SEMIGROUP STRUCTURES FOR FAMILIES OF FUNCTIONS, I

SOME HOMOMORPHISM THEOREMS

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1. Introduction

This is the first of several papers which grew out of an attempt to provide $C(X, Y)$, the family of all continuous functions mapping a topological space X into a topological space Y, with an algebraic structure. In the event Y has an algebraic structure with which the topological structure is compatible, pointwise operations can be defined on $C(X, Y)$. Indeed, this has been done and has proved extremely fruitful, especially in the case of the ring $C(X, R)$ of all continuous, real-valued functions defined on X [**3**]. Now, one can provide $C(X, Y)$ with an algebraic structure even in the absence of an algebraic structure on Y. In fact, each continuous function from Y into X determines, in a natural way, a semigroup structure for $C(X, Y)$. To see this, let $\mathfrak{f}$ be any continuous function from Y into X and for f and g in $C(X, Y)$, define fg by

$$(fg)(x) = f(\mathfrak{f}(g(x))) \text{ for each } x \text{ in } X.$$

In a similar manner, one can provide semigroup structures for other families of functions on topological spaces, and this will also be done subsequently.

We note that if $X = Y$ and $\mathfrak{f}$ is the identity function on X, we obtain the semigroup structure on $C(X, X)$ which was studied in [**6**], [**7**], and [**8**]. Consequently, some of the results we obtain here generalize some of those in the latter papers.

Any time an algebraic structure is imposed on $C(X, Y)$, one would want to know to what extent the algebraic structure determines the topological structures of X and Y and conversely, to what extent the topological structures of X and Y determine the algebraic structure imposed on $C(X, Y)$. We begin our investigation of this particular problem in this paper by defining a class of semigroups and considering homomorphisms from one such semigroup into another. We will be working with semigroups of functions whose domains are contained in a given set X and whose

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ranges are contained in a given set Y. Initially, however, X and Y will not have topologies. We will consider applications to semigroups of functions on topological spaces in a later paper.

And now, a few words about some conventions which will be used. If f is a function, the statement $f(x) = y$ is equivalent to the statement $(x, y) \in f$ and for two functions f and g, we define the composition $f \circ g$ by

$$(*) \qquad f \circ g = \{(x, y) : (x, z) \in g \text{ and } (z, y) \in f \text{ for some } z\}.$$

This is the way composition is defined in [4]. In [1], however, xf is written in place of $f(x)$ and composition is defined by

$$f \circ g = \{(x, y) : (x, z) \in f \text{ and } (z, y) \in g \text{ for some } z\}.$$

Our choice of definitions is based on the fact that the results we obtain will eventually be applied to families of functions on topological spaces and in such cases it has been traditional to write operators to the left of the element and define composition as in (*). Finally, for any function f, we define $f^{\leftarrow}$ (introduced in [3]) by