GENERAL RELATIVISTIC DESCRIPTION OF CELESTIAL REFERENCE FRAMES

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ABSTRACT. The astonomical consequences of recently developed theoretical methods of relativistic astrometry are discussed. The set of practically important reference systems is described. These reference systems generalize the locally inertial frames of general relativistic test observer, the hierarchy of Jacoby coordinates for dynamical problems and the dynamically inertial reference systems of fundamental astrometry. In practical application of this formalism much attention is paid for relativistic transformation functions relating the ecliptical coordinates corresponding to the barycenters of the Solar system, the Earth-Moon subsystem and the Earth. Solutions to several kinds of relativistic precession are also presented.

The ultimate aim of astrometry is to set up an inertial reference frame. Traditionally, the problem is to introduce a coordinate system which does not move and rotate with respect to very remote light emitters. Within the framework of classical mechanics such a kinematical construction immediately provides the nesessary dynamical properties of an inertial system - namely the absence of translatory, centripetal and Coriolise inertial forces.

General Relativity prohibits the classical inertiality. Only in the case of weak gravitation one may construct a system which retains some particular properties of an inertial one. Thus, if a system moves, it cannot be sumutaneously dynamically and kinematically inertial.

If we consider the Solar system as a whole, we can use its *Barycentric Reference System* (*BRS*). It may be regarded as completely inertial at the sufficient level of accuracy.

On the other hand, most of astronomical techniques and applications are concerned with the Earth, its close vicinity and the Earth-Moon subsystem. Therefore it is reasonable to consider a set of quasi-inertial reference frames which are related to these bodies. Evidently, most important of them would be the *Geocentric Reference System* (*GRS*) and *Terrestrial- lunar Reference System (TRS)*. The latter

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is related to the Earth-Moon barycenter.

Let us consider in more detail the *dynamically inertial* terrestrial-lunar reference system *(TRS)* $(x^{\tilde{0}}, x^{\tilde{i}})$ in its relation to the *BRS* (x^{0}, x^{i}) of the Solar system.

To define a coordinate system in General Relativity is to maintain the corresponding metric tensor.

Since the Earth-Moon subsystem is compact with respect to its distance to the Sun, we may take an advantage to treat separately the gravitation of the internal bodies (the Earth and the Moon) and of those external (the Sun and planets). Then the required metric in *TRS* may be found as a post-Newtonian solution to the Einstein equations in harmonic coordinates where the boundary conditions are used to account for the dynamical inertiality of the spatial axes of *TRS*. Detailed description of this techniques may be found in [1,2,3,4].

As a result we obtain some general form of metric tensor both in *BRS* and in *TRS*:

$$g_{00} = 1 - 2\varphi + 2\varphi^{2} - 2\chi_{,00} - 2\psi, \quad g_{0i} = 4\varphi_{i} \quad , \quad g_{ij} = -\delta_{ij}(1 + 2\varphi)$$

where all the "potentials" φ, φ_{i}, χ and ψ are represented as sums of its internal and background parts. Internal components define the gravitational field of the Earth-Moon subsystem in the post-Newtonian limit. The background field is produced by the Sun and the planets. The background potentials are "direct" in *BRS* and "tidal" in *TRS*. Thus, the background solar potential takes the form:

$$w_{\tilde{k}}x^{\tilde{k}} + \frac{1}{2}w_{\tilde{k}\tilde{l}}x^{\tilde{k}}x^{\tilde{l}} + \frac{1}{6}w_{\tilde{k}\tilde{l}\tilde{m}}x^{\tilde{k}}x^{\tilde{l}}x^{\tilde{m}} + \dots ,$$

where

$w_{\tilde{i}}$ is the covariant acceleration of the *TRS* origin (point T),

$$w_{\tilde{i}\tilde{j}} = E_{\tilde{i}\tilde{j}} + 3w_{\tilde{i}}w_{\tilde{j}} - w_{\tilde{m}}w_{\tilde{m}}\delta_{ij} \ , \ w_{\tilde{i}\tilde{j}\tilde{k}} = E_{\tilde{i}\tilde{j}\tilde{k}} - \frac{10}{3}\delta_{(ij}w_{\tilde{k})\tilde{n}}w_{\tilde{n}} + 4w_{(\tilde{i}}w_{\tilde{j}\tilde{k})}$$

$E_{\tilde{i}\tilde{j}}$ is the "elecrtic" part of background curvature, which leading terms are:

$$R_{\tilde{0}\tilde{i}\tilde{0}\tilde{j}}x^{\tilde{i}}x^{\tilde{j}} = E_{\tilde{i}\tilde{j}}x^{\tilde{i}}x^{\tilde{j}} = \frac{m_{S}}{R_{T}^{3}} P_{2}[\cos(\mathbf{R}_{T}\cdot\tilde{\mathbf{r}})] + \dots..$$

The structure of this equation is similar to that of traditional expansion on powers of parallax.

As a side product of these techniques we immediately obtain the transformation functions which relate *TRS* to *BRS*:

$$x^{\alpha} = x^{\tilde{\alpha}} + L^{\tilde{\alpha}}(x^{\tilde{\nu}}) + P^{\tilde{\alpha}}(x^{\tilde{\nu}}) + T^{\tilde{\alpha}}(x^{\tilde{\nu}})$$

It contains the Lorentz bust (L), the relativistic precession (P) and the terms (T), which are nesessary to reduce the background potentials to the tidal ones.

