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WEAK*-CLOSED DERIVATIONS FROM $C[0,1]$ INTO $L^\infty[0,1]$

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ABSTRACT. We show that every weak*-closed derivation from $C[0,1] \subset L^\infty[0,1]$ into $L^\infty[0,1]$ is the inverse of integration against a function in $L^1[0,1]$.

1. **Introduction.** By a weak*-closed derivation from $C[0,1]$ into $L^\infty[0,1]$ we mean a linear map δ from a sup norm-dense self-adjoint subalgebra of $C[0,1]$ into $L^\infty[0,1]$ which satisfies

$$\delta(fg) = f\delta(g) + \delta(f)g$$

for all $f, g \in \mathrm{dom}(\delta)$, and whose graph is a weak*-closed subspace of $L^\infty[0,1]^2$. (We consider $C[0,1] \subset L^\infty[0,1]$, so the graph of δ is contained in $C[0,1] \times L^\infty[0,1] \subset L^\infty[0,1]^2$.)

In this paper we show that for every such derivation there exists a function $\phi \in L^1[0,1]$ such that δ satisfies

$$\delta(f) = f'/\phi,$$

with the domain of δ consisting of all absolutely continuous functions f with the property that f'/ϕ is essentially bounded. An equivalent form of this statement which avoids concern with the zeros of ϕ is that δ satisfies

$$\delta(\alpha + \int_0^{\cdot} g\phi) = g,$$

with domain all such expressions with $\alpha \in \mathbf{C}$ and $g \in L^\infty[0,1]$. In this sense we say that δ is the inverse of integration against an L^1 function. It it also easy to see that any $\phi \in L^1[0,1]$ which does not vanish on any open interval determines such a derivation.

One easy consequence of our result is that every such derivation has as a core a derivation from $C[0,1]$ into itself. Thus, we have a simple classification of those derivations of $C[0,1]$ with a certain closability property. This may be contrasted with two previous partial characterizations of derivations of $C[0,1]$, given in ([1], Theorems 3.2 and 3.7) and ([10], Theorem 6). Both of these results are powerful and decisive, but also seem to involve a fair amount of technical detail even to state precisely. Obviously, weak*-closability is a strong assumption, and from the simplicity of our result it appears that

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it may be a useful assumption for studying derivations. We remark in passing that with some minor additional hypotheses our result can be derived from ([10], Theorem 6). However, this does not involve significantly less work than the proof from scratch.

The main technical tool used to prove our result is the analysis of domains of derivation of abelian von Neumann algebras given in [16], in particular the fact that any such domain equals the algebra of Lipschitz functions for some metric. However, we have tried to write this paper so that it can be read independently, provided a small number of results on Lipschitz algebras are taken on faith.

Derivations of $C[0,1]$ have been considered in [1], [7], [8], [9], [10], [11], [13]. Two good general references on derivations of C*-algebras are [2] and [12]. Derivations of von Neumann algebras are discussed in [3], [4], [5], [6], and [16].

I wish to thank the referee for a number of improvements on the exposition.

2. **Results.** Throughout Lemmas 0–9 fix a weak*-closed derivation δ from $C[0,1]$ into $L^\infty[0,1]$.

We need the concept of a "measurable pseudometric" on [0,1] (see [15]). This is a map $\rho\colon \mathcal{B}\times\mathcal{B}\longrightarrow \mathbf{R}^+$ ($\mathcal{B}$ denoting the Lebesgue subsets of [0,1] of positive measure) with the following properties: $\rho(A,B)=\rho(A',B)$ if A and A' differ by a (Lebesgue) null set; and for all $A,B,C,A_n\in\mathcal{B}$

$$\begin{aligned}\rho(A,A)&=0,\\ \rho(A,B)&=\rho(B,A),\\ \rho(\bigcup A_n,B)&=\inf\rho(A_n,B),\\ \rho(A,C)&\le\sup_{B'\subset B}\big(\rho(A,B')+\rho(B',C)\big).\end{aligned}$$

A complex-valued function f on [0,1] is considered Lipschitz with respect to ρ if its Lipschitz number,

$$L(f)=\sup\{\rho_f(A,B)/\rho(A,B):A,B\in\mathcal{B}\text{ and }\rho(A,B)>0\}$$

is finite, where $\rho_f(A,B)$ is the distance (in $\mathbf{C}$) between the essential ranges of $f|_A$ and $f|_B$. (The essential range of f is the set of $x\in\mathbf{C}$ such that $f^{-1}(U)$ has positive measure, for every open neighborhood U of x.)

