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AN IMPROVED RESULT ON GROUND STATE SOLUTIONS OF QUASILINEAR SCHRÖDINGER EQUATIONS WITH SUPER-LINEAR NONLINEARITIES

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Abstract

By using variational and some new analytic techniques, we prove the existence of ground state solutions for the quasilinear Schrödinger equation with variable potentials and super-linear nonlinearities. Moreover, we establish a minimax characterisation of the ground state energy. Our result improves and extends the existing results in the literature.

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1. Introduction

We consider the quasilinear Schrödinger equation

$$-\triangle u + V(x)u - \tfrac{1}{2}\triangle(u^2)u = f(u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $N \geq 3$, $V : \mathbb{R}^N \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ satisfy the following basic assumptions:

(V1) $V \in C(\mathbb{R}^N, [0, \infty))$ and $V_\infty := \lim_{|y|\to\infty} V(y) \geq V(x)$ for all $x \in \mathbb{R}^N$;

(F1) $f \in C(\mathbb{R}, \mathbb{R})$, $\lim_{|t|\to 0} f(t)/t = 0$ and $\lim_{|t|\to\infty} |f(t)|/|t|^{2\cdot 2^*-1} = 0$, where $2 \cdot 2^*$ is the critical exponent for (1.1);

(F2) $\lim_{|t|\to\infty} F(t)/|t|^2 = \infty$, where $F(t) = \int_0^t f(s)\,ds$.

This type of equation was introduced in [1, 7] to study a model of self-trapped electrons in quadratic or hexagonal lattices. After the work of Poppenberg [11], equations like (1.1) have received much attention in mathematical analysis and its applications.

Let $X = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} u^2|\nabla u|^2\,dx < +\infty\} = \{u \in H^1(\mathbb{R}^N) : u^2 \in H^1(\mathbb{R}^N)\}$. Define the energy functional

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} [|\nabla u|^2 + V(x)u^2 + u^2|\nabla u|^2]\,dx - \int_{\mathbb{R}^N} F(u)\,dx \quad \text{for } u \in X. \tag{1.2}$$

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Although X is not a vector space (it is not closed under the sum), it is a complete metric space with distance $d_X(u, v) = \|u - v\| + \|\nabla u^2 - \nabla v^2\|_2$. It is easy to check that I is continuous on X. For any $\varphi \in C_0^\infty(\mathbb{R}^N)$, $u \in X$ and $u + \varphi \in X$, we can compute the Gâteaux derivative

$$\langle I'(u), \varphi\rangle = \int_{\mathbb{R}^N} [(1 + u^2)\nabla u \cdot \nabla\varphi + |\nabla u|^2 u\varphi + V(x)u\varphi - f(u)\varphi]\, dx. \tag{1.3}$$

As in [10, 12], $u \in X$ is a solution of (1.1) if and only if the Gâteaux derivative of I along any direction in $C_0^\infty(\mathbb{R}^N)$ vanishes. A solution is called a ground state solution if its energy is minimal among all nontrivial solutions.

Since the term $\int_{\mathbb{R}^N} u^2|\nabla u|^2\, dx$ is not convex and X is not even a vector space, the usual minimax techniques cannot be applied directly to I. To overcome this difficulty, the authors in [6, 9] introduced a new variable replacement $v = h^{-1}(u)$ and transformed (1.1) into a related semilinear problem

$$-\triangle v + V(x)h(v)h'(v) = f(h(v))h'(v), \quad x \in \mathbb{R}^N,$$

where $h'(t) = 1/\sqrt{1 + 2|h(t)|^2}$ on $[0, +\infty)$ and $h(-t) = -h(t)$ on $(-\infty, 0]$ and this idea has been used extensively. A typical way to deal with this semilinear problem is to use the mountain-pass theorem. For this purpose, one usually assumes that f is superlinear at $t = 0$ and super-cubic at $t = \infty$ and satisfies the Ambrosetti–Rabinowitz-type condition

(AR) there exists $\mu \geq 4$ such that $f(t)t \geq \mu F(t) \geq 0$ for all $t \in \mathbb{R}$.

The condition (AR) plays a crucial role in getting a bounded Palais–Smale sequence.

