# FINITE DIRECT SUMS OF COMPLETE MATRIX RINGS OVER PERFECT COMPLETELY PRIMARY RINGS

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The main theorem of this paper names seven ring classifications which coincide with the class of rings named in the title. Three of them are (1) rings over which every module is rationally complete, (2) left and right perfect rings over which every module is corationally complete, and (3) right perfect rings over which every right module is rationally complete. Corational completeness (introduced in this paper) and rational completeness are generalizations of projectivity and injectivity, respectively. One recalls that a proper subclass of the rings investigated in this paper, the artinian rings with zero radical, is known to have one-sided characterizations in terms of the injectivity (projectivity) of modules. An example proves, however, that the class of rings over which every right module is rationally complete properly contains the rings of the title. It is also proved that the rings of this paper are not characterized by the corational completeness of all right modules together with the right-perfectness of the rings.

Another result states that if $R$ is a right perfect ring, then every right $R$-module is corationally complete if and only if no right $R$-module is a corational extension by a proper factor module. The known analogue concerning rational completeness and rational extensions makes no restriction on the ring and avoids global formulations.

**1.** We make the following conventions to be used without comment: All rings will have an identity element; the radical of a ring $R$ (frequently written $(\text{rad}\, R)$) will be the Jacobson radical; all modules will be unital.

Some needed definitions appear after the following theorem.

THE MAIN THEOREM. *The following statements on a ring $R$ are equivalent:*

(1A) *$R$ is the direct sum of a finite set of ideals $R_i$, where each $R_i$ is a left and right perfect ring and $R_i/(\text{rad}\, R_i)$ is a simple artinian ring;*

(1B) *Every $R$-module is rationally complete;*

(1C) *$R$ is left and right perfect and no $R$-module is a corational extension by a proper factor module;*

(1D) *$R$ is left and right perfect and every $R$-module is corationally complete;*

(1E) *$R$ is left and right perfect and if $\{u_1, \ldots, u_k\}$ is the full set of primitive orthogonal central idempotents of $R/(\text{rad}\, R)$ whose sum is the identity modulo*

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(rad $R$), *then each* $u_i$ *can be lifted to a primitive central idempotent* $e_i$ *of* $R$ *such that* $\sum e_i$ *is the identity of* $R$ *and the set* $\{e_1, \ldots, e_k\}$ *is orthogonal;*

(1F) $R$ *is the direct sum of a finite set of ideals* $R_i$ *each of which satisfies: There is a unique positive integer* $n_i$ *and, up to isomorphism, a unique left and right perfect completely primary ring* $S_i$ *such that* $R_i$ *is isomorphic to the* $n_i \times n_i$ *matrix ring over* $S_i$*;*

(1G) $R$ *is right perfect and every right* $R$*-module is rationally complete;*

(1H) $R$ *is left perfect and every left* $R$*-module is rationally complete.*

*Definitions.* For homological terms and for ring and module terminology, see (**2**; **6**), respectively.

(1) *Socle.* The socle of a module is the sum of its irreducible submodules, where a module is irreducible if and only if it has exactly two submodules. The right (left) socle of a ring $R$ is the socle of $R$ as a right (left) $R$-module.

(2) *Rational completeness.* A right $R$-module $M$ is rationally complete if and only if in every case where $N$ is a right $R$-module with submodule $V$ such that $\operatorname{Hom}_R(X, M) = 0$ for every submodule $X$ of $N/V$ the induced map
$$\operatorname{Hom}_R(N, M) \to \operatorname{Hom}_R(V, M)$$
is onto.

(3) *Corational completeness.* A right $R$-module $M$ is corationally complete if and only if in every case where $N$ is a right $R$-module with submodule $V$ such that $\operatorname{Hom}_R(M, \bar{V}) = 0$ for every factor module $\bar{V}$ of $V$ the induced map
$$\operatorname{Hom}_R(M, N) \to \operatorname{Hom}_R(M, N/V)$$
is onto.

(4) *Essential, rational, coessential, and corational extensions.* Let $V$ be a submodule of a right $R$-module $M$. We say that $V$ is essential or large in $M$ and that $M$ is an essential extension of $V$ if $V$ has non-zero intersection with every non-zero submodule of $M$. $M$ is a rational extension of $V$ if for every submodule $Y$ of $M/V$, $\operatorname{Hom}_R(Y, M) = 0$. $M$ is a coessential extension by its factor module $M/V$ and $V$ is called small in $M$ if $W + V = M$ ($W$ a submodule of $M$) implies $W = M$. $M$ is a corational extension by $M/V$ if $\operatorname{Hom}_R(M, \bar{V}) = 0$ for every epimorph $\bar{V}$ of $V$. (Using an appropriate injection or surjection, these definitions can be extended to arbitrary modules.)

(5) *Projective cover; right perfect rings.* If $P$ is a projective right $R$-module and is a coessential extension by its epimorph $P/K$, then $P$ is called a projective cover of any right $R$-module $R$-isomorphic with $P/K$. The ring $R$ is right perfect if and only if every right $R$-module has a projective cover.

(6) *Primitive idempotent.* An idempotent is called primitive if it is not the sum of two orthogonal idempotents.

(7) *Completely primary.* A ring $R$ is completely primary if and only if $R/(\operatorname{rad} R)$ is a division ring.

As immediate consequences of the definitions, we have that

(1I) *Every injective right module is rationally complete;*

(1J) *Every projective right module is corationally complete.*

*Notation*. We shall write (1D; left) for the statement "$R$ is left perfect and every left $R$-module is corationally complete". The meaning of (1D; right) and of similarly modified references will be clear.

Sections 2–6 are devoted to the proof of the main theorem. The equivalence of (1C) and (1D) is obtained in § 2 by proving the equivalence of (1C; right) and (1D; right). In § 3 we prove that (1B; right) implies (1A; left) and in § 4 that (1A; left) implies (1C; left). Thus, (1B) implies (1A), and (1A) implies (1C). In § 5 we prove that if $R$ is a left perfect ring and if no proper corational extensions by right $R$-modules exist, then every right $R$-module is rationally complete. This theorem and the theorem obtained from it by interchanging the words right and left prove that (1C) implies (1B). Thus, the first four statements of the main theorem are proved equivalent in §§ 2–5.

