^2$, $\kappa = F(\mathbf{p})$ and

$$p_{00} = \frac{\alpha_0}{t_{00}},$$

we obtain

$$D(g) = \frac{1}{p_0^2} D(f) \cdot F(\mathbf{p}).$$

5. The form $S_n(r_1, r_2, \cdots, r_k)$

5.1. *Definition, Minimum and Conditions for Perfection.* For convenience, in this section we write $m = n+1$.

We define $S_n = S_n(r_1, r_2, \cdots, r_k)$ to be the section of $R_m(r_1, r_2, \cdots, r_k)$ given by

$$f(x) = \sum_{t=1}^{k} A_{r_t}(x^{(t)}), \qquad (r_1 \geqq r_2 \geqq \cdots \geqq r_k \geqq 1, \sum_1^k r_t = m), \tag{5.1}$$

where

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(5.2) $$\sum_1^m x_i = 0.$$

We shall show that

$$M(S_n) = 2, D(S_n) = \Delta(S_n) = \frac{1}{2^{n+1}}\left(\prod_{i=1}^{k} (r_i+1)\right)\left\{\tfrac{1}{6}\sum_{j=1}^{k} r_j(r_j+1)(r_j+2)\right\}.$$

Since the values taken by S_n form a subset of the values taken by the corresponding R_m, it follows that $M(S_n) = 2$, and the minimal vectors of S_n are just those minimal vectors of R_m which satisfy (5.2).

We have an immediate analogue of Lemma 3.1 which we merely state.

LEMMA 5.1. *If the form S_n defined by* (5.1) *and* (5.2) *is perfect, then so is the form $S_{n+r_0}(r_0 \leqq r_1)$:*

$$f_0(x, x^{(0)}) = f(x)+A_{r_0}(x^{(0)})$$

where

$$\sum_1^{m+r_0} x_i = 0.$$

Now we need only consider those forms which cannot be obtained in this way.

By applying Theorem 4.1 to the forms R_m, we find that the corresponding section S_n is perfect if and only if either

(i) S_n contains a single term A_m, and $m \geqq 8$; or

(ii) S_n contains just two terms A_{r_1}, A_{r_2} $(r_1 \geqq r_2 \geqq 2,$

$$r_1+r_2 = m) \quad \text{and} \quad r_1 \geqq 5; \quad \text{or}$$

(iii) S_n contains three terms $A_{r_1}, A_{r_2}, A_{r_3}$, $(r_1 \geqq r_2 \geqq r_3 \geqq 2, \sum_1^3 r_t = m)$, (or S_n can be obtained from one of these using Lemma 5.1).

5.2. *Calculation of the Determinant of S_n*. From § 4.3 we see that the determinant D of S_n is given by

$$D = \frac{1}{p_m^2} D(f)\cdot F(p),$$

where here

$$p = (p_1, p_2, \cdots, p_m) = (1, 1, \cdots, 1);$$

f is the form of the corresponding R_m, and F its adjoint.

Now

$$F(x) = \sum_{t=1}^{k} A_{r_t}^*(x^{(t)})$$

where

$$\tfrac{1}{2}(r+1)A_r^*(x) = \sum_1^r x_i^2 + \sum_1^{r-1}(x_i+x_{i+1})^2 + \cdots + (\sum_1^r x_i)^2.$$

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Hence

$$\tfrac{1}{2}(r+1)A_r^*(1,1,\cdots,1) = \sum_{i=1}^{r} i^2(r-i+1)$$
$$= (r+1)\left\{\frac{r(r+1)(2r+1)}{6}\right\} - \left\{\frac{r(r+1)}{2}\right\}^2$$
$$= \tfrac{1}{12}r(r+1)^2(r+2).$$