If f further satisfies

(MN) $f(t)/|t|^3$ is nondecreasing for $t \in \mathbb{R} \setminus \{0\}$,

after the change of variables, the authors in [18, 19] found ground state solutions of (1.1) by the Nehari technique. Without any change of variables, Liu *et al.* [10] found ground state solutions of (1.1) with special form $f(t) = |t|^{p-1}t$ for $4 \leq p < 2 \cdot 2^*$ by using a minimisation on a Nehari-type constraint for I. These methods do not work for (1.1) in case $f(t) = |t|^{p-1}t$ with $2 \leq p < 4$ due to the competing effect between $\triangle(u^2)u$ and $f(u)$. To overcome this difficulty, Ruiz and Siciliano [12] introduced a constrained minimisation procedure for I, using a constraint related to the Pohožaev identity, and proved for the first time that (1.1) has a ground state solution when $f(t) = |t|^{p-1}t$ with $2 \leq p < 4$ and V satisfies (V1) and

(V2′) $V \in C^1(\mathbb{R}^N)$, $\inf_{\mathbb{R}^N} V > 0$ and $t \mapsto t^{(N+2)/(N+p)}V(t^{1/(N+p)}x)$ is concave on $(0, \infty)$ for any $x \in \mathbb{R}^N$.

Recently, Wu and Wu [17] obtained a similar result by using the change of variables, Jeanjean's monotonicity trick [8] and a Pohožaev-type identity, where V satisfies (V1) and

(V2″) $V \in C^1(\mathbb{R}^N)$, $\inf_{\mathbb{R}^N} V > 0$, $V(x) = V(|x|)$ and $t^{3-p}\nabla V(tx) \cdot x$ is nonincreasing on $t \in (0, \infty)$ for any $x \in \mathbb{R}^N$.

The strategies used in [12, 17] rely heavily on the condition $\inf_{\mathbb{R}^N} V > 0$ and the algebraic form $f(t) = |t|^{p-2}t$ (see [12, Proposition 3.3 and the proof of Theorem 2.1] and [17, Lemma 2.6]).

Motivated by [12, 17], we shall establish the existence of ground state solutions for (1.1) with more general super-linear nonlinearities and give a minimax characterisation of the ground state energy. To state our results, we introduce the following new assumptions modelled on the approach taken in [12–14]:

(F3) there exists $p > 2$ such that $(f(t)t + NF(t))/|t|^{p-1}t$ is nondecreasing on both $(-\infty, 0)$ and $(0, \infty)$;

(V2) $V \in C^1(\mathbb{R}^N)$ and there exists $\theta \in [0, 1)$ such that

$$t \mapsto \frac{(N+2)V(tx) + \nabla V(tx)\cdot(tx)}{t^{p-2}} + \frac{\theta N(N-2)^2}{8t^p|x|^2}$$

is nonincreasing on $(0, \infty)$ for any $x \in \mathbb{R}^N$, where $p > 2$ is given by (F3).

Similarly to [10, 12], we define the Pohožaev functional of (1.1) by

$$\begin{aligned}\mathcal{P}(u) := \frac{N-2}{2}\|\nabla u\|_2^2 + \frac{N-2}{2}\int_{\mathbb{R}^N} u^2|\nabla u|^2\,dx \\ + \frac{1}{2}\int_{\mathbb{R}^N}[NV(x) + \nabla V(x)\cdot x]\,u^2\,dx - N\int_{\mathbb{R}^N} F(u)\,dx. \qquad (1.4)\end{aligned}$$

It is well known that any solution u of (1.1) satisfies $\mathcal{P}(u) = 0$. Motivated by this fact, we define the functional $J(u) := \langle I'(u), u\rangle + \mathcal{P}(u)$ for all $u \in X$ and define a constraint manifold of Pohožaev–Nehari type $\mathcal{M} := \{u \in X \setminus \{0\} : J(u) = 0\}$. Then every nontrivial solution of (1.1) is contained in $\mathcal{M}$. The assumptions (F3) and (V2) will play a crucial role in proving that $m := \inf_{\mathcal{M}} I$ is achieved and the minimiser of m is a critical point of I. We can now state our main result.

THEOREM 1.1. *Assume* (V1), (V2) *and* (F1)–(F3) *hold. Then problem (1.1) possesses a ground state solution* $\bar{u} \in X$ *such that* $I(\bar{u}) = \inf_{\mathcal{M}} I = \inf_{u\in X\setminus\{0\}} \max_{t>0} I(tu_t) > 0$, *where* $u_t(x) := u(x/t)$.