For example, any ordinary pseudometric on [0,1] gives rise to a measurable pseudometric in the following way. Let ρ' be an ordinary pseudometric and let $\rho'(A,B)$ denote the usual distance between sets, $\rho'(A,B)=\inf\{\rho'(x,y):x\in A,y\in B\}$. Then we can define a measurable pseudometric by

$$\rho(A,B)=\sup\{\rho'(A',B'):A'\sim A,B'\sim B\},$$

where $A'\sim A$ means that A' and A differ by a null set. There do exist measurable pseudometrics which do not arise in this way from ordinary pseudometrics; however, one of our first goals is to show that the measurable pseudometrics of interest to us in this paper do arise in this way.

An important fact that we will use repeatedly is that if ρ is any measurable pseudometric and A and B are positive measure sets with $\rho(A,B) > 0$, then there exists a real-valued function f with $L(f) \leq 1$ such that $\rho_f(A,B) = \rho(A,B)$. Such a function can be defined by taking the L^∞ supremum of all the functions $\rho(A,C)\cdot\chi_C$ as C ranges over positive measure sets. (χ denotes characteristic function.) Essentially this argument is made in the proof of Theorem 10 of [15].

The relevance of these concepts to the present situation is that according to ([16], Theorem 16), given δ there exists a measurable pseudometric ρ on [0,1] such that the domain of δ consists of precisely those bounded measurable functions which are Lipschitz with respect to ρ, *i.e.*dom(δ) = $\mathrm{Lip}_\rho[0,1]$; and furthermore, we have

$$\|f\|_L \equiv \max\big(L(f), \|f\|_\infty\big) = \max\big(\|\delta(f)\|_\infty, \|f\|_\infty\big)$$

for all such f. (This is a general fact about weak*-closed derivations of abelian von Neumann algebras.) In particular ([15], Theorem 6) this tells us that the real part of the unit ball of dom(δ) (using the norm $\|\cdot\|_L$) is a complete lattice.

To clarify the preceding paragraph, consider the ordinary derivative map $f \longmapsto f'$ from $C[0,1]$ into $L^\infty[0,1]$. With domain $C^1[0,1]$ its graph is not weak*-closed, but it is weak*-closable; taking its weak*-closure yields a derivation of the form we are considering here. Its domain then becomes Lip[0,1]. (This follows from [14], Theorem B.)

The subsequent argument relies heavily on the fact that the domain of δ is a Lipschitz algebra. Indeed, Lemmas 0–5 use only this fact. The derivation property of δ only comes in explicitly in the proof of Theorem 10.

LEMMA 0. *Let J be a closed subset of $[0,1]$ and let $y \in [0,1] - J$. Then there exists a function in* dom(δ) *which vanishes on J but not in a neighborhood of y.*

The preceding lemma follows from ([12], Lemma 3.5.12).

LEMMA 1. *Let $f \in$* dom(δ) *and $x_0 \in (0,1)$. Then the function*

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \leq x_0 \\ f(x_0) & \text{if } x \geq x_0 \end{cases}$$

is also in dom(δ) *and satisfies $\|\tilde{f}\|_L \leq \|f\|_L$.*

PROOF. Let $n \in \mathbf{N}$ and choose $x_1 > x_0$ such that

$$f(x_0) - 1/n \leq f(x) \leq f(x_0) + 1/n$$

for all $x_0 \leq x \leq x_1$.

Define

$$g_0 = \bigvee\{g \in \mathrm{dom}(\delta) : g[0,1] \subset \mathbf{R}, \|g\|_L \leq 1, \text{ and } g|_{[x_1,1]} = 0\}.$$

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Then $g_0 \in \mathrm{dom}(\delta)$ by ([15], Theorem 6), and g_0 is strictly positive on $[0, x_0]$ by Lemma 0. Thus, scaling g_0 and adding a constant gives us a function $g_1 \in \mathrm{dom}(\delta)$ with the property that $g_1 \geq f$ on $[0, x_1]$ and $g_1 = f(x_0) + 1/n$ on $[x_1, 1]$. Similarly, we can find a function $g_2 \in \mathrm{dom}(\delta)$ such that $g_2 \leq f$ on $[0, x_1]$ and $g_2 = f(x_0) - 1/n$ on $[x_1, 1]$.

