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ON VECTOR SPACES OF CERTAIN MODULAR FORMS OF GIVEN WEIGHTS: ADDENDUM

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The statement $f = \lim_{\rightarrow} g_t$ used in proving Theorem 2 of [1] needs explanation. This was pointed out to us by Professor S. Raghwan of Tata Institute of Fundamental Research, Bombay, and we gave the explanation of this in [2]. For the sake of completeness we give here the full proof of the theorem; filling the gap in the proof. We use the same notations and definitions as those of [1]. Also for simplicity of notation we write k_{mn}, g_{mn} and a_{mn} to mean $k_{m,n}$, $g_{m,n}$ and $a_{m,n}$ respectively. We need the following lemma.

LEMMA. *Let $p \geq 5$ be a prime number and u an even integer such that $0 \leq u < p-1$. Further let $\{g_m\}$ be a family of all modular forms over $SL_2(Z)$ such that*

$$k_m \equiv u \bmod (p-1)$$

for all m, where k_m denotes the weight of the modular form g_m. Then for each $n = 0, 1, 2, \ldots$, there exist non-negative integers $a(n)$ and $b(n)$, satisfying

(i) $4a(n) + 6b(n) + 12n \equiv u \bmod (p-1)$, *and*

(ii) $k_{mn} = \{k_m - (4a(n)+6b(n)+12n)\}/(p-1)$ *are non-negative integers for $0 \leq n < d_m$, where d_m denotes the dimension of vector space of modular forms over $SL_2(Z)$ of weight k_m.*

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Proof. Arrange $\{d_m\}$ in ascending order of magnitude. Let k_{m_i} be the least among the weights of modular forms of dimension d_i in the family $\{g_m\}$. Then $\{k_{m_i}\}$ are also in ascending order.

For $0 \le n < d_0$, we have $k_{m_0} - 12n \ge 0$ and not equal to 2. Therefore for these n, we can choose non-negative integers $a(n)$ and $b(n)$ such that $4a(n) + 6b(n) = k_{m_0} - 12n$. Then

$$4a(n) + 6b(n) + 12n = k_{m_0} \equiv u \bmod (p-1) .$$

As $k_{m_0} \le k_m$ for all m, therefore k_{mm} are non-negative integers for $0 \le n < d_0$ and for all m. Proceeding as above we can find non-negative integers $a(n)$ and $b(n)$ for $d_i \le n < d_{i+1}$, $i = 0, 1, 2, \ldots$, satisfying the required conditions.

REMARK. Let S be any subfamily of the family of all modular forms $\{g_m\}$ with weights k_m satisfying $k_m \equiv u \bmod (p-1)$. Then the same choice of $a(n)$ and $b(n)$, as is done for this family of all modular forms in the lemma, will work for the subfamily S also.

For a given prime $p \ge 5$ and $k = (s, u)$, u as above, construct $a(n)$ and $b(n)$ for each n. Consider

$$f_m = Q^{a(n)} R^{b(n)} \Delta^n E_{p-1}^{s_n} ,$$

where $s_n = \{s-(4a(n)+6b(n)+12n)\}/(p-1)$. It follows from Theorem 1 of [1] that f_n is a p-adic modular form of weight k. Writing 'q' series expansion for f_n, that is,

$$f_n = \sum_{m=0}^{\infty} a_m^{(n)} q^m , \text{ with } a_m^{(n)} \text{ in } Q_p ,$$

we see that $a_m^{(n)} = 0$ for $0 \le m < n$ and $a_n^{(n)} = 1$. We now give a complete proof of Theorem 2 of [1].