REMARK 1.2. Note that (V2′) or (V2″) implies (V2) with $\theta = 0$, and $f(u) = |u|^{p-2}u$ with $2 < p < 4$ satisfies (F1)–(F3). Moreover, $\inf_{\mathbb{R}^N} V > 0$ is not required in (V2). From this point of view, our result generalises and improves the results in [12, 17] and extends some other results of a similar type.

Since the approaches used in [12, 17] are not applicable to (1.1) with more general functions f, some new ideas and more careful analytical techniques are introduced to prove Theorem 1.1. More precisely, by using some new techniques and inequalities related to $I(u), I(tu_t)$ and $J(u)$, we prove that a minimising sequence $\{u_n\} \subset X$ of $\inf_{\mathcal{M}} I$ weakly converges to some nontrivial $\hat{u}$ in X (after a translation and extraction of a subsequence) and $\hat{t}\hat{u}_{\hat{t}} \in \mathcal{M}$ is a minimiser of $\inf_{\mathcal{M}} I$ for some $\hat{t} > 0$ provided

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$m = \inf_{\mathcal{M}} I \leq m^\infty := \inf_{\mathcal{M}^\infty} I^\infty$. Then, following [12, pages 1231–1232], we obtain $I'(\hat{t}\hat{u}) = 0$, where

$$I^\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left[|\nabla u|^2 + V_\infty u^2 + u^2|\nabla u|^2\right] dx - \int_{\mathbb{R}^N} F(u)\, dx \quad \text{for all } u \in X \tag{1.5}$$

and $\mathcal{M}^\infty = \{u \in X \setminus \{0\} : J^\infty(u) = 0\}$ with $J^\infty(u) = (d/dt)I^\infty(tu_t)|_{t=1}$.

Remark 1.3. Our approach is totally different from that used in [17] and can be used to treat more general quasilinear problems because it does not depend on any change of variables and compactness of the Sobolev embedding.

Throughout this paper, we denote the usual norms of $L^s(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$ by $\|\cdot\|_s$ and $\|\cdot\|$, respectively.

2. Preliminaries

Lemma 2.1. *Assume that* (V1), (V2), (F1) *and* (F3) *hold. For* $t \geq 0, x \in \mathbb{R}^N \setminus \{0\}$ *and* $\tau \in \mathbb{R}$,

$$\begin{aligned} h_1(t,x) := V(x) - t^{N+2}V(tx) - \frac{1-t^{N+p}}{N+p}[(N+2)V(x) + \nabla V(x)\cdot x] \\ + \frac{\theta(N-2)^2}{4(N+p)|x|^2}[Nt^{N+p} - (N+p)t^N + p] \geq 0, \end{aligned} \tag{2.1}$$

$$h_2(t,\tau) := t^N F(t\tau) - F(\tau) + \frac{1-t^{N+p}}{N+p}[f(\tau)\tau + NF(\tau)] \geq 0. \tag{2.2}$$

Proof. For $x \in \mathbb{R}^N \setminus \{0\}$, by (V2),

$$\begin{aligned} \frac{d}{dt}h_1(t,x) = t^{N+p-1}\bigg\{\bigg[(N+2)V(x) + \nabla V(x)\cdot x + \frac{\theta N(N-2)^2}{4|x|^2}\bigg] \\ - \bigg[\frac{(N+2)V(tx) + \nabla V(tx)\cdot(tx)}{t^{p-2}} + \frac{\theta N(N-2)^2}{4t^p|x|^2}\bigg]\bigg\} \end{aligned}$$

and this expression is greater than or equal to zero for $t \geq 1$ and less than or equal to zero for $0 < t < 1$. Together with the continuity of $h_1(\cdot, x)$, this yields (2.1). For $\tau \in \mathbb{R} \setminus \{0\}$, by (F4),

$$\frac{d}{dt}h_2(t,\tau) = t^{N+p-1}|\tau|^p\bigg[\frac{f(t\tau)t\tau + NF(t\tau)}{|t\tau|^p} - \frac{f(\tau)\tau + NF(\tau)}{|\tau|^p}\bigg]$$

and this expression is greater than or equal to zero for $t \geq 1$ and less than or equal to zero for $0 < t < 1$. Together with the continuity of $h_2(\cdot, \tau)$, this yields (2.2). □