Now let $f_n = (f \wedge g_1) \vee g_2$. Again by ([15], Theorem 6) f_n is in $\mathrm{dom}(\delta)$. Also, f_n agrees with f on $[0, x_1]$ and with

$$f_{(n)} = \Big(f \wedge \big(f(x_0) + 1/n\big)\Big) \vee \big(f(x_0) - 1/n\big)$$

on $[x_1, 1]$. By ([16], Corollary 4) this implies that $\delta(f_n)$ agrees with $\delta(f)$ on $[0, x_1]$ and with $\delta\big(f_{(n)}\big)$ on $[x_1, 1]$. Observe that $\|f_{(n)}\|_\infty \leq \|f\|_\infty$ and $L\big(f_{(n)}\big) \leq L(f)$, the latter because the Lipschitz number of a constant function is zero and in general

$$L(f \vee g) \leq \max\big(L(f), L(g)\big).$$

It follows that $\|f_n\|_L \leq \|f\|_L$ for all n. Since $f_n \longrightarrow \tilde{f}$ in norm and the graph of δ is weak*-closed, it now follows that $\tilde{f} \in \mathrm{dom}(\delta)$ and $\|\tilde{f}\|_L \leq \|f\|_L$. ∎

The next lemma may be compared with ([10], Lemma 2).

LEMMA 2. *Let* $f, g \in \mathrm{dom}(\delta)$ *and suppose* $f(x_0) = g(x_0)$. *Then the function*

$$h(x) = \begin{cases} f(x) & \text{if } x \leq x_0 \\ g(x) & \text{if } x \geq x_0 \end{cases}$$

also belongs to $\mathrm{dom}(\delta)$ *and satisfies*

$$\delta(h)(x) = \begin{cases} \delta(f)(x) & \text{if } x \leq x_0 \\ \delta(g)(x) & \text{if } x \geq x_0. \end{cases}$$

In particular, $\|h\|_L \leq \max(\|f\|_L, \|g\|_L)$.

PROOF. By Lemma 1 the functions

$$h_1(x) = \begin{cases} f(x) & \text{if } x \leq x_0 \\ f(x_0) & \text{if } x \geq x_0 \end{cases}$$

and

$$h_2(x) = \begin{cases} g(x_0) & \text{if } x \leq x_0 \\ g(x) & \text{if } x \geq x_0 \end{cases}$$

both belong to $\mathrm{dom}(\delta)$. Therefore $h = h_1 + h_2 - f(x_0)$ also belongs to $\mathrm{dom}(\delta)$. Agreement of $\delta(h)$ with $\delta(f)$ and $\delta(g)$ on $[0, x_0]$ and $[x_0, 1]$, respectively, now follows from ([16], Corollary 4). ∎

Define an ordinary (*i.e.*, not measurable) pseudometric ρ' by $\rho'(x, y) = \sup \rho(U, V)$, taking the supremum over all open neighborhoods U of x and V of y. An alternative definition of ρ' which more obviously satisfies the triangle inequality is $p'(x, y) = \sup |f(x) - f(y)|$, taking the supremum over all functions f which are constant on neighborhoods of

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x and y and whose Lipschitz number with respect to ρ is at most one. (By assumption, all such functions are continuous.)

Our next goal is to show that we can replace ρ by ρ'.

LEMMA 3. *Let $f \in C[0,1]$. Then the Lipschitz number of f with respect to ρ is the same as its Lipschitz number with respect to ρ'.*

PROOF. First we show $L_{\rho'}(f) \leq L_\rho(f)$. To see this choose $x, y \in [0,1]$ and let $\epsilon > 0$. Then we can find open neighborhoods U of x and V of y such that the distance between $f(U)$ and $f(V)$ is at least $|f(x) - f(y)| - \epsilon$. This implies that

$$\frac{\rho_f(U,V)}{\rho(U,V)} \geq \frac{|f(x) - f(y)| - \epsilon}{p'(x,y)}$$

(since $\rho'(x,y) \geq \rho(U,V)$ automatically). Taking ϵ to zero shows that $L_\rho(f) \geq L_{\rho'}(f)$.

