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ON COMPACT PERTURBATIONS OF OPERATORS

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Recently R. G. Douglas showed [**4**] that if V is a nonunitary isometry and U is a unitary operator (both acting on a complex, separable, infinite dimensional Hilbert space $\mathscr{H}$), then $V - K$ is unitarily equivalent to $V \oplus U$ (acting on $\mathscr{H} \oplus \mathscr{H}$) where K is a compact operator of arbitrarily small norm. In this note we shall prove a much more general theorem which seems to indicate "why" Douglas' theorem holds (and which yields Douglas' theorem as a corollary). Our theorem is based on the Calkin algebra analogue of the following well-known fact: If λ is an eigenvalue for the operator T which lies in the boundary of the numerical range of T, then the eigenspace determined by λ reduces T.

If T and S are operators acting on Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ respectively, we shall write $T \approx S$ if for each $\epsilon > 0$ there is a compact operator K such that $T - K$ is unitarily equivalent to S, and the norm of K is $< \epsilon$. We shall show that a large class of operators have the property that $T \approx T \oplus N$ where N is any normal operator such that $\sigma(N)$ the spectrum of N lies in a certain set (determined by T). In particular, if T_φ is a Toeplitz operator and N is a normal operator such that $\sigma(N)$ lies in the set of extreme points of the convex hull of $\sigma(T_\varphi)$ then $T_\varphi \approx T_\varphi \oplus N$.

In what follows $\mathscr{B}(\mathscr{H})$ will denote the algebra of bounded linear operators (henceforth, simply "operators") acting on a fixed complex, separable, infinite dimensional Hilbert space $\mathscr{H}$. The Calkin algebra $\mathscr{C}$ is the C^*-algebra which results from forming the quotient space: $\mathscr{B}(\mathscr{H})$ modulo the ideal of compact operators. For an operator T we shall let T_e denote the coset in $\mathscr{C}$ which contains T. Recall that the spectrum of T is by definition the set $\sigma(T) = \{\lambda : T - \lambda I \text{ is not invertible}\}$ and the numerical range of T is by definition the set $W(T) = \{(Tf, f) : f \text{ is a unit vector in } \mathscr{H}\}$. The analogous objects for $\mathscr{C}$ are the essential spectrum $\sigma_e(T) = \{\lambda : T_e - \lambda I_e \text{ is not invertible in } \mathscr{C}\}$ and the essential numerical range $W_e(T) = \{p_e(T_e) : p_e \text{ is a state on the Calkin algebra}\}$. It is well-known that $W(T)$ is convex, bounded and that $W(T)^-$ contains $\sigma(T)$. Similarly $W_e(T)$ is convex, compact, and $W_e(T)$ contains $\sigma_e(T)$. Further, it is clear from the definitions that $W_e(T + K) = W_e(T)$ and $\sigma_e(T + K) = \sigma_e(T)$ for all compact operators K. (The reader is referred to [**5**; **10**] for proofs of these and other basic facts concerning $W_e(T)$ and $\sigma_e(T)$.)

The main theorem. We begin with the Calkin algebra analogue of the result mentioned in the introduction.

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LEMMA 1. *Let T be an operator and suppose that there is a complex number λ in $\partial W_e(T)$ ($\partial W_e(T)$ denotes the boundary of $W_e(T)$) and an infinite rank projection P such that $TP - \lambda P$ is a compact operator. Then $PT - \lambda P$ is also a compact operator.*

Proof. By translating and rotating we may assume that $\lambda = 0$ and that $W_e(T)$ lies in the closed right half plane. Let $T = A + iB$ where A and B are the real and imaginary parts of T. Then $PAP = \frac{1}{2}(PTP + PT^*P)$ is compact. Further, since $W_e(T)$ is contained in the closed right half plane A_e is a positive element of $\mathscr{C}$. Let C_e be the positive square root of A_e in $\mathscr{C}$. Then for any state p_e on $\mathscr{C}$

$$|p_e(A_eP_e)|^2 \leqq p_e(C_e^2)p_e(P_eA_eP_e) = 0$$

and it follows that AP is compact. Hence, $iBP = TP - AP$ is compact and, therefore, PT is compact. (This argument is a slight generalization of a proof due to Stampfli [**9**].)

THEOREM 2. *Let T be an operator and let λ belong to $\sigma_e(T) \cap \partial W_e(T)$. Then $T \approx T_1 \oplus \lambda I$, where the I is infinite dimensional, $W_e(T_1) = W_e(T)$, and $\sigma_e(T_1) = \sigma_e(T)$.*

Proof. Since λ is in $\sigma_e(T)$ by a theorem of Fillmore, Stampfli and Williams [**5**] there is an infinite rank projection P such that either $TP - \lambda P$ or $PT - \lambda P$ is compact. Taking adjoints if necessary, we may assume that $TP - \lambda P$ is compact. Hence, by Lemma 1, $PT - \lambda P$ is compact. Now, given $\epsilon > 0$ we may replace P by a smaller infinite rank projection so that $PT - \lambda P$ and $TP - \lambda P$ both have norm less than $\epsilon/3$ and so that the operator $T_1 = (I - P)T(I - P)$ restricted to $(I - P)\mathscr{H}$ has the same essential spectrum and essential numerical range as T. Letting $K = TP + PT(I - P) - \lambda P$, it is easy to check that the norm of K is less than ϵ and that $T - K$ has the desired form.