For $t \geq 0$, let

$$\beta_1(t) := Nt^{N+p} - (N+p)t^N + p, \tag{2.3}$$

$$\beta_2(t) := (N+2)t^{N+p} - (N+p)t^N + p - 2. \tag{2.4}$$

Since $p > 2$, for all $t \in [0,1) \cup (1,\infty)$,

$$\beta_1(t) > \beta_1(1) = 0, \quad \beta_2(t) > \beta_2(1) = 0. \tag{2.5}$$

Lemma 2.2. *Assume that* (V1), (V2), (F1) *and* (F3) *hold. Then, for all* $u \in H^1(\mathbb{R}^N)$ *and* $t > 0$,

$$I(u) \geq I(tu_t) + \frac{1 - t^{N+p}}{N + p} J(u) + \frac{(1 - \theta)\beta_1(t)}{2(N + p)} \|\nabla u\|_2^2 + \frac{\beta_2(t)}{2(N + p)} \int_{\mathbb{R}^N} u^2 |\nabla u|^2 \, dx. \quad (2.6)$$

Proof. Note that

$$I(tu_t) = \frac{t^N}{2} \|\nabla u\|_2^2 + \frac{t^{N+2}}{2} \int_{\mathbb{R}^N} V(tx) u^2 \, dx + \frac{t^{N+2}}{2} \int_{\mathbb{R}^N} u^2 |\nabla u|^2 \, dx - t^N \int_{\mathbb{R}^N} F(tu) \, dx. \quad (2.7)$$

Since $J(u) = \langle I'(u), u\rangle + \mathcal{P}(u)$ for $u \in X$, (1.3) and (1.4) imply that

$$J(u) = \frac{N}{2} \|\nabla u\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^N} [(N + 2)V(x) + \nabla V(x) \cdot x] u^2 \, dx + \frac{N + 2}{2} \int_{\mathbb{R}^N} u^2 |\nabla u|^2 \, dx - \int_{\mathbb{R}^N} [f(u)u + NF(u)] \, dx. \quad (2.8)$$

By Hardy's inequality, for $u \in H^1(\mathbb{R}^N)$,

$$\|\nabla u\|_2^2 \geq \frac{(N - 2)^2}{4} \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} \, dx. \quad (2.9)$$

Thus, by (1.2), (2.1)–(2.5) and (2.7)–(2.9),

$$\begin{aligned}
I(u) - I(tu_t) &= \frac{1 - t^N}{2} \|\nabla u\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^N} \left[V(x) - t^{N+2} V(tx)\right] u^2 \, dx \\
&\quad + \frac{1 - t^{N+2}}{2} \int_{\mathbb{R}^N} u^2 |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} [t^N F(tu) - F(u)] \, dx \\
&= \frac{1 - t^{N+p}}{N + p} J(u) + \frac{\beta_2(t)}{2(N + p)} \int_{\mathbb{R}^N} u^2 |\nabla u|^2 \, dx + \frac{\beta_1(t)}{2(N + p)} \|\nabla u\|_2^2 \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^N} \Big\{ V(x) - t^{N+2} V(tx) \\
&\quad - \frac{1 - t^{N+p}}{N + p} [(N + 2)V(x) + \nabla V(x) \cdot x] \Big\} u^2 \, dx \\
&\quad + \int_{\mathbb{R}^N} \Big\{ t^N F(tu) - F(u) + \frac{1 - t^{N+p}}{N + p} [f(u)u + NF(u)] \Big\} \, dx \\
&\geq \frac{1 - t^{N+p}}{N + p} J(u) + \frac{(1 - \theta)\beta_1(t)}{2(N + p)} \|\nabla u\|_2^2 + \frac{\beta_2(t)}{2(N + p)} \int_{\mathbb{R}^N} u^2 |\nabla u|^2 \, dx
\end{aligned}$$

for all $u \in H^1(\mathbb{R}^N)$ and $t > 0$. This shows that (2.6) holds. □

From Lemma 2.2, we have the following corollary.

Corollary 2.3. *Assume that* (V1), (V2), (F1) *and* (F3) *hold. Then, for* $u \in \mathcal{M}$,

$$I(u) = \max_{t>0} I(tu_t).$$

Lemma 2.4. *Assume that* (V1), (V2) *and* (F1)–(F3) *hold. Then, for any* $u \in X \setminus \{0\}$, *there exists a unique* $t_u u_{t_u} > 0$ *such that* $t_u u_{t_u} \in \mathcal{M}$.