The remaining steps in the proof of the main theorem are in § 6, with some help from earlier results. In § 6 we prove that (1E) implies (1F) and that (1F) implies (1A), by proving the left versions. Thus, (1F) implies any of the first four (equivalent) statements. Proof that (1B) implies (1E) is obtained from a result in § 3: (1B; right) implies (1E; left). Thus, the first six statements have been proved equivalent. As the final step in the proof of the main theorem, (1G) and (1H) are proved equivalent with the preceding six statements.

Some by-products of the results in §§ 2–6 are mentioned in § 7, as evidence of right-left asymmetry *qua* rational and corational completeness of modules and perfectness of rings. We record one of them (Theorem (7C)).

THEOREM. *If R is a left perfect ring over which every right module is corationally complete, then every left R-module is corationally complete.*

The question is open as to the possibility, suggested by the quoted theorem, that the corational completeness of all right modules together with the left-perfectness of the rings characterizes the rings of this paper. It is shown in § 7, however, that the rational completeness of all right modules does not characterize these rings. This is accomplished by displaying a ring $R$ over which all right modules, but not all left modules, are rationally complete. The same example shows that the rings of this paper do not coincide with the class of left perfect rings over which every left module is corationally complete.

Concerning (1B), Findlay and Lambek proved (**5**, p. 156) that a right $R$-module is rationally complete if and only if it has no proper rational extensions. In § 2 it is proved that no proper corational extensions by a right module $M$ exist if $M$ is corationally complete, but at the time of writing we do not have the converse (and conjecture against its existence). To obtain the equivalence of (1C; right) and (1D; right), we prove, in § 2, that a right $R$-module $M$ is corationally complete if, for every factor module $F$ of $M$, $F$ has a projective cover and no proper corational extensions by $F$ exist.

A sense in which corational extension is dual to rational extension is mentioned in (**3**, p. 953, following Definition 2).

## 2. Proof that (1C) and (1D) of the main theorem are equivalent.

*Notation* (2A). If $A$ and $B$ are right $R$-modules, we shall write $f \in \mathrm{Hom}_R(A, B/\cdot)$ if, for some epimorph $\bar{B}$ of $B$, $f \in \mathrm{Hom}_R(A, \bar{B})$. By $\mathrm{Hom}_R(A, B/\cdot) = 0$ we mean that $\mathrm{Hom}_R(A, \bar{B}) = 0$ for every epimorph $\bar{B}$ of $B$. Otherwise, we write $\mathrm{Hom}_R(A, B/\cdot) \neq 0$.

PROPOSITION (2B). *If an R-module M is a corational extension by its epimorph M/V, then M is a coessential extension by M/V.*

*Proof.* If $M$ is not coessential over $M/V$, then for a submodule $Y \neq M$ we have that $M = V + Y$. Thus, $M/Y \cong V/(V \cap Y) \neq 0$, proving that $M$ is not a corational extension by $M/V$.

*Remark* (2C). In the commutative diagram

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & & \\ A \arrow[rr, "\mathrm{nat}"'] & & A/B \arrow[r] & 0 \end{tikzcd}$$

let $Y = f(S)$. Then, if $g$ is onto $A/B$, $A = Y + B$. For $a$ belongs to the coset $f(s) + B$, if $a + B = g(s)$.

THEOREM (2D). *Let M be a corationally complete right R-module. Then no proper corational extension by M exists.*

*Proof.* Suppose, on the contrary, that there is a right $R$-module $C$ with submodule $K$, such that $C$ is corational over $C/K$, and that $C/K \cong M$. By the definition of corational extension, $\mathrm{Hom}_R(C, K/\cdot) = 0$. Evidently,

$$\mathrm{Hom}_R(C/K, K/\cdot) = 0. \tag{1}$$

The corational completeness of $C/K \cong M$ and (1) imply that there is a map $\sigma$: $C/K \to C$ making commutative the diagram

$$\begin{tikzcd} & C/K \arrow[dl, "\sigma"'] \arrow[d, "g"] & \\ C \arrow[r, "\alpha"'] & C/K \arrow[r] & 0 \end{tikzcd}$$

where $g$ is the identity map on $C/K$ and $\alpha$ is the natural map. Since $g$ is an onto map, Remark (2C) implies that $C = K + Y$, where $Y = \sigma(C/K)$. Let $k \in K \cap Y$. Thus, $k = \sigma(t + K)$ for some $t \in C$. From

$$g(t + K) = \alpha\sigma(t + K) = \alpha(k) = (0 + K)$$

we have that $t \in K$, since $g$ is the identity map of $C/K$. Then

$k = \sigma(0 + K) = 0$. Thus, $C$ is the direct sum of $Y$ and $K$. Since $K$ is small in $C$ by Proposition (2B), $C = Y$ and, in consequence, $K = 0$. Thus, the extension is not proper.

THEOREM (2E). *If $R$ is a right perfect ring, every right $R$-module is corationally complete if and only if no right $R$-module is a proper corational extension by an $R$-module.*

Theorem (2E) is implied by Theorem (2D) and the following proposition.

PROPOSITION (2F). *Let $M$ be a right $R$-module such that*

(i) *No proper corational extension by an epimorph of $M$ exists;*

(ii) *Every epimorph of $M$ has a projective cover.*

*Then $M$ is corationally complete.*

*Proof.* Suppose that $M$ is not corationally complete. Then there is an $R$-module $A$ with submodule $B$ such that

$$\mathrm{Hom}_R(M, B/\cdot) = 0 \tag{2}$$

and that, for some $R$-homomorphism $f: M \to A/B$, there is no completing map $M \to A$ making

$$\begin{tikzcd} & M \arrow[d, "f"] & \\ A \arrow[r, "\mathrm{nat}"'] & A/B \arrow[r] & 0 \end{tikzcd} \tag{3}$$

commutative. For convenience we assume that $f$ is onto $A/B$. We proceed to obtain a contradiction to (2).