Now for the converse. Let A and B be Lebesgue subsets of [0,1] and let $\epsilon > 0$. Choose a nested sequence of compact subsets A_n of A such that $A - \bigcup A_n$ is null; then since $\rho(A,B) = \inf \rho(A_n, B)$, we have $\rho(A', B) \leq \rho(A,B) + \epsilon$ for A' equal to some A_n. In a similar way we can find a compact subset B' of B such that $\rho(A', B') \leq \rho(A,B) + 2\epsilon$. Without loss of generality we may assume that A' has positive measure intersection or empty intersection with every open subset of [0,1], and similarly for B'; consequently, for any $x \in A'$, $f(x)$ belongs to the essential range of $f|_{A'}$, and similarly for B'. It will suffice to show that

$$L_{\rho'}(f) \geq \frac{\rho_f(A', B')}{\rho(A', B')}.$$

Thus we may replace A and B with A' and B'.

Now suppose that $\rho'(x,y) \geq \rho(A,B) + \epsilon$ for all $x \in A, y \in B$. Then for every such x and y we can find open neighborhoods U_x and V_y such that $\rho(U_x, V_y) \geq \rho(A,B) + \epsilon$, and by a standard trick involving subcovers (since A and B are now compact) we can then find open sets U containing A and V containing B such that $\rho(U,V) \geq \rho(A,B) + \epsilon$. But this is absurd, so we conclude that there exist $x \in A, y \in B$ such that $\rho'(x,y) \leq \rho(A,B) + \epsilon$. We then have

$$\frac{|f(x) - f(y)|}{\rho'(x,y)} \geq \frac{\rho_f(A,B)}{\rho(A,B) + \epsilon},$$

which is good enough. We conclude that $L_{\rho'}(f) \geq L_\rho(f)$ as well. ∎

LEMMA 4. *Every bounded scalar-valued function on* $[0,1]$ *which is Lipschitz with respect to ρ' is continous.*

PROOF. It will suffice to show that $x_n \longrightarrow x$ implies $\rho'(x_n, x) \longrightarrow 0$. Without loss of generality suppose $x_n \leq x$ for all n.

Let $\epsilon > 0$ and define

$$g_0 = \bigvee \{g \in \mathrm{dom}(\delta) : g[0,1] \subset \mathbf{R}, \|g\|_L \leq 1, \text{ and } g|_{[x,1]} = 0\}.$$

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Then $g_0 \in \mathrm{dom}(\delta)$, and hence it is continuous. Therefore there exists $N \in \mathbf{N}$ such that $g_0(x_n) \leq \epsilon$ for all $n \geq N$.

We claim that $\rho'(x_n, x) \leq \epsilon$ for all $n \geq N$. For suppose this fails for some $n \geq N$. Then there exists a function $g_1 \in \mathrm{dom}(\delta)$ which vanishes on a neighborhood of x and satisfies $\|g_1\|_L \leq 1$ and $g_1(x_n) > \epsilon$. But by Lemma 1, the function

$$g_2(y) = \begin{cases} g_1(y) & \text{if } y \leq x \\ 0 & \text{if } y \geq x \end{cases}$$

is also in $\mathrm{dom}(\delta)$ and also satisfies $\|g_2\|_L \leq 1$. Thus g_2 is in the collection whose supremum defines g_0, and it is greater than ϵ in a neighborhood of x_n; this implies that $g_0(x_n) > \epsilon$, a contradiction. This establishes that $\rho'(x_n, x) \leq \epsilon$ for all $n \geq N$ and we conclude that $\rho'(x_n, x) \longrightarrow 0$, as desired. ∎

From Lemmas 3 and 4 it follows that a function is Lipschitz with respect to ρ if and only if it is Lipschitz with respect to ρ', and its Lipschitz numbers with respect to ρ and ρ' agree. Therefore, without loss of generality we will henceforth assume ρ is actually a pseudometric, not just a measurable pseudometric.

By compactness of [0,1] and continuity of ρ (Lemma 4), it follows that the diameter of ρ is finite. Equivalently, if $f(x) = 0$ for any $x \in [0,1]$ then $\|f\|_\infty \leq K \cdot L(f)$, hence $\|f\|_\infty \leq K\|\delta(f)\|_\infty$ where K is the diameter of ρ. Now suppose we multiply δ by K; this would give us that $\|f\|_\infty \leq \|\delta(f)\|_\infty$, hence $\|f\|_\infty \leq L(f)$, if $f(x) = 0$ for some $x \in [0,1]$. Thus, multiplying δ by K, without loss of generality we may assume that the diameter of ρ is at most one.