Relativistic precession (especially - geodetic) determines the difference between the kinematically and dynamically inertial orientations of the moving reference systems. This precession may be expressed in terms of two angular quantities (γ and ε), which define correspondingly the relativistic precession in longitude and inclination. The laws of Fermi-Walker transport in the background metric provide two equations for these angles:

$$\frac{d\gamma}{dx^{\tilde{0}}} = \omega_G^3 + \omega_T^3 \quad , \frac{d\varepsilon}{dx^{\tilde{0}}} = (\omega_G^1 + \omega_T^1)\cos\gamma + (\omega_G^2 + \omega_T^2)\sin\gamma \ .$$

where

$$\omega_G = \frac{3}{2}V_T \times \nabla \overline{b} + 2\,\overline{rot b} \ , \quad \text{(geodetic + Lense-Thirring)}$$

$$\omega_T = \frac{1}{2}V_T \times w \ . \quad \text{(Thomas)}$$

$$V_T = dr_T/dx^0, \qquad w = dV_T/dx^0$$

In virtue of a perturbation method we find the solutions for these equations:

$$\gamma = \gamma_p + \gamma_N \ ,$$

$$\gamma_p = \left[\frac{3}{2}\frac{n'^3 a'^2}{1-c'^2} + \sum_{b=1}^{N} \nu_b \left(-\frac{3}{2} n_b^2 n' a'^2 - 2n_b^3 a_b - \frac{27}{32} n_b^2 n' \frac{a'^4}{a_b^2} - \right.\right.$$
$$\left.\left. - \frac{3}{2}\nu_b n_b^3 a_b^2 \right) + \dots \right] x^{\tilde{0}} = (19''.192996/1000\text{years})\ x^{\tilde{0}} \ ,$$

$$\gamma_N = N_0 \sin(E-\pi') + 0.00192\sin(2E-2\pi') -- 0.00106\sin(E-J) + \dots \ ,$$

$$\varepsilon = \varepsilon_0 + 0.00011\cos(E+J) + 0.00010\cos(E-J) + \dots \ ,$$

$$N_0 = -^2 a'^2 c' \left(\frac{9}{2} + \frac{45}{16}c'^2 + \dots + \frac{39}{8}\sum_{b=1}^{N} \nu_b \frac{n_b^2}{n'^2} + \dots \right) = 0.15321 \ ,$$

$$\nu_b = \frac{m_b}{m_s} \ ,$$

Here the "primed" quantities describe the heliocentric motion of the Earth-Moon barycenter. Besides, n_b is the mean motion of the planet b, a_b is the semimajor axis of the planet b ($a_b > a'$), E is the mean longitude of T, J is the mean longitude of Jupiter, π' is the longitude of perihelion of T. ε_0 is the inclination constant. For example,

$\varepsilon_0 = 0$ means ecliptical orientation,

$\varepsilon_0 = 23^0 27' + \dots$ *may* mean the equatorial one.

Hereafter all the angular coefficients are expressed in milli arc seconds.

We can construct analogous analytical or semianalytical expansions for transformation between *BRS* and *TRS*. It seems convenient to express them in terms of spherical coordinates.

Let us adopt for the sake of simplicity the ecliptical orientation of both *BRS* and *TRS* and introduce the following:

$$\mathbf{r} - \mathbf{r}_T = r\,(\cos\lambda\,\cos\beta,\ \sin\lambda\,\cos\beta,\ \sin\beta),$$

$$\tilde{\mathbf{r}} = \tilde{r}\,(\cos\tilde{\lambda}\,\cos\tilde{\beta},\ \sin\tilde{\lambda}\,\cos\tilde{\beta},\ \sin\tilde{\beta}),$$

Then the relativistic transformation mentioned above is reduced to (note that $\varepsilon_0=0$):

$$\lambda = \tilde{\lambda} - \gamma + \delta\lambda, \qquad \beta = \tilde{\beta} + \delta\beta, \qquad r = \tilde{r} + \delta r,$$

where

$$\cos\tilde{\beta}\ \delta\lambda = L_1\cos\tilde{\beta} + L_2\frac{\tilde{r}}{a'},$$

$$\delta\beta = B_1\cos\tilde{\beta}\ \sin\tilde{\beta} + B_2\frac{\tilde{r}}{\tilde{a}'}\cos\tilde{\beta} + B_3\frac{\tilde{r}}{\tilde{a}'}\sin\tilde{\beta},$$

$$\delta r = \tilde{r}(R_1 + R_2\cos^2\tilde{\beta} + R_3\frac{r}{a'}\cos\tilde{\beta} + R_4\frac{r}{\tilde{a}'}\sin\tilde{\beta}),$$

$$L_1 = 0.50884\sin(2\tilde{\lambda}-2E) + 0.01701\sin(2\lambda-3E+\pi') +$$

$$+ 0.00046\sin(2\tilde{\lambda}-4E+2\pi') - 0.00041\sin(2\tilde{\lambda}-E-J) + \ldots,$$

Analogous expansions may be written for the other coefficients in above formulae.

Since this transformation is relativistic, it must contain the appropriate time component (see e.g.[ ]).

Expression for λ contains explicitly the relativistic precession and nutation in longitude. That for the inclination occurs completely negligible. Therefore we can easily obtain a more practical reference system, which is related to *BRS* with only the periodic terms of the above transformation:

$$\lambda = \lambda - \gamma_N + \delta\lambda, \qquad \beta = \tilde{\beta} + \delta\beta, \qquad r = \tilde{r} + \delta r,$$

This system is seen to be kinematically inertial in average. It also meets the modern IAU standards which combine the secular part of geodetic precession with that Newtonian.

Nevertheless, it should be noted, that initial definition of *TRS* is more theoretically consistent from the point of view of General Relativity.

References

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