Proof. Inspired by [2–5, 13, 15], we let $u \in X \setminus \{0\}$ be fixed and define the function $\zeta(t) := I(tu_t)$ on $(0, \infty)$. Clearly, by (2.8) and (2.7),

$$\zeta'(t) = 0 \Leftrightarrow \frac{N}{2} t^{N-1} \|\nabla u\|_2^2 + \frac{t^{N+1}}{2} \int_{\mathbb{R}^N} [(N+2)V(tx) + \nabla V(tx) \cdot (tx)] u^2 \, dx \\ - t^{N-1} \int_{\mathbb{R}^N} [f(tu)tu + NF(tu)] \, dx = 0 \\ \Leftrightarrow J(tu_t) = 0 \Leftrightarrow tu_t \in \mathcal{M}.$$

It is easy to verify, using (V1), (V2), (F1) and (2.7), that $\lim_{t\to 0} \zeta(t) = 0$, $\zeta(t) > 0$ for $t > 0$ small and $\zeta(t) < 0$ for t large. Therefore, $\max_{t\in(0,\infty)} \zeta(t)$ is achieved at some $t_u > 0$, so that $\zeta'(t_u) = 0$ and $t_u u_{t_u} \in \mathcal{M}$.

Next we claim that t_u is unique for any $u \in X \setminus \{0\}$. In fact, for some $u \in X \setminus \{0\}$, if there exist two positive constants $t_1 \neq t_2$ such that both $t_1 u_{t_1}, t_2 u_{t_2} \in \mathcal{M}$, that is, $J(t_1 u_{t_1}) = J(t_2 u_{t_2}) = 0$, then (2.5) and (2.6) imply that

$$I(t_1 u_{t_1}) > I(t_2 u_{t_2}) + \frac{t_1^{N+p} - t_2^{N+p}}{(N+p) t_1^{N+p}} J(t_1 u_{t_1}) = I(t_2 u_{t_2}) \\ > I(t_1 u_{t_1}) + \frac{t_2^{N+p} - t_1^{N+p}}{(N+p) t_2^{N+p}} J(t_2 u_{t_2}) = I(t_1 u_{t_1}).$$

This contradiction shows that $t_u > 0$ is unique for any $u \in X \setminus \{0\}$. □

Lemma 2.5. *Assume that* (V1) *and* (V2) *hold. Then there exists* $\gamma_1 > 0$ *such that, for all* $u \in H^1(\mathbb{R}^N)$,

$$N\|\nabla u\|_2^2 + \int_{\mathbb{R}^N} [(N+2)V(x) + \nabla V(x) \cdot x] u^2 \, dx \geq \gamma_1 \|u\|^2. \tag{2.10}$$

Proof. Arguing by contradiction, suppose that there exists a sequence $\{u_n\} \subset H^1(\mathbb{R}^N)$ such that

$$\|u_n\| = 1, \quad N\|\nabla u_n\|_2^2 + \int_{\mathbb{R}^N} [(N+2)V(x) + \nabla V(x) \cdot x] \, u_n^2 \, dx = o(1). \tag{2.11}$$

Then there exists $\bar{u} \in H^1(\mathbb{R}^N)$ such that, going to a subsequence, $u_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^N)$, $u_n \to \bar{u}$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$ for $2 \leq s < 2^*$ and $u_n \to \bar{u}$ almost everywhere in $\mathbb{R}^N$. Let $t \to +\infty$ in (2.1). Then

$$(N+2)V(x) + \nabla V(x) \cdot x + \frac{\theta N(N-2)^2}{4|x|^2} \geq 0. \tag{2.12}$$

From (2.9), (2.11) and (2.12), the weak semicontinuity of the norm and Fatou's lemma,

$$0 = \lim_{n\to\infty} \left[N\|\nabla u_n\|_2^2 + \int_{\mathbb{R}^N} [(N+2)V(x) + \nabla V(x) \cdot x] u_n^2 \, dx \right] \\ \geq (1-\theta) N \|\nabla \bar{u}\|_2^2 + \int_{\mathbb{R}^N} \left[(N+2)V(x) + \nabla V(x) \cdot x + \frac{\theta N(N-2)^2}{4|x|^2} \right] \bar{u}^2 \, dx,$$

which implies that $\bar{u} = 0$. Using (V1) and (V2), it is easy to check that

$$\int_{\mathbb{R}^N} \{(N+2)[V(x) - V_\infty] + \nabla V(x) \cdot x\} u_n^2 \, dx = o(1). \tag{2.13}$$