Therefore

$$A_r^*(1,1,\cdots,1) = \tfrac{1}{6}r(r+1)(r+2),$$

and

$$F(p) = \tfrac{1}{6}\sum_{j=1}^{k} r_j(r_j+1)(r_j+2).$$

Also, it is easily verified that

$$D(f) = \prod_{t=1}^{k}\left(\frac{r_t+1}{2^{r_t}}\right) = \frac{1}{2^m}\prod_{t=1}^{k}(r_t+1).$$

Hence

$$D = \frac{1}{2^m}\prod_{t=1}^{k}(r_t+1)\left\{\tfrac{1}{6}\sum_{j=1}^{k} r_j(r_j+1)(r_j+2)\right\}.$$

5.3. *Equivalences amongst the Forms* S_n. We have the following equivalences:

(i) $S_7\,(4,2,2) \sim S_7\,(6,2)$, under the transformation

$$x \to \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & 1 & 1 \\ -1 & \cdot & -1 & \cdot & -1 & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix} x$$

(ii) $S_7\,(8) \sim S_7\,(5,3)$ under the transformation

$$x \to \begin{pmatrix} \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & -1 & -1 & \cdot & -1 & -1 & -1 \\ -1 & -1 & \cdot & -1 & -1 & -1 & -1 \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \cdot & 1 & \cdot & \cdot & 1 & 1 & \cdot \end{pmatrix} x$$

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(iii) $S_8\,(9) \sim S_8\,(5,4)$ under the transformation

$$x \to \begin{pmatrix} \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & 1 & 1 & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot \\ \cdot & -1 & -1 & \cdot & -1 & -1 & \cdot & -1 \end{pmatrix} x.$$

5.4 *The Adjoint and Eutaxy of* S_n. We generally take S_n to be the form obtained by eliminating x_m between (5.1) and (5.2). Then from § 4.2, we find that the adjoint of S_n is given by a multiple of

$$\omega(y) = \kappa F(y, y_m) - \left(\sum_1^m q_i y_i\right)^2$$

where $y_m = 0$, $F(y, y_m)$ is the inverse of $R_m(r_1, \cdots, r_k)$, and

$$\kappa = F(p) = \tfrac{1}{6}\sum_{j=1}^{k} r_j(r_j+1)(r_j+2).$$

Also

$$q_i = \sum_{j=1}^{m} A_{ij} p_j \qquad (1 \leqq i \leqq m)$$

$$= \sum_{j=1}^{m} A_{ij}.$$

Now the (i, j)th component of an arbitrary A_r^* from the adjoint of R_m, is found to be for $j \geqq i$

$$A_{ij} = \frac{2}{r+1}\, i(r-j+1).$$

Hence

$$\begin{aligned} \sum_{j=1}^{r} A_{ij} &= \sum_{j=1}^{i-1} A_{ji} + \sum_{j=i}^{r} A_{ij} \\ &= \frac{2}{r+1}\left\{\sum_{j=1}^{i-1}(r-i+1)j + \sum_{j=i}^{r} i(r-j+1)\right\} \\ &= \frac{2}{r+1}\left[\tfrac{1}{2}i(i-1)(r-i+1) + i\{(r+1)(r-i+1) - (\tfrac{1}{2}r(r+1) - \tfrac{1}{2}i(i-1))\}\right] \\ &= i(r-i+1). \end{aligned}$$

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Thus the q_i corresponding to the ith variable of an arbitrary A_r is given by

$$q_i = i(r-i+1).$$

Having identified the adjoint ω_n of S_n, we now apply Voronoi's criterion for eutactic forms, and test whether or not ω_n can be expressed as

$$\omega_n = \sum_1^s \rho_k \lambda_k^2, \qquad (\rho_k > 0,\ k = 1, \cdots, s). \tag{5.3}$$

This is in general difficult; we have however the following simple case. Suppose

$$r_1 > r_2 > \left[\frac{r_1}{2}\right] - 1. \tag{5.4}$$

Now, subject to (5.4), the only terms λ_k^2 in (5.3) which give rise to the product $y_1 y_{r_2+1}$, contain the square of the difference $y_1 - y_{r_2+1}$. Thus if S_n is eutactic, the coefficient of $y_1 y_{r_2+1}$ in ω_n must be negative.

