CORRESPONDENCE.

UNIFORM SENIORITY AND LAST SURVIVOR ANNUITIES:—
(PROOF THAT, IF $a_{xy} = a_{w.w}$, $a_{xy} \neq a_{\overline{ww}}$).

To the Editor of the Journal of the Institute of Actuaries.

DEAR SIR,—It is well known that when a mortality table follows Makeham's Law we can find an age w such that $a_{x:y} = a_{w:w}$, and beginners sometimes fall into the error of supposing from this that in such circumstances $a_{x:y}$ and $a_{\overline{w:w}}$ are similarly related.

The following demonstration may therefore be of interest to some of your readers :—

If $a_{x:y} = a_{w:w}$ then ${}_t p_{x:y} = {}_t p_{w:w}$ for all values of t

and
$$\bar{a}_{x:y} = \bar{a}_{w:w} . \qquad . \qquad . \qquad . \qquad . \qquad (1)$$

Similarly, if $a_{\overline{x:y}}$ were equal to $a_{\overline{w:w}}$, we should have $\bar{a}_{x:y} = \bar{a}_{\overline{w:w}}$

or
$$\bar{a}_x + \bar{a}_y - \bar{a}_{x:y} = 2\bar{a}_w - \bar{a}_{w:w}$$

or, (since $\bar{a}_{x:y} = \bar{a}_{w:w}$)
$$\bar{a}_x + \bar{a}_y = 2\bar{a}_w . \qquad . \qquad . \qquad . \qquad . \qquad (2)$$

It is fairly obvious that as a rule these two different relationships between x, y and w, cannot hold good, but a formal proof can be furnished as follows :

Differentiating (1) with respect to age, we get

$$\bar{a}_{x\,:\,y}(\mu_{x\,:\,y} + \delta) - 1 = \bar{a}_{w\,:\,w}(\mu_{w\,:\,w} + \delta) - 1 \; ;$$

whence (remembering that $\bar{a}_{x\,:\,y} = \bar{a}_{w\,:\,w}$) $\mu_{x\,:\,y} = \mu_{w\,:\,w}$

or $$\mu_x + \mu_y = 2\mu_w . \qquad . \qquad . \qquad . \qquad . \quad (3)$$

Differentiating (2) we get

$$\bar{a}_x(\mu_x + \delta) - 1 + \bar{a}_y(\mu_y + \delta) - 1 = 2\bar{a}_w(\mu_w + \delta) - 2$$

and substituting for $\bar{a}_w$ and μ_w their values from (2) and (3)

$$\bar{a}_x\mu_x + \bar{a}_y\mu_y = (\bar{a}_x + \bar{a}_y)\left(\frac{\mu_x + \mu_y}{2}\right)$$

multiplying out, this reduces to

$$\bar{a}_x\mu_x + \bar{a}_y\mu_y = \bar{a}_x\mu_y + \bar{a}_y\mu_x$$

or $$\bar{a}_x(\mu_x - \mu_y) = \bar{a}_y(\mu_x - \mu_y)$$

which cannot be true unless $\bar{a}_x = \bar{a}_y$, or $\mu_x = \mu_y$

and neither of these relations can hold unless $x = y$, or μ is constant. Now μ is not constant, but increases with the age, the increase being very rapid at advanced ages, therefore the equations

$$\bar{a}_{x\,:\,y} = \bar{a}_{w\,:\,w} \text{ and } \bar{a}_{x\,:\,y} = \bar{a}_{w\,:\,w} \text{ (or } a_{\overline{x\,:\,y}} = a_{\overline{w\,:\,w}} \text{ and } a_{x\,:\,y} = a_{w\,:\,w})$$

cannot both hold good unless $x = y$, when the two sides of each become identical. That the amount of the error may be serious is shown by the following figures taken from the $O^{M(5)}$ Table at 3 per-cent, the values of w being calculated to one decimal place :—

x	y	w	$a_{\overline{w\,:\,w}}$	$a_{\overline{x\,:\,y}}$	Error
20	40	34	22·463	23·890	1·427
...	60	52·6	16·390	22·736	6·346
...	80	72·3	8·363	22·224	13·861
40	60	54	15·849	18·366	2·517
...	80	72·6	8·245	17·455	9·210
60	80	74	7·702	10·857	3·155
80	90	86·1	3·716	4·395	·679

It will be noticed that $\bar{a}_{\overline{w\,:\,w}}$ is less than $\bar{a}_{\overline{x\,:\,y}}$ in all cases, and that the difference increases rapidly as the difference between x and y becomes larger. This also appears from the following considerations. The joint life annuity cannot continue longer than the "complement of life" of the older of the two lives, however young the other may be ; therefore the age of the older life is the preponderating factor in determining the joint life annuity value $a_{x\,:\,y}$; and this preponderance increases with the difference between x and y, until, in the limiting case, when the older life (y, say) reaches the limit of

life, $a_{x:y} = a_y$ (both of them being zero), and the age of the younger
life has no effect whatever on the value. Accordingly, the
substituted age w must be nearer the older than the younger of the
two lives, or $w > (x+y):2$ and the excess increases with the difference
between x and y. On the other hand the last survivor annuity may
continue for the "complement of life" of the younger of the two
lives, however old the other may be ; and the age of the younger
is therefore the preponderating factor in this case, and the
preponderance increases with the difference between x and y until,
in the limit, when y reaches the limit of life, $a_{\overline{x:y}} = a_x$, and the
value depends solely on the age of the younger life. Therefore the
age z at which $a_{\overline{x:y}} = a_{\overline{z:z}}$ must be nearer the younger than the older
life, or $z < (x+y):2$. Hence z must be less than w, and accordingly
$a_{\overline{x:y}}$ greater than $a_{\overline{w:w}}$, and the difference between them increases
with the difference between x and y.

<div align="center">Yours faithfully,</div>

<div align="right">A. E. SPRAGUE.</div>

Edinburgh 26 *August*, 1910.

RELATION BETWEEN RATES OF SICKNESS AND SIZE OF LODGE.

To the Editor of the Journal of the Institute of Actuaries.

SIR,—One of the most striking features of the monumental
work of Mr. Alfred W. Watson, as shown in his paper in *J.I.A.*,
xxxv, is the apparent proof that the rates of sickness increase with
the size of the lodge, vide the tables shown on pages 289 *et seq.*,
and the accompanying remarks. I reproduce Table 16 (p. 291).

<div align="center">TABLE 16.</div>

Number of Members in the Lodge	Total No. of Lodges in the group	Percentage of the whole number of Lodges	SICKNESS COST					
			HIGH		NORMAL		LOW	
			No. of Lodges	Percentage of the number of Lodges in the group	No. of Lodges	Percentage of the number of Lodges in the group	No. of Lodges	Percentage of the number of Lodges in the group
				Per cent.		Per cent.		Per cent
Under 80	823	23	337	41	255	31	231	28
80–149	1,059	30	543	51	314	30	202	19
150–249	877	24	504	57	212	24	161	19
250–349	425	12	278	66	87	20	60	14
350–489	279	8	206	74	47	17	26	9
500 and over	118	3	97	82	11	9	10	9
...	3,581	100	1,965	55	926	26	690	19