Let $M'$ be the epimorph of $M$ such that $f$ takes $M'$ isomorphically onto $A/B$. Let projective module $P$ be a projective cover of $M'$: $P/K \cong M'$, where $K$ is a small submodule of $P$. We define $g: P \to A/B$ by the composition of maps

$$P \xrightarrow[\mathrm{nat}]{} P/K \longrightarrow M' \xrightarrow[f]{} A/B,$$

where $P/K \to M'$ is an isomorphism. We have a commutative diagram

$$\begin{tikzcd} & P \arrow[dl, "\sigma"'] \arrow[dr, "g"] & & \\ A \arrow[rr, "\mathrm{nat}"'] & & A/B \arrow[r] & 0 \end{tikzcd} \tag{4}$$

$\sigma$ exists since $P$ is projective. Let $L$ be the kernel of $\sigma$. Then $K$ contains $K \cap L$ properly; otherwise, (4) would provide a completion for diagram (3). Thus,

$P/(K \cap L)$ is a proper extension by $P/K \cong M'$ and, by hypothesis, cannot be a corational one. This proves that a non-zero element $h$ of $\mathrm{Hom}_R(P/(K \cap L), (K/(K \cap L))/\cdot)$ exists. Since $K$ is small in $P$, $K + (\ker h) \neq P$, so that $h$ induces a non-zero homomorphism defined on $P/K \cong M'$, and we have that

$$(5) \qquad \mathrm{Hom}_R(M', (K/(K \cap L))/\cdot) \neq 0.$$

Since $\sigma(t) \in B$, if $t \in K$, $\sigma$ restricted to $K/(K \cap L)$ is an isomorphism onto $B \cap \sigma(P)$:

$$(6) \qquad K/(K \cap L) \cong B' = B \cap \sigma(P).$$

Considering (5) and (6), there exists a non-zero homomorphism of $M'$ into an epimorph of $B'$; say to $B'/V$. Since $V$ is a submodule of $B$, a non-zero homomorphism of $M$ into an epimorph of $B$ has been found, in contradiction of (2). This completes the proof that $M$ is a corationally complete module. We have proved (2F) and (2E).

Since (1C; right) and (1D; right) are equivalent, the equivalence of (1C) and (1D) is evident.

**3. Proof that (1B) of the main theorem implies (1A) and (1E).** We plan to prove that if every right module for $R$ is rationally complete, then the ring $R$ is the direct sum of ideals $R_i$, each a left perfect ring which is artinian and simple modulo its radical. Findlay and Lambek's theorem mentioned in § 1 permits us to interchange "there are no rational extensions of the module $M$" with "$M$ is rationally complete". We shall do so without comment.

*Definition.* An ideal $N$ is called left $T$-nilpotent if, given a sequence $\{u_i\}$ of elements of $N$, the product $u_1u_2 \ldots u_n = 0$ for some $n$.

We extract the following from a theorem of Bass (**1**, pp. 467–468, Theorem P).

THEOREM (3A). *The following statements on a ring $R$ are equivalent:*

(1) *$R$ is left perfect;*

(2) *The radical $P$ of $R$ is left $T$-nilpotent and $(R/P)$ is artinian;*

(3) *$R$ has no infinite sets of orthogonal idempotents and every non-zero right $R$-module has non-zero socle.*

PROPOSITION (3B). *If $R/(\mathrm{rad}\, R)$ has no infinite sets of orthogonal idempotents, then $R$ also has this property.*

*Proof.* If the proposition is false, the hypothesis implies the existence of non-zero orthogonal idempotents $e$ and $e + j$ with $j \in (\mathrm{rad}\, R)$. From $0 = e(e + j) = e + ej$, $e$ has a quasi-inverse $h$: $e + h = eh$. Then $e + eh = eh$. We have the contradiction that $e = 0$.

We call a module irreducible if it has precisely two submodules.

PROPOSITION (3C). *Let a ring $R$ be the direct sum of some ideals $R_i$ $(1 \leqq i \leqq k)$. Then* (1) *every non-zero right $R$-module has an irreducible submodule if and only if* (2) *for each $i$ and each $(R_i)$-ideal $X \neq R_i$ the ring $(R_i/X)$ has non-zero right socle.*

*Proof.* Let $\{e_i | 1 \leqq i \leqq k\}$ be the central idempotents whose sum is the identity of $R$ such that $R_i = e_iR = Re_i$. If $X$ is an $(R_i)$-ideal, then $Y$ is an irreducible right $(R_i)$-submodule of $(R_i/X)$ if and only if $Y$ is an irreducible right $R$-submodule. Thus (1) implies (2). We assume that (2) holds and that $M$ is a right $R$-module with $Me_i \neq 0$ for some $i$, and that $X$ is the $(R_i)$-ideal such that $Me_i$ is a faithful $(R_i/X)$-module. By assumption, $(R_i/X)$ has an irreducible right $R$-submodule $T/X$. If $m \in Me_i$ is such that $mT \neq 0$, then clearly $mT \cong T$. Thus (1) holds.

PROPOSITION (3D). *Let $R$-module $M$ be a rational extension of its submodule $V$. Then $M$ is an essential extension of $V$.*

*Proof.* If $V$ were not essential in $M$, then for some non-zero submodule $T$ of $M$, $V + T$ is a direct sum. Then $f(t + V) = t$ is a non-zero homomorphism of $(T + V)/V$ to $M$ so that $M$ is not a rational extension of $V$.

*Definition and Remark.* If $M$ is a right $R$-module, $t \in M$ is a singular element if and only if $tE = 0$ for some essential right ideal $E$ of $R$. Clearly, $t$ is non-singular if and only if there is a non-zero right ideal $V$ of $R$ such that the obvious map $V \to tV$ is a monomorphism. The singular elements of $M$ form a submodule, the singular submodule of $M$, and the singular submodule of the right $R$-module $R$ is an ideal, the right singular ideal of $R$ (**4**, p. 47, Proposition 4).

PROPOSITION (3E). *Let a right $R$-module $M$ have submodules $N$ and $W \supseteq N$, where $W$ is essential over $N$. If $f \in \mathrm{Hom}_R((W/N), M)$, then $f(W/N)$ is contained in the singular submodule of $M$.*

*Proof.* If, on the contrary, $f(t + N) = s$ for some $t \in W$, where $s$ is not a singular element, then for some non-zero right ideal $J$ of $R$ the obvious map $J \to sJ$ is a monomorphism of $J$ into $M$. From $sj \neq 0$ for all non-zero $j \in J$ and from $sj = f((t + N)j)$, $M/N$ contains the isomorphic copy of $J$, $(t + N)J = tJ + N$, so that $tJ \cap N = 0$. This contradicts the essentiality of $N$ in $W$, and completes the proof.