Together (2.11) and (2.13) imply that

$$\begin{aligned} o(1) &= N\|\nabla u_n\|_2^2 + \int_{\mathbb{R}^N} [(N+2)V(x) + \nabla V(x) \cdot x] \, u_n^2 \, dx \\ &= N\|\nabla u_n\|_2^2 + (N+2)V_\infty \|u_n\|_2^2 + o(1) \\ &\geq \min\{N, (N+2)V_\infty\}\|u_n\|^2 + o(1) \\ &= \min\{N, (N+2)V_\infty\} + o(1). \end{aligned}$$

This contradiction shows that there exists $\gamma_1 > 0$ such that (2.10) holds. □

LEMMA 2.6. *Under the assumptions of Theorem 1.1,*

$$\inf_{u \in \mathcal{M}} I(u) = m = \inf_{u \in H^1(\mathbb{R}^N)\setminus\{0\}} \max_{t>0} I(tu_t) > 0.$$

PROOF. From Corollary 2.3 and Lemma 2.4,

$$\mathcal{M} \neq \emptyset \quad \text{and} \quad m = \inf_{u \in H^1(\mathbb{R}^N)\setminus\{0\}} \max_{t>0} I(tu_t).$$

Next we prove that $m > 0$. Since $J(u) = 0$ for $u \in \mathcal{M}$, it follows from (F1), (2.8), (2.10) and the Sobolev embedding inequality that

$$\begin{aligned} &\gamma_1 \|u\|^2 + (N+2) \int_{\mathbb{R}^N} |\nabla u|^2 u^2 \, dx \\ &\quad \leq N\|\nabla u\|_2^2 + \int_{\mathbb{R}^N} [(N+2)V(x) + \nabla V(x) \cdot x] u^2 \, dx + (N+2) \int_{\mathbb{R}^N} |\nabla u|^2 u^2 \, dx \\ &\quad = 2 \int_{\mathbb{R}^N} [f(u)u + NF(u)] \, dx \qquad (2.14) \\ &\quad \leq \frac{\gamma_1}{2} \|u\|^2 + C_1 S^{-2^*/2} \Big(\int_{\mathbb{R}^N} |\nabla u|^2 u^2 \, dx \Big)^{2^*} \qquad (2.15) \end{aligned}$$

for $u \in \mathcal{M}$. This implies that there exists $\rho_0 > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla u|^2 u^2 \, dx \geq \rho_0 \quad \text{for } u \in \mathcal{M}. \tag{2.16}$$

From (2.6) with $t \to 0$,

$$I(u) - \frac{1}{N+p} J(u) \geq \frac{(1-\theta)p}{2(N+p)} \|\nabla u\|_2^2 + \frac{p-2}{2(N+p)} \int_{\mathbb{R}^N} u^2 |\nabla u|^2 \, dx. \tag{2.17}$$

Combining (2.16) with (2.17) yields $m = \inf_{\mathcal{M}} I > 0$. □

Following [12, pages 1231–1232], we can obtain the following lemma.

LEMMA 2.7. *Under the assumptions of Theorem 1.1, if $\bar{u} \in \mathcal{M}$ and $I(\bar{u}) = m$, then $\bar{u}$ is a critical point of I.*

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3. Proof of Theorem 1.1

LEMMA 3.1. *Under the assumptions of Theorem 1.1, $m^\infty \geq m$.*

PROOF. Note that the conclusions in Section 2 on I and J hold for I^∞ and J^∞, where

$$J^\infty(u) = \frac{N}{2}\|\nabla u\|_2^2 + \frac{N+2}{2}\int_{\mathbb{R}^N} V_\infty u^2\,dx + \frac{N+2}{2}\int_{\mathbb{R}^N} u^2|\nabla u|^2\,dx - \int_{\mathbb{R}^N}[f(u)u + NF(u)]\,dx. \tag{3.1}$$