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THEOREM. *f is a p-adic modular form of weight $k = (s, u)$ if and only if $f = \sum_{n=0}^{\infty} a_n f_n$ with $\nu_p(a_n) \to \infty$ as $n \to \infty$.*

Proof. Suppose first that f is a p-adic modular form of weight k. Write

$$f = \sum_{m=0}^{\infty} b_m^{(0)} q^m , \quad b_m^{(0)} \text{ in } Q_p .$$

The Fourier series expansion of $f - b_0^{(0)} f_0$ has no constant term, therefore we can write

$$f - b_0^{(0)} f_0 = \sum_{m=1}^{\infty} b_m^{(1)} q^m \text{ with } b_m^{(1)} \text{ from } Q_p .$$

Similarly, we can write

$$f - b_0^{(0)} f_0 - b_1^{(1)} f_1 = \sum_{m=2}^{\infty} b_m^{(2)} q^m \text{ with } b_m^{(2)} \text{ from } Q_p .$$

Repeating this process t times, we can write

$$f - \sum_{n=0}^{t} b_n^{(n)} f_n = \sum_{m=t+1}^{\infty} b_m^{(t+1)} q^m \text{ with } b_m^{(t+1)} \text{ from } Q_p .$$

This means that we can find $b_n^{(n)}$ in Q_p, $n = 0, 1, 2, \ldots$, such that as a formal series in q, we have

$$f = \sum_{n=0}^{\infty} b_n^{(n)} f_n .$$

Writing a_n for $b_n^{(n)}$, we see that

$$f = \sum_{n=0}^{\infty} a_n f_n , \text{ with } a_n \text{ in } Q_p .$$

Now we shall prove that $v_p(a_n) \to \infty$ as $n \to \infty$. Choose a sequence $\{g_m\}$ of modular forms converging to f. Then the sequence $\{k_m\}$ of their weights converges to k in X. Therefore $\{k_m\}$ converges to s in

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Z_p . For the family $\{g_m\}$ we construct $a(n)$, $b(n)$ and k_{mn} as in the lemma. For each m , define

$$g_{mn} = Q^{a(n)}R^{b(n)}\Delta^n E_{p-1}^{k_{mn}} \quad \text{for} \quad n = 0, 1, \ldots, d_m-1 ,$$

where d_m is the dimension of the vector space $M(k_m)$ of modular forms of weight k_m . Then g_{mn} are modular forms of weight k_m and constitute a basis for $M(k_m)$. Therefore for each m , we can write

$$g_m = \sum_{n=0}^{\infty} a_{mn}g_{mn} , \text{ with } a_{mn} = 0 \text{ for } n \geq d_m .$$

Now $k_{mn} = (k_m-4a(n)-6b(n)-12n)/(p-1)$ are integers (may be negative for large n) for all m and all n . Define, for each n ,

$$s_n = (s-(4a(n)+6b(n)+12n))/(p-1) .$$

Then $s_n \in Z_p$, as $k_m \to s$ in Z_p . Therefore $k_{mn} \to s_n$ for each n and this convergence is uniform in n . Therefore $E_{p-n}^{k_{mn}} \to E_{p-1}^{s_n}$ uniformly in n . Hence $g_{mn} \to f_n$ uniformly in n .

Now

$$(*) \qquad f - g_m = \sum_{n=0}^{\infty} (a_nf_n-a_{mn}g_{mn})$$
$$= \sum_{n=0}^{\infty} a_n(f_n-g_{mn}) + (a_n-a_{mn})g_{mn} .$$

As $g_m \to f$ and $g_{mn} \to f_n$ uniformly in n , therefore given a positive integer N , we can find a positive integer m_0 such that for each $m \geq m_0$, we have

$$v_p(f-g_m) > N ,$$

and

$$v_p(f_n-g_{mn}) > N \text{ for all } n \geq 0 .$$

Therefore, for $m \geq m_0$ and $n \geq 0$, it follows from (*), that

$$v_p(a_n - a_{mn}) > N .$$

In particular,

$$v_p(a_n - a_{m_0 n}) > N \text{ for all } n \geq 0 .$$

But $a_{m_0 n} = 0$ for $n \geq d_{m_0}$. Hence $v_p(a_n) > N$ for $n \geq d_{m_0}$. This shows that $v_p(a_n) \to \infty$ as $n \to \infty$.

Conversely suppose now that $f = \sum_{n=0}^{\infty} a_n f_n$, with $v_p(a_n) \to \infty$ as $n \to \infty$. Take $g_t = \sum_{n=0}^{t} a_n f_n$. Then g_t is a p-adic modular form of weight k. Since $v_p(a_n) \to \infty$ as $n \to \infty$ and $v_p(f_n) = 0$, so g_t is a convergent sequence with its limit equal to f. Hence f is a p-adic modular form of weight k.

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