THEOREM (3F). *Let $M$ be an $R$-module with vanishing singular submodule and let $N$ be a submodule of $M$. Then $M$ is an essential extension of $N$ if and only if $M$ is a rational extension of $N$.*

*Proof.* Proposition (3D) proves one direction. If $M$ is essential over $N$ and if $f \in \mathrm{Hom}_R(W/N, M)$, where $W$ is a submodule of $M$, then Proposition (3E) implies that $f = 0$. Thus, $M$ is rational over $N$, completing the proof.

PROPOSITION (3G). *Let $R$ be a ring whose right singular ideal is zero and over which every right module is rationally complete. Then $R$ is a semi-simple artinian ring.*

*Proof.* We show that every right $R$-module is injective; the desired conclusion is then immediate (**7**, p. 12). If $T$ is an essential right ideal of $R$, then by Theorem (3F) $T$ is a rational submodule of the right $R$-module $R$. It follows that the rationally complete $R$-module $T$ equals $R$. Thus, an element $m$ of an $R$-module $M$ is a singular element if and only if $mR = 0$, which is equivalent to $m = 0$, since $M$ is unital. Thus, the singular submodule of $M$ is zero for every right $R$-module $M$. By Theorem (3F), then, every essential extension is a rational extension. Thus, by hypothesis, there can be no proper essential extensions, and every right module is injective.

PROPOSITION (3H). *Let $M$ be an $R$-module with submodule $A$ and let $B$ be a submodule of $M$ maximal with respect to $A \cap B = 0$ ($B$ exists by Zorn's lemma). Then $A + B$ is an essential submodule of $M$.*

The proof is straightforward. It appears in (**4**, p. 16).

LEMMA (3I). *Let $R$ be a ring over which every right module is rationally complete and let $R$ have non-zero right singular ideal $Z$. Then $Z$ contains an irreducible right ideal whose square is zero.*

*Proof.* Let $H$ be a right ideal which is maximal with respect to $H \cap Z = 0$; by Proposition (3H), $Z + H$ is an essential right ideal of $R$. If the identity element of $R$ were in $Z + H$,

$$1 = z + h, \qquad z \in Z, h \in H,$$

we would have for each $k \in Z$, $k = zk + hk = zk$, since $hk \in Z \cap H$. Since $z \in Z$, $z$ annihilates an essential right ideal $E$ of $R$. Thus, there is a non-zero $t \in Z \cap E$. We have the contradiction that

$$t = zt = 0,$$

proving that $1 \notin Z + H$. A maximal proper right ideal $V$ containing $Z + H$ exists and $V$ is an essential right ideal. The rational completeness of $V$ implies that $f(V) = 0$ for some non-zero $f \in \mathrm{Hom}_R(Y, R)$, $V \subseteq Y \subseteq R$. Since $V \neq Y$ and since $R/V$ is irreducible, $Y = R$. Now $T = f(R) \subseteq Z$ by Proposition (3E), and $T \cong R/V$ is irreducible. From $f(V) = 0$ and from $RZ \subseteq Z \subseteq V$, we see that $TZ = 0$, and have $T^2 = 0$.

THEOREM (3J). *Let $R$ be a ring with radical $P$ over which every right module is rationally complete. Then*

(1) *$R/P$ is an artinian ring and*

(2) *Every non-zero right $R$-module $M$ has non-zero socle.*

*Proof.* If $Z$, the right singular ideal of $R/P$ is zero, $R/P$ is artinian by Proposition (3G) (since every right $(R/P)$-module is rationally complete).

If $Z \neq 0$, then by Lemma (3I), $R/P$ has a non-zero nilpotent right ideal $T/P$. We have a contradiction since every nil right ideal is contained in the radical (**6**, p. 8) and the radical of $R/P$ is zero. Thus, (1) has been proved.

Now let $m$ be a non-zero element of a right $R$-module $M$. Let $H$ be a submodule of $M$ maximal with respect to the property of not containing $\{m\}$. Then $m \in T$, the intersection of the submodules which properly contain $H$. Clearly, $T/H$ is irreducible and is contained in every submodule of $M/H$. Since $H$ is rationally complete, there is a non-zero homomorphism $g$ of $T/H$ into $T$. Clearly, $g$ is an isomorphism and $M$ has an irreducible submodule: $g(T/H)$. This completes the proof of the theorem.

THEOREM (3K). *Let $R$ be a ring over which every right $R$-module is rationally complete. Then $R$ is a left perfect ring.*

*Proof.* By Theorem (3J), every right $R$-module has non-zero socle and $R/(\text{rad } R)$ is artinian. By Proposition (3B), $R$ inherits from $R/(\text{rad } R)$ the non-existence of infinite sets of orthogonal idempotents. Thus, by Bass' theorem (see Theorem (3A)), $R$ is a left perfect ring.

*Notation* (3L). If $S$ and $T$ are subsets of an $R$-module, $(S:T) = \{r \in R| \ Tr \subseteq S\}$. We write $(S:t)$ if $T$ is the set $\{t\}$.

*Remark* (3M). If $t$ is a non-zero element of an irreducible $R$-module $T$, the map $f(r) = tr$ is an $R$-homomorphism of $R$ onto $T$ with kernel $(0:t)$. Thus, $T \cong R/(0:t)$ and $(0:t)$ is a maximal right ideal of $R$.

LEMMA (3N). *Let $e$ be an idempotent of a ring $R$ and let $ye = 0$ for some non-zero element $y$ of $R$. Let $H$ and $H'$ be right ideals such that $(eR + H')/H'$ is an irreducible right $R$-module and is isomorphic with $(yR + H)/H$. Then $e$ is not central modulo the radical $P$ of $R$.*

*Proof.* We assume the contrary: $er - re \in P$ for all $r \in R$. Since $(eR + H')/H'$ is irreducible, $(H':e)$ is a maximal right ideal by Remark (3M) and contains the intersection $P$ of the maximal right ideals. From $er - re \in P \subseteq (H':e)$ we have, for each $r \in R$, that

$$er - ere \in H'.$$

Evidently, $(er + H') = (er + H')e$; $e$ acts as right identity on the right $R$-module $(eR + H')/H'$. On the other hand, $(y + H)$ is a non-zero element of $(yR + H)/H$ and $(y + H)e = (ye + H) = 0$ by hypothesis. This is a contradiction of the isomorphism in the hypothesis, completing the proof.