By Lemma 2.5, $\mathcal{M}^\infty \neq \emptyset$. Arguing by contradiction, we assume that $m > m^\infty$. Let $\varepsilon := m - m^\infty$. Then there exists u_ε^∞ such that

$$u_\varepsilon^\infty \in \mathcal{M}^\infty \quad \text{and} \quad m^\infty + \frac{\varepsilon}{2} > I^\infty(u_\varepsilon^\infty). \tag{3.2}$$

In view of Lemma 2.5, there exists $t_\varepsilon > 0$ such that $t_\varepsilon(u_\varepsilon^\infty)_{t_\varepsilon} \in \mathcal{M}$. Since $V_\infty \geq V(x)$ for all $x \in \mathbb{R}^N$, it follows from (1.2), (1.5), (3.2) and Corollary 2.3 that

$$m^\infty + \frac{\varepsilon}{2} > I^\infty(u_\varepsilon^\infty) \geq I^\infty(t_\varepsilon(u_\varepsilon^\infty)_{t_\varepsilon}) \geq I(t_\varepsilon(u_\varepsilon^\infty)_{t_\varepsilon}) \geq m = m^\infty + \varepsilon.$$

This contradiction shows that $m^\infty \geq m$. □

LEMMA 3.2. *Under the assumptions of Theorem 1.1, the infimum m is achieved.*

PROOF. Let $\{u_n\} \subset \mathcal{M}$ be such that $I(u_n) \to m$. By (2.17), $\{\|\nabla u_n\|_2\}$ and $\{\|\nabla(u_n^2)\|_2\}$ are bounded. Passing to a subsequence, we have $u_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^N)$, $u_n^2 \rightharpoonup \bar{u}^2$ in $H^1(\mathbb{R}^N)$, $u_n \to \bar{u}$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$ for $2 \leq s < 2 \cdot 2^*$ and $u_n \to \bar{u}$ almost everywhere in $\mathbb{R}^N$. There are two possible cases: (i) $\bar{u} = 0$ and (ii) $\bar{u} \neq 0$.

Case (i). $\bar{u} = 0$, that is, $u_n \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$ and $u_n^2 \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$. Then $u_n \to 0$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$ for $2 \leq s < 2 \cdot 2^*$ and $u_n \to 0$ almost everywhere in $\mathbb{R}^N$. Using (V1), it is easy to show that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}[V_\infty - V(x)]u_n^2\,dx = \lim_{n\to\infty}\int_{\mathbb{R}^N}\nabla V(x)\cdot x u_n^2\,dx = 0. \tag{3.3}$$

From (1.2), (1.5), (2.8) and (3.3),

$$I^\infty(u_n) \to m, \qquad J^\infty(u_n) \to 0.$$

By (F1), (2.14) and (2.16),

$$\gamma_1\|u_n\|^2 + (N+2)\rho_0 \leq 2\int_{\mathbb{R}^N}[f(u_n)u_n + NF(u_n)]\,dx \leq \varepsilon(\|u_n\|^2 + \|u_n^2\|_{2^*}^{2^*}) + C_\varepsilon\|u_n\|_q^q \tag{3.4}$$

for some $q \in (2, 2\cdot 2^*)$. By (3.4) and Lions' concentration compactness principle [16, Lemma 1.21], there exist $\delta > 0$ and $\{y_n\} \subset \mathbb{R}^N$ such that $\int_{B_1(y_n)} |u_n|^2\,dx > \delta$. Let $\hat{u}_n(x) = u_n(x + y_n)$; then

$$J^\infty(\hat{u}_n) = o(1), \quad I^\infty(\hat{u}_n) \to m \quad \text{and} \quad \int_{B_1(0)} |\hat{u}_n|^2\,dx > \delta. \tag{3.5}$$

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Therefore, there exists $\hat{u} \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that, passing to a subsequence,

$$\begin{cases} \hat{u}_n \rightharpoonup \hat{u} & \text{in } H^1(\mathbb{R}^N) \quad \text{and} \quad \hat{u}_n^2 \rightharpoonup \hat{u}^2, \text{ in } H^1(\mathbb{R}^N); \\ \hat{u}_n \to \hat{u} & \text{in } L^s_{\text{loc}}(\mathbb{R}^N), \text{ for all } s \in [1, 2 \cdot 2^*); \\ \hat{u}_n \to \hat{u} & \text{almost everywhere on } \mathbb{R}^N. \end{cases} \tag{3.6}$$