LEMMA 5. *For any $x < y < z$ in $[0,1]$, we have $\rho(x,z) = \rho(x,y) + \rho(y,z)$.*

Proof. Define $f(w) = \rho(x, w)$ and $g(w) = \rho(x,y) + \rho(y, w)$. These functions are clearly Lipschitz, hence belong to $\mathrm{dom}(\delta)$, and satisfy $\|f\|_L, \|g\|_L \leq 1$. Thus the function

$$h(w) = \begin{cases} f(w) & \text{if } w \leq y \\ g(w) & \text{if } w \geq y \end{cases}$$

also belongs to $\mathrm{dom}(\delta)$ and satisfies $\|h\|_L \leq 1$, by Lemma 2.

It follows that

$$\rho(x,z) \geq h(z) - h(x) = g(z) - f(x) = \rho(x,y) + \rho(y,z).$$

The reverse inequality is automatic. ∎

LEMMA 6. *If $f \in \mathrm{dom}(\delta)$ and $\delta(f) = 0$ then f is a constant function.*

PROOF. Suppose $f \in \mathrm{dom}(\delta)$ is not constant but $\delta(f) = 0$. By taking real and imaginary parts we may assume f is real, and by adding a constant and multiplying by a scalar we may suppose $f(x_0) = 0$ for some $x_0 \in (0,1)$ but $f(x) = 1$ for some $x < x_0$. Finally, applying Lemma 1 we may assume $f(x) = 0$ for all $x \geq x_0$.

For each $n \in \mathbf{N}$ let x_n be the greatest real number $< x_0$ such that $f(x_n) = 1/n$. Define

$$f_n(x) = \begin{cases} 1 & \text{if } x \leq x_n \\ nf(x) & \text{if } x \geq x_n. \end{cases}$$

Then $\delta(f_n) = 0$ for all n by Lemma 1, and (f_n) converges weak* to a discontinuous function. This contradicts weak*-closure of the graph of δ and the fact that $\mathrm{dom}(\delta) \subset C[0,1]$. We conclude that $\delta(f) = 0$ implies f is constant. ∎

Define a measure ν on [0,1] by $\nu[x,y] = \rho(x,y) = \rho(y,0) - \rho(x,0)$ (*i.e.* ν is the derivative of the function $\rho(\cdot,0)$). Also denote Lebesgue measure on [0,1] by μ.

LEMMA 7. $\nu \ll \mu$.

PROOF. Suppose ν is not absolutely continuous with respect to Lebesgue measure. Then there exists a set $S \subset [0,1]$ of Lebesgue measure zero such that $\nu(S) \neq 0$.

For each $n \in \mathbf{N}$ let S_n be a countable union of closed intervals such that $S \subset S_n$ and $\mu(S_n) \leq 1/n$. Then define

$$f_n(x) = \int_0^x \chi_{S_n} d\nu$$

(where χ denotes characteristic function); we have that $\|f_n\|_L \leq 1, f_n(0) = 0$, and $f_n(1) = \nu(S_n)$. Also $\delta(f_n) = 0$ on $[0,1] - S_n$. Then (f_n) has a weak*-cluster point f which satisfies $f(0) = 0, f(1) = \nu(S)$, and $\delta(f) = 0$ on $[0,1] - S$, *i.e.* $\delta(f) = 0$ almost everywhere. This contradicts Lemma 6, so we are done. ∎

LEMMA 8. *The function* $\tau(x) = \rho(x,0)$ *satisfies* $|\delta(\tau)(x)| = 0$ *or* 1 *almost everywhere.*

PROOF. Note first that $\|\delta(\tau)\|_\infty \leq \|\tau\|_L = 1$. Suppose $0 < |\delta(\tau)(x)| \leq 1 - \epsilon$ on a set S of positive (Lebesgue) measure, and define $\alpha = 1/(1-\epsilon)$. For each $n \in \mathbf{N}$ let S_n be a finite union of intervals which differs from S on a set of measure at most $1/n$. Then define $f_n \in \mathrm{dom}(\delta)$ by setting $f_n(0) = 0$ and patching together τ and $\alpha\tau$ (via Lemma 2) in such a way that $f_n - \tau$ is locally constant on $[0,1] - S_n$, while $f_n - \alpha\tau$ is locally constant on S_n.