THEOREM (3O). *Let $R$ be a ring over which every right $R$-module is rationally complete. Let $\{e_1, \ldots, e_k\}$ be a set of orthogonal idempotents of $R$ whose sum is the identity of $R$. Let $P$ denote the radical of $R$ and assume, for each $r \in R$ and for $i = 1, \ldots, k$, that*

$$e_i r - re_i \in P.$$

*Then the idempotents $e_i$ are central idempotents (thus $e_iR$ is an ideal) $i = 1, \ldots, k$.*

*Proof.* If the conclusion is not true, then $xe_i \neq e_ix$ for some $i \in \{1, \ldots, k\}$ and some $x \in R$. We must have that $e_ixe_i \neq e_ix$ or $e_ixe_i \neq xe_i$ and we assume the latter with no loss of generality. Thus, there exists $\beta \in \{1, \ldots, k\}$ such that

$$y = e_\beta xe_i \neq 0, \qquad \beta \neq i.$$

Using Zorn's lemma, let $H$ be a right ideal maximal with respect to $y \notin H$. Clearly, $y$ belongs to the intersection $F$ of those right ideals which properly contain $H$. We have

$$F/H \text{ irreducible}, \qquad F = H + yR.$$

Evidently, $e_\beta \notin H$. We claim that $e_\beta \notin F$. For, if $e_\beta \in F$, then $(e_\beta + H)$, as well as $(y + H)$, generates the irreducible module $F/H$ and, since $ye_\beta = 0$, Lemma (3N) implies that $e_\beta r - re_\beta \notin P$ for some $r \in R$, a contradiction.

Since $e_\beta \notin F$, let $H'$ be a right ideal containing $F$ and maximal with respect to $e_\beta \notin H'$. Then we must have $F'/H'$ irreducible, where $F' = H' + e_\beta R$. Since $H'/H$ is rationally complete and is a proper submodule of $F'/H$, a non-zero homomorphism of the irreducible module $F'/H'$ to $F'/H$ exists; its image is necessarily the unique irreducible submodule $F/H$ and we have that

$$F/H \cong F'/H'. \tag{7}$$

Since $ye_\beta = 0$ and since the isomorphic irreducible modules of (7) are generated by $y$ and $e_\beta$, respectively, Lemma (3N) again implies that $e_\beta r - re_\beta \notin P$ for some $r \in R$. This contradiction completes the proof.

*Remark* (3P). A ring having nil Jacobson radical is an SBI ring (ring suitable for building idempotents), that is, a ring such that idempotents modulo the radical can be lifted (**6**, pp. 53–54). More precisely, for such a ring (**6**, p. 54), if $P$ denotes the radical and if $\{u_1 + P, \ldots, u_k + P\}$ is a set of orthogonal idempotents of $R/P$, then a set $\{e_1, \ldots, e_k\}$ of orthogonal idempotents of $R$ exists such that $e_i + P = u_i + P$, $i = 1, \ldots, k$. Furthermore, if $\sum u_i - 1 \in P$, where 1 is the identity of $R$, then the $e_i$ can be chosen so that their sum is 1.

THEOREM (3Q). *Let $R$ be a ring over which every right $R$-module is rationally complete. Then $R$ is the direct sum of a finite set of ideals $R_i$ such that for each $i$, $R_i/(\text{rad } R_i)$ is a simple artinian ring and $R_i$ is a left perfect ring.*

*Proof.* Let $P$ denote the radical of $R$. We use the notation $\bar{x}$ for the coset $x + P$ and $\bar{S}$ to denote $\{\bar{s} | s \in S\}$ if $S$ is a subset of $R$. By Theorems (3K) and (3A), $P$ is a nil ideal and by Theorem (3J), $R/P$ is artinian. Thus, $R/P$ is a direct sum of simple rings (each an ideal of $R/P$) so that for some positive integer $k$, $R/P$ has central orthogonal idempotents $\bar{u}_1, \ldots, \bar{u}_k$ whose sum is the coset $1 + P$, such that $\bar{u}_i\bar{R}$ is an $(R/P)$-ideal and a simple artinian ring. By Remark (3P), $\bar{u}_i$ can be replaced in the preceding sentence by $\bar{e}_i$, where

$\{e_1, \ldots, e_k\}$ is a set of orthogonal idempotents of $R$ and $\sum e_i = 1$. The $e_i$ are central idempotents by Theorem (3O). Thus, $R_i = e_i R$ is an ideal for each $i$ and $R$ is the direct sum of the $R_i$. Then each $(R_i)$-module is an $R$-module and is rationally complete. By Theorem (3K), $R_i$ is left perfect, $i = 1, \ldots, k$.

Since the radical is the intersection of all maximal right ideals, it is easy to verify that the radical of $R_i$ is $R_i \cap P$. By the ring isomorphism

$$e_i R/(e_i R \cap P) \cong (e_i R + P)/P = \overline{e_i R},$$

$R_i/(\text{rad } R_i)$ is a simple artinian ring, $i = 1, \ldots, k$.

*Remark* (3R). In Theorem (3Q), (1A; left) has been obtained from (1B; right). (1E; left) has been obtained also since $R$ is a left perfect ring and since the primitive orthogonal central idempotents of the artinian ring $R/P$ have been lifted to orthogonal central idempotents (necessarily primitive) of $R$. We have proved that (1B) implies (1A) and (1E).

**4. Proof that (1A) of the main theorem implies (1C).** We shall prove that (1A; left) $\Rightarrow$ (1C; left). Let $R_i$ $(1 \leqq i \leqq k)$ be left perfect rings and let $R = R_1 \oplus \ldots \oplus R_k$ be the ring made from the direct sum of their additive groups. We use Theorem (3A) to show that $R$ is left perfect. If $P_i = (\text{rad } R_i)$, $i = 1, \ldots, k$, then it is easy to verify that $P = (\text{rad } R) = \sum P_i$, using the definition of radical as the intersection of maximal right ideals. The left $T$-nilpotence of $P$ is obvious since each $P_i$ is left $T$-nilpotent, and $R/P$ is artinian as the direct sum of the artinian rings $R_i/P_i$. Thus, $R$ is left perfect.