COROLLARY 3. *Let T be an operator and suppose $\{\lambda_n\}$ is a sequence of complex numbers which belongs to $\sigma_e(T) \cap \partial W_e(T)$. Then $T \approx T' \oplus D$, where T' is an operator and D is the diagonal operator $\sum \lambda_n P_n$ determined by an orthogonal family of infinite rank projections $\{P_n\}$.*

Proof. By Theorem 2, there is a compact operator K_1 of norm less than $\epsilon/2$ such that $T - K_1$ is unitarily equivalent to $T_1 \oplus \lambda_1 I$, and T_1 has the same essential spectrum and essential numerical range as T. Thus, applying Theorem 2 to T_1 we may find a compact operator K_2 of norm less than $\epsilon/4$ and such that $T_1 - K_2$ is unitarily equivalent to $T_2 \oplus \lambda_2 I$ where T_2 again preserves the essential spectrum and the essential numerical range. Clearly we may now proceed inductively to obtain the desired infinite sequence.

LEMMA 4. *Let N be a normal operator and suppose D is a diagonal operator such that $N - D$ is a compact operator of norm less than ϵ. Then there exists a diagonal operator D_1 such that $N - D_1$ is a compact operator of norm less than 2ϵ and such that $\sigma(D_1)$ is contained in $\sigma(N)$.*

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Proof. Let $\{\lambda_n\}$ be the sequence of eigenvalues associated with D. Let $E_k = \{n:\text{dist}(\lambda_n, \sigma(N)) \geqq 1/k\}$. If any E_k were infinite then $N - D$ would not be compact. Now let D_1 be a diagonal operator obtained by shifting each λ_n by the smallest length needed to move it into $\sigma(N)$. Clearly $D - D_1$ is a compact operator of norm less than ϵ and the result follows.

THEOREM 5. *Let T be an operator. If N is a normal operator such that $\sigma(N)$ is contained in $\sigma_e(T) \cap \partial W_e(T)$, then $T \approx T \oplus N$.*

Proof. By a theorem of I. D. Berg [**2**], $N \approx D$ where D is a diagonal operator. Hence, by the lemma, $N \approx D_1$ where $\sigma(D_1)$ is contained in $\sigma(N)$. Thus, it suffices to show $T \approx T \oplus D$ where D is a diagonal operator whose spectrum is contained in $\sigma_e(T) \cap \partial W_e(T)$. Now by Corollary 3

$$T \approx T' \oplus D \oplus D \oplus \cdots = D \oplus T' \oplus D \oplus D \oplus \cdots \approx D \oplus T = T \oplus D.$$

Applications. Recall that a *convexoid* operator is by definition an operator such that conv $\sigma(T) = W(T)^-$. In what follows we shall use the fact (due to Putnam [**8**]) that the set $\pi_\infty(T) \cup \sigma_e(T)$ contains $\partial\sigma(T)$ for each operator T, where $\pi_\infty(T)$ denotes the set of isolated eigenvalues of finite multiplicity in $\sigma(T)$.

In what follows the convex hull of a set and the extreme points of a set shall be denoted by conv$(\cdot)$ and ext$(\cdot)$ respectively.

COROLLARY 6. *Let T be a convexoid operator and let δ denote the set* ext(conv $\sigma(T))\backslash\pi_\infty(T)$. *If N is a normal operator such that $\sigma(N)$ is contained in δ then $T \approx T \oplus N$.*

Proof. If λ is in δ then λ is in $\partial\sigma(T)$ and, hence, by Putnam's theorem λ is in $\sigma_e(T)$. Thus, λ is in $W_e(T)$. But since T is convexoid λ is in $\partial W(T)^-$ and, hence, λ is in $\partial W_e(T)$.

Note that if δ contained only a finite number of points, it would follow that $\sigma_e(T)$ contained only a finite number of points. Thus, if $\sigma_e(T)$ is infinite we are assured that δ is an infinite set.

Let L^∞ and L^2 denote the sets of all (equivalence classes) of essentially bounded functions and square integrable functions on the unit circle respectively and let H^2 denote the Hardy space of functions analytic in the unit disk with square integrable boundary values. Then H^2 may be viewed as being contained in L^2 so we may let P denote the projection from L^2 to H^2. If φ is an element of L^∞ the Toeplitz operator induced by φ is the operator acting on H^2 determined by the equation $T_\varphi g = P(\varphi g)$. We shall use the following facts about Toeplitz operators:

(i) T_φ is convexoid [**3**, p. 99].

(ii) $\pi_\infty(T_\varphi) \cap \partial$conv $\sigma(T_\varphi) = \emptyset$ [**3**, Theorem 10].

(iii) conv $\sigma(T_\varphi) =$ conv $R(\varphi)$ where $R(\varphi)$ is the essential range of φ [**3**; **6**].

COROLLARY 7. *Let T_φ be a Toeplitz operator and let* ext(T_φ) be the set of

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extreme points of conv $\sigma(T_\varphi)$ (= the set of extreme points of conv $R(\varphi)$). *If* N *is a normal operator such that* $\sigma(N)$ *is contained in* $\operatorname{ext}(T_\varphi)$, *then* $T \approx T \oplus N$.

Proof. This follows from Corollary 6 and (i), (ii) and (iii) above.

COROLLARY 8 [**4**]. *Let* T_z *denote the (simple) unilateral shift. Then* $T_z \approx T_z \oplus U$ *for all unitary operators* U.

Proof. The extreme points of conv $\sigma(T_z)$ are the points on the unit circle. The corollary now follows from the fact that a normal operator is unitary if and only if its spectrum is contained in the unit circle.

In closing we remark that Pearcy and Salinas [**7**] have (independently) obtained results similar to those given here for a different class of operators.

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