Hence we must have

$$\kappa \frac{2}{r_1+1}(r_1-r_2) - r_1(r_2+1)(r_1-r_2) < 0;$$

that is

$$2\kappa < r_1(r_1+1)(r_2+1). \tag{5.5}$$

We find that the following forms S_n do not satisfy (5.5):

$S_7(3,2,2,1)$, $S_8(3,2,2,2)$, $S_8(3,2,2,1,1)$, $S_9(4,3,3)$,
$S_9(4,2,2,2)$, $S_9(4,2,2,1,1)$, $S_9(3,2,2,2,1)$, $S_9(3,2,2,1,1,1)$.

It follows that these forms are not eutactic, and so not extreme.

In Table 2 are listed the new forms $S_n(r_1, \cdots, r_k)$ for $n = 7, 8, 9$. The columns give respectively the value of n; the values of the parameters $r_1, \cdots, r_k$ as a partition of $n+1$; the quantity $\Delta = (2/M)^n D$; and the number s of pairs of opposite minimal vectors. All these forms have been shown to be perfect; those known to be non-extreme are denoted by a (P).

TABLE 2

The forms $S_n(r_1, \ldots, r_k)$ for $n = 7, 8, 9$.

n	Partition of m	Δ	s
7	6+2	$3^2 . 5 . 7/2^6$	30
	5+3	$3^3 . 5/2^5$	34
	5+2+1	$3^2 . 5/2^3$	28
	3+2+2+1	$3^2 . 19/2^5$	29(*P*)
8	8+1	$3^2 . 11/2^5$	42
	7+2	$3 . 11/2^3$	42
	6+3	$3 . 7 . 11/2^6$	46
	5+4	$3 . 5^2 . 11/2^6$	50
	6+2+1	$3 . 7 . 61/2^5$	38
	5+3+1	$3 . 23/2^4$	42
	5+2+2	$3^3 . 43/2^6$	40
	4+3+2	$3 . 5 . 17/2^6$	43
	3+3+3	$3 . 5/2^3$	45
	5+2+1+1	$3^2 . 41/2^6$	36
	4+2+2+1	$3^2 . 5 . 29/2^5$	38
	3+2+2+2	$3^3 . 11/2^6$	40(*P*)
	3+2+2+1+1	$3^2 . 5/2^7$	37(*P*)
9	10	$5 . 11^2/2^6$	60
	9+1	$5 . 83/2^7$	59
	8+2	$3^3 . 31/2^8$	57
	7+3	$47/2^4$	61
	6+4	$5 . 7 . 19/2^8$	66
	5+5	$3^2 . 5 . 7/2^7$	69
	8+1+1	$3^2 . 61/2^7$	51
	7+2+1	$3 . 89/2^6$	51
	6+3+1	$7 . 67/2^7$	55
	6+2+2	$3^2 . 7/2^4$	52
	5+4+1	$3 . 5 . 7/2^5$	59
	5+3+2	$3^2 . 7^2/2^7$	56
	4+4+2	$3 . 5^2 . 11/2^8$	58
	4+3+3	$5^2/2^3$	59(*P*)
	6+2+1+1	$3 . 7 . 31/2^7$	47
	5+3+1+1	$3 . 47/2^5$	51
	5+2+2+1	$3^2 . 11/2^6$	49
	4+3+2+1	$3 . 5^2 . 7/2^7$	52
	4+2+2+2	$5 . 3^3/2^5$	51(*P*)
	3+3+3+1	$31/2$	54
	3+3+2+2	$3^2 . 7/2^4$	45
	4+2+2+1+1	$3^3 . 5^2/2^7$	47(*P*)
	3+3+2+1+1	$3 . 13/2^3$	49
	3+2+2+2+1	$3^2 . 23/2^7$	49(*P*)
	3+2+2+1+1+1	$3^3 . 7/2^5$	46(*P*)

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