THEOREM (4A). *Let $R$ be a ring which is the direct sum of some ideals $R_1, \ldots, R_k$, such that for each $i$*

(i) *$R_i/P_i$ is a simple artinian ring, where $P_i = (\text{rad } R_i)$;*

(ii) *$R_i$ is a left perfect ring.*

*Then* (1) *$R$ is a left perfect ring and* (2) *there are no proper corational extensions by left $R$-modules.*

*Proof.* The opening remarks prove conclusion (1). A theorem which appears in (1, p. 474) states that for any ring $R$, non-zero projective $R$-modules always have maximal proper submodules. If $M$ is any non-zero left $R$-module for the ring $R$ of this theorem, we claim that $M$ has a maximal proper submodule. Since $R$ is left perfect, there is a projective left $R$-module $P$ with a small submodule $K$ such that $M \cong P/K$. If $H$ is a maximal proper submodule of $P$, we have that $K + H \neq P$ so that $K \subseteq H$, since $K$ is small. Thus, $H/K$ is a maximal proper submodule of $P/K$; $M$ has a maximal proper submodule.

Now assume that conclusion (2) is false, that a left $R$-module $G$ exists with submodule $N \neq 0$ such that $G$ is a corational extension by $G/N$. Thus, $\text{Hom}_R(G, \bar{N}) = 0$ for every epimorph $\bar{N}$ of the left module $N$.

Let $e_1, e_2, \ldots, e_k$ be the central idempotents of $R$ whose sum is the identity of $R$ (whose existence the hypothesis implies). For any left module $H$, we use

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the notation $H_{(j)} = e_j H$. Thus, $H$ is the direct sum of the $H_{(i)}$ and $e_j$ is the identity map on $H_{(j)}$ and is the zero map on $H_{(i)}$ for $i \neq j$. We remark that from the definition of the $e_i$ and from (i) in the hypothesis, exactly one (up to $R$-isomorphism) irreducible left $R$-module $T_i$ exists on which $e_i$ is the identity map and $e_j$ induces the zero map, if $j \neq i$. Let $\beta$ be such that $N_\beta \neq 0$. Let $F_1$ be a maximal proper submodule of $G_{(\beta)}$ and let $F = F_1 + \sum_{i \neq \beta} G_{(i)}$. Let $V_1$ be a maximal proper submodule of $N_{(\beta)}$ and let $V = V_1 + \sum_{i \neq \beta} N_{(i)}$. Then $G/F \cong T_\beta \cong N/V$ as left $R$-modules. This contradicts the corationality of $G$ over $G/N$, proving the theorem.

Clearly, (1A) $\Rightarrow$ (1C), since (1A; left) $\Rightarrow$ (1C; left).

**5. Proof that (1C) of the main theorem implies (1B).** Our plan is to prove that if $R$ is a left perfect ring and if no proper corational extensions by right $R$-modules exist, then every right $R$-module is rationally complete.

LEMMA (5A). *Let $R$ be a ring such that no right module is a proper corational extension by another. Let $P$ be the radical of $R$ and let $\{e_1, \ldots, e_k\}$ be a set of orthogonal idempotents with $\sum e_i$ equal to the identity of $R$, such that for each $i$ and for every $r \in R$,*

$$e_i r - r e_i \in P.$$

*Then for $i = 1, \ldots, k$, $e_i$ is a central idempotent. (Thus, the right ideals $e_i R$ are ideals.)*

*Proof.* We assume the contrary. Thus, $xe_i \neq e_i x$ for some $i \in \{1, \ldots, k\}$ and some $x \in R$. We must have that $e_i x e_i \neq e_i x$ or $e_i x e_i \neq x e_i$ and we assume the latter with no loss of generality. Thus, there exists $\alpha \in \{1, \ldots, k\}$ such that

$$y = e_\alpha x e_i \neq 0, \qquad \alpha \neq i.$$

Let $V = \sum_{j \neq \alpha} e_j R$. Then $y = e_\alpha y \notin V$. Let $H \supseteq V$ be a right ideal maximal with respect to $y \notin H$. The intersection $F$ of right ideals properly containing $H$ satisfies

$$F/H \text{ irreducible}, \qquad F = H + yR.$$

Since $R/H$ is a proper extension by $R/F$, it cannot be a corational extension. Thus, there is a non-zero homomorphism on $R/H$ to the irreducible right module $F/H$. Let $H'$ be the kernel of that homomorphism. Then

$$R/H' \cong F/H.$$

Since $H' \supseteq H \supseteq V$ and since $R \neq H'$, $e_\alpha \notin H'$ so that

$$R = H' + e_\alpha R.$$

Considering the displayed statements and $ye_\alpha = 0$, Lemma (3N) implies that $e_\alpha$ is not central modulo $P$, a contradiction. This completes the proof.

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THEOREM (5B). *Let $R$ be a left perfect ring such that no proper corational extensions by right $R$-modules exist. Then every right $R$-module is rationally complete.*

*Proof.* Since $R$ is left perfect, Theorem (3A) implies that $R/P$ is artinian, where $P = \operatorname{rad} R$. Thus, $R/P$ has central orthogonal idempotents $\{e_1 + P, \ldots, e_k + P\}$ whose sum is $1 + P$, each of which generates one of the indecomposable ideals (= simple artinian rings) into which $R/P$ decomposes uniquely. By Theorem (3A), the radical is nil and, by Remark (3P), we may suppose that the $e_i$ are orthogonal idempotents whose sum is 1. By Lemma (5A) they are central idempotents.

We show that a right $R$-module $M$ cannot be a proper rational submodule of an $R$-module $G$. For $i = 1, \ldots, k$, let