Let $w_n = \hat{u}_n - \hat{u}$. Then, from (3.6) and the Brezis–Lieb lemma (see [14, Lemma 2.7], [12, (12)] and [16]),

$$I^\infty(\hat{u}_n) = I^\infty(\hat{u}) + I^\infty(w_n) + o(1), \qquad J^\infty(\hat{u}_n) = J^\infty(\hat{u}) + J^\infty(w_n) + o(1). \tag{3.7}$$

By (1.5), (3.1), (3.5) and (3.7),

$$J^\infty(w_n) = -J^\infty(\hat{u}) + o(1), \tag{3.8}$$

$$I^\infty(w_n) - \frac{1}{N+p} J^\infty(w_n) = m - \Big[I^\infty(\hat{u}) - \frac{1}{N+p} J^\infty(\hat{u})\Big] + o(1). \tag{3.9}$$

If there exists a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ such that $w_{n_i} = 0$, then

$$I^\infty(\hat{u}) = m, \quad J^\infty(\hat{u}) = 0 \tag{3.10}$$

and so the proof is completed. Next we assume that $w_n \neq 0$. We claim that $J^\infty(\hat{u}) \leq 0$. Otherwise, if $J^\infty(\hat{u}) > 0$, then (3.8) implies that $J^\infty(w_n) < 0$ for large n. Applying 2.4 to I^∞, there exists $t_n > 0$ such that $t_n(w_n)_{t_n} \in \mathcal{M}^\infty$ for large n. Applying Lemma 2.2 to I^∞, from (1.5), (3.1), (3.9) and Lemma 3.1,

$$\begin{aligned} m - \Big[I^\infty(\hat{u}) - \frac{1}{N+p} J^\infty(\hat{u})\Big] + o(1) &= I^\infty(w_n) - \frac{1}{N+p} J^\infty(w_n) \\ &\geq I^\infty\left(t_n(w_n)_{t_n}\right) - \frac{t_n^{N+p}}{N+p} J^\infty(w_n) \geq m^\infty \geq m, \end{aligned}$$

which is a contradiction because $I^\infty(\hat{u}) - J^\infty(\hat{u})/(N+p) > 0$ by (2.17). This shows that $J^\infty(\hat{u}) \leq 0$. From Lemma 2.2, there exists $t_\infty > 0$ such that $t_\infty \hat{u}_{t_\infty} \in \mathcal{M}^\infty$. Moreover, it follows from (1.5), (3.1), (2.6), (3.5), (3.6), Fatou's lemma and Lemma 3.1 that

$$\begin{aligned} m &= \lim_{n\to\infty} \Big[I^\infty(\hat{u}_n) - \frac{1}{N+p} J^\infty(\hat{u}_n)\Big] \geq I^\infty(\hat{u}) - \frac{1}{N+p} J^\infty(\hat{u}) \\ &\geq I^\infty\left(t_\infty \hat{u}_{t_\infty}\right) - \frac{t_\infty^{N+p}}{N+p} J^\infty(\hat{u}) \geq m^\infty \geq m, \end{aligned}$$

which implies that (3.10) holds. In view of Lemma 2.4, there exists $\hat{t} > 0$ such that $\hat{t}\hat{u}_{\hat{t}} \in \mathcal{M}$. Applying Corollary 2.3 to I^∞, we deduce from (V1), (1.2), (1.5) and (3.10) that

$$m \leq I(\hat{t}\hat{u}_{\hat{t}}) \leq I^\infty(\hat{t}\hat{u}_{\hat{t}}) \leq I^\infty(\hat{u}) = m.$$

This shows that m is achieved at $\hat{t}\hat{u}_{\hat{t}} \in \mathcal{M}$.

Case (ii): $\bar{u} \neq 0$. In this case, analogous to the proof of (3.10), by using I and J instead of I^∞ and J^∞, we can deduce that $I(\bar{u}) = m$ and $J(\bar{u}) = 0$. □

Proof of Theorem 1.1. In view of Lemmas 2.7 and 3.2, there exists $\bar{u} \in \mathcal{M}$ such that $I'(\bar{u}) = 0$ and $I(\bar{u}) = m = \inf_{u \in X\setminus\{0\}} \max_{t>0} I(tu_t)$. This completes the proof. □

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