Now $\|\delta(f_n)\|_\infty \leq \alpha$ for all n (hence $\|f_n\|_\infty \leq \alpha$, since $f_n(0) = 0$), while f_n converges pointwise to a function g which satisfies $g(0) = 0$ and $g \geq \tau$. This implies that $g \in \mathrm{dom}(\delta)$. Also since $\mu(S_n - S) \longrightarrow 0$, we have $\|\delta(g)\|_\infty \leq 1$. But $g \neq \tau$ since $\delta(g) = \alpha\delta(\tau) \neq 0$ on S, so for some $x \in [0,1]$ we have $g(x) - g(0) > \tau(x) - \tau(0) = \rho(x,0)$, a contradiction. This establishes the lemma. ∎

It is clear that ν is a finite measure since ρ has finite diameter. Therefore the Radon-Nikodym theorem implies that $d\nu = \phi_0\, d\mu$ for some $\phi_0 \in L^1[0,1]$. Define $\phi \in L^1[0,1]$ by $\phi = \phi_0\overline{\delta(\tau)}$.

It is also clear that ϕ cannot vanish on any interval of positive length, for this would imply that $\rho(x,y) = 0$ for all x,y in this interval, and hence that every $f \in \mathrm{dom}(\delta)$ is constant on this interval, contradicting the norm-density of $\mathrm{dom}(\delta)$ in $C[0,1]$.

Now define a derivation δ_ϕ by

$$\delta_\phi(\alpha + \int_0^{\cdot} g\phi\, d\mu) = g,$$

with domain all such expressions such that $\alpha \in \mathbf{C}$ and $g \in L^\infty[0,1]$.

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LEMMA 9. *δ_ϕ is a weak*-closed derivation from $C[0,1]$ into $L^\infty[0,1]$. Its domain equals the domain of δ.*

PROOF. That δ_ϕ is a derivation follows from integration by parts. Its graph is weak*-closed by the dominated convergence theorem, and its domain is obviously contained in $C[0,1]$. Finally, $\mathrm{dom}(\delta) \subset C[0,1]$ separates points since ϕ does not vanish on any interval of positive length.

If $f \in \mathrm{dom}(\delta_\phi)$, say $f(x) = \alpha + \int_0^x g\phi\, d\mu$, then

$$\begin{aligned} |f(x) - f(y)| &\leq \|g\|_\infty \int_x^y |\phi|\, d\mu \\ &= \|g\|_\infty \int_x^y d\nu \\ &= \|g\|_\infty\big(\rho(y,0) - \rho(x,0)\big) \\ &= \|g\|_\infty \rho(x,y). \end{aligned}$$

Thus f is Lipschitz with respect to ρ and so $f \in \mathrm{dom}(\delta)$. Conversely, if $f \in \mathrm{dom}(\delta)$ then f is Lipschitz with respect to ρ, so the measure $\nu_f[x,y] = f(y) - f(x)$ is absolutely continuous with respect to ν and satisfies $g = d\nu_f/d\nu \in L^\infty[0,1]$. It follows that $f \in \mathrm{dom}(\delta_\phi)$ and $\delta_\phi(f) = g\delta(\tau)$. ∎

We are finally ready to prove our main result.

THEOREM 10. *Let $\delta\colon C[0,1] \longrightarrow L^\infty[0,1]$ be a weak*-closed derivation. Then there exists a function $\phi \in L^1[0,1]$ such that $\delta = \delta_\phi$.*

PROOF. Define ϕ as above. By Lemma 9, δ and δ_ϕ are weak*-closed derivations from $C[0,1]$ into $L^\infty[0,1]$ with the same domain. Furthermore, we have $\delta(\tau) = \delta_\phi(\tau)$ where τ is the function $\tau(x) = \rho(x,0)$. From the derivation identity it follows that δ agrees with δ_ϕ on the algebra generated by τ.

The function τ separates points uniformly in the sense that $|\tau(x) - \tau(y)| = \rho(x,y)$ for all $x, y \in [0,1]$. According to ([14], Theorem B) the algebra generated by τ is dense in $\mathrm{dom}(\delta) = \mathrm{Lip}_\rho[0,1]$ in its weak* topology. This is just the weak*-topology that $\mathrm{dom}(\delta)$ inherits from $L^\infty[0,1] \times L^\infty[0,1]$, identifying $\mathrm{dom}(\delta)$ with the graph of δ by the map $f \mapsto \big(f, \delta(f)\big)$. Thus, $f_\alpha \longrightarrow f$ weak* in $\mathrm{Lip}_\rho[0,1]$ if and only if $f_\alpha \longrightarrow f$ and $\delta(f_\alpha) \longrightarrow \delta(f)$, both weak* in $L^\infty[0,1]$. Thus, density of the algebra generated by τ and the fact that δ and δ_ϕ agree on this algebra imply that $\delta = \delta_\phi$. ∎

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