$$M_{(i)} = Me_i, \qquad G_{(i)} = Ge_i.$$

Let $W = \{i \mid 1 \leqq i \leqq k,\ M_{(i)} \neq 0\}$. If $\sum_{i \in W} M_{(i)} = \sum_{i \in W} G_{(i)}$, then $M$ is a direct summand of $G$, so that there is a projection onto some complement of $M$ and $\operatorname{Hom}_R(G/M, G) \neq 0$, whence $M$ is not a rational submodule of $G$. Taking the alternate case, $M_{(\beta)}$ is properly contained in $G_{(\beta)}$ for some $\beta \in W$. Let $y \in G_{(\beta)}$, $y \notin M_{(\beta)}$. Let $H$ be a submodule of $G$ containing $M$ and maximal with respect to $y \notin H$. Then $y \in F$, where $F$ is the intersection of all the submodules of $G$ properly containing $H$, and $F/H$ is an irreducible right $R$-module. Since $ye_i = 0$ for $i \in \{1, \ldots, k\}$, $i \neq \beta$, $F/H$ is $R$-isomorphic with the unique irreducible module afforded by the simple ring $e_\beta R/e_\beta P$. Let $X$ be the ideal of $e_\beta R$ such that $G_{(\beta)}$ is faithful for $(e_\beta R/X)$. By statement (3) of Theorem (3A), $(e_\beta R/X)$ has an irreducible right $R$-submodule $T$. Since $e_\beta R$ is an ideal direct summand of $R$, $T = Te_\beta$ is an irreducible $(e_\beta R)$-module. From $0 = TP = Te_\beta P$, $T$ is an irreducible module for the simple ring $(e_\beta R/e_\beta P)$, so that $T \cong F/H$. Since $G_\beta$ is a faithful $(e_\beta R/X)$-module, there is an element $x \in G_\beta$ such that $xT \neq 0$. Clearly,

$$xT \cong T \cong F/H.$$

Since $M \subseteq H \subseteq F \subseteq G$, a non-zero homomorphism of $F/M$ into $G$ exists. $M$ is not rational in $G$.

THEOREM (5C). *Statement* (1C) *of the main theorem implies* (1B).

*Proof.* This is clear from Theorem (5B) and the theorem obtained from (5B) by interchanging the words right and left.

*Remark* (5D). The first four statements of the main theorem are equivalent by (5C) and the concluding sentences of §§ 2, 3, and 4.

**6. Conclusion of the proof of the main theorem.** For a ring $R$ (with radical $P$) over which every right module is rationally complete, the requirements of (1E; left) were deduced, incidentally, in Theorems (3K), (3O), and (3Q) and Remark (3P). $R$ is a left perfect ring and the full set of primitive orthogonal central idempotents of the semi-simple ring $R/P$ can be lifted to central orthogonal idempotents of $R$ whose sum is the identity of $R$. Thus (1B; right) implies (1E; left) and we have the following result.

THEOREM (6A). (1B) *of the main theorem implies* (1E).

*Definition and Remark.* For a ring with identity, $R/(\text{rad } R)$ is a division ring if and only if the non-units of $R$ form an ideal (**6**, p. 56 and p. 58, Proposition 1). Such a ring is called a completely primary ring.

*Remark.* As mentioned in Remark (3P), a ring having nil radical is an SBI ring. From (**6**, p. 56, Theorem 1 and p. 59, Theorem 3) we record the following result.

THEOREM (6B). *Let $T$ be an SBI ring such that $T/(\text{rad } T)$ is a simple artinian ring. Then $T$ is isomorphic with an $n \times n$ matrix ring over a completely primary ring. The integer $n$ and, up to isomorphism, the completely primary ring are unique.*

THEOREM (6C). (1E) *of the main theorem implies* (1F).

*Proof.* We show that (1E; left) implies (1F; left). Let $R$ be a left perfect ring with radical $P$ and let $e_1, \ldots, e_k$ be central orthogonal idempotents of $R$ such that $\sum e_i$ equals the identity of $R$ and for each $i$ $(e_i + P)(R/P) \cong (e_iR/e_iP)$ is a simple artinian ring. For $i = 1, \ldots, k$, $R_i = e_iR$ is an ideal, since $e_i$ is central, and $R$ is the direct sum of the $R_i$, since $\{e_i\}$ is a set of orthogonal idempotents whose sum is the identity of $R$. Since $R$ is left perfect, $P$ is left $T$-nilpotent by Theorem (3A). Since the radical of $R_i$ is $e_iP$, it is left $T$-nilpotent, and $R_i/(\text{rad } R_i)$ is a simple artinian ring. Thus, $R_i$ is a left perfect ring. By the remark preceding Theorem (6B), $R_i$ is an SBI ring, $i = 1, \ldots, k$. By Theorem (6B), there is a unique positive integer $n_i$ such that $R_i$ is an $n_i \times n_i$ matrix ring over a completely primary ring $S_i$, which is unique up to isomorphism. Since $S_i$ is a division ring modulo its radical, $S_i$ is left perfect if its radical $J$ is left $T$-nilpotent. If $J$ is not left $T$-nilpotent, there exists a sequence $\{u_i\}$, $u_i \in J$, such that the product $u_1u_2 \ldots u_m \neq 0$ for every positive integer $m$. The left $T$-nilpotence of $J_{n_i}$ is then contradicted by the sequence $u_1I, u_2I, \ldots$, where $I$ is the identity of $(S_i)_{n_i}$. But this contradicts the fact that $R_i$ is left perfect since $(\text{rad } R_i) = J_{n_i}$ (**6**, p. 11). Thus, $J$ is left $T$-nilpotent and $S_i$ is a left perfect, completely primary ring. Thus, (1F; left) has been obtained from (1E; left), completing the proof of the theorem.

PROPOSITION (6D). (1F) *of the main theorem implies* (1A).

*Proof.* We shall obtain (1A; left) from (1F; left). We need to prove that a ring $R$ is left perfect and, modulo its radical $P$, is simple artinian, given that $R$ is an $n \times n$ matrix ring over a left perfect, completely primary ring $S$. If

$J = (\text{rad}\, S)$, then by the definition of completely primary, $S/J$ is a division ring. Since $(\text{rad}\, R) = J_n$ (**6**, p. 11), $R/P = S_n/J_n \cong (S/J)_n$ is a simple artinian ring. Thus, by (3B), $R$ has no infinite sets of orthogonal idempotents. $R$ will be left perfect if every right $R$-module has an irreducible submodule. By Proposition (3C), it is sufficient to show that $R/X$ has non-zero right socle for every ideal $X \neq R$. It is easy to see, for every matrix unit $E_{ij}$, that $kE_{ij} \in X$ if $k \in S$ is a non-zero entry in a matrix belonging to $X$, so that $X = Y_n$ for some ideal $Y$ of $S$, $Y \neq S$. Since $S$ is left perfect, $S/Y$ has an irreducible right ideal $T$ by Theorem (3A). Then $TE_{11} + TE_{12} + \ldots + TE_{1n} = F$ is an irreducible right ideal of $R/X$ (since, for $i = 1, \ldots, n$ and for any non-zero $t \in T$, $tE_{1i} \in fR$ if $f$ is a non-zero element of $F$). Thus, $R$ is left perfect and the proposition has been proved.

*Remark* (6E). The first six statements of the main theorem are equivalent.

*Proof.* By Remark (5D), the first four statements of the main theorem are equivalent and by Theorem (6A) any one of them implies (1E). By Theorem (6C), (1E) implies (1F), and by Theorem (6D), the first four statements of the main theorem are implied by (1F), completing the proof.

(6F) *Final steps of the proof of the main theorem.* We have just proved the equivalence of statements (1A)–(1F). Clearly, a ring $R$ which satisfies these statements satisfies (1G): $R$ is a right perfect ring and every right $R$-module is rationally complete. If a ring $R$ satisfies (1G), then by the remarks at the beginning of this section, $R$ is a left perfect ring and the full set of primitive orthogonal central idempotents of $R/(\text{rad}\, R)$ can be lifted to orthogonal central idempotents of $R$ whose sum is the identity of $R$. Since $R$ is right perfect, (1E) holds, whence $R$ satisfies the first six statements. Thus, the equivalence of statements (1A)–(1G) has been proved. By symmetry, the proof of the theorem is complete, since (1H) is the left analogue of (1G).

**7. Evidence of asymmetry.** To obtain a view of left-right asymmetry, *qua* perfectness of rings and rational and corational completeness of modules, we mention some theorems which are implied by the results in §§ 2–6. Theorem (2D) states that no proper corational extensions by corationally complete right modules exist. Together with Theorem (5B), this implies the following theorem.

THEOREM (7A). *Let $R$ be a left perfect ring over which every right module is corationally complete. Then every right $R$-module is rationally complete.*

(1C; left) is implied by (1A; left), which is implied by (1B; right) (Theorems (4A) and (3Q), respectively). Since, by the left analogue of Theorem (2E), (1D; left) and (1C; left) are equivalent, we conclude that (1B; right) implies (1D; left).

THEOREM (7B). *If every right $R$-module is rationally complete, then $R$ is a left perfect ring and every left $R$-module is corationally complete.*

(7A) and (7B) yield the following result.

THEOREM (7C). *If $R$ is a left perfect ring over which every right module is corationally complete, then every left $R$-module is corationally complete.*

*Remark.* If it can be proved that the hypotheses of (7C) imply that the ring $R$ is right perfect, then the rings of this paper are the left perfect rings over which every right module is corationally complete. This is an open question.

(7D) *Example of a ring over which every right module, but not every left module, is rationally complete.*[1] Let $K$ be a field and let $S$ be the ring of linear transformations on a vector space of countable dimension over $K$. Using matrix notation, let $\{E_{ij} | i > j\}$ generate over $K$ a subalgebra $P$ of $S$. Then $P$ is nil and is the radical of the algebra $R$ which $P$ and the identity transformation generate over $K$. Since $R/P$ is a field, $P$ is the only maximal right or left ideal of $R$. We claim that $P$ is a rational submodule of the left $R$-module $R$. If, on the contrary, a non-zero homomorphism existed from the left module $R/P$ to $R$, $R$ would have an irreducible left ideal. This cannot be, since $Px = 0$, $x \in R$, implies that $x = 0$. Thus, if $x = \sum k_j E_{i\alpha_j}$ for some $i$, then $E_{i+1,i}x = \sum k_j E_{i+1,\alpha_j} = 0$ proving that $k_1 = k_2 = \ldots = 0$ and $x = 0$. Obviously, $x = 0$ can be proved if $x$ is a $K$-sum of matrix units with varying first subscript.

We next prove that $R/X$ has non-zero right socle for every ideal $X \neq R$ of $R$. It is easy to verify that (for $m = 1, 2, \ldots$) $P^m$ is generated over $K$ by $\{E_{ij} | i \geqq j + m\}$ and that the left annihilator $L(P^m)$ is generated by $\{E_{ij} | 1 \leqq j \leqq m, j < i\}$; also, that $L(P^0) = 0$, where $P^0 = R$. Evidently, $P$ is the union of the $L(P^m)$. Now, $R/X$ is irreducible if $X = P$ and we show that $R/X$ has irreducible right modules when $P$ properly contains $X$. Let $m$ be the unique non-negative integer such that $L(P^m) \subseteq X$ and $L(P^{m+1}) \not\subseteq X$. Thus, for some $t \notin X$, $t \in L(P^{m+1})$, so that $tP \subseteq L(P^m) \subseteq X$. Thus, $(t + X)/X$ generates an irreducible submodule of the right $R$-module $R/X$. As observed in Proposition (3C), every right $R$-module then has an irreducible submodule. Furthermore, if $T$ is an irreducible $R$-module, then by Remark (3M), $T \cong R/P$, since $P$ is the only maximal right ideal of $R$. If an $R$-module $M$ has a proper non-zero submodule $N$, then $M/N$ and $N$ have irreducible submodules which are $R$-isomorphic with $R/P$. This proves that $N$ is not a rational submodule of $M$. Every right $R$-module is rationally complete.

*Remark.* The ring of example (7D) does not satisfy (1B), proving that the rings of the main theorem are not those for which every right module is rationally complete. Nor are they the left perfect rings over which every left module is corationally complete, since the ring of example (7D) is such a ring by Theorem (7B).

[1]This example is mentioned, without proof, in (**1**, p. 476).

## REFERENCES

1. H. Bass, *Finitistic dimension and a homological generalization of semi-primary rings*, Trans. Amer. Math. Soc. *95* (1960), 466–488.
2. H. Cartan and S. Eilenberg, *Homological algebra* (Princeton University Press, Princeton, N.J., 1956).
3. R. Courter, *The maximal corational extension by a module*, Can. J. Math. *18* (1966), 953–962.
4. C. Faith, *Lectures on injective modules and quotient rings* (Springer-Verlag, New York, 1967).
5. G. D. Findlay and J. Lambek, *A generalized ring of quotients*. II, Can. Math. Bull. *1* (1958), 155–167.
6. N. Jacobson, *Structure of rings*, Amer. Math. Soc. Colloq. Publ., Vol. 37 (Amer. Math. Soc., Providence, R.I., 1956).
7. J. P. Jans, *Rings and homology* (Holt, Rinehart and Winston, New York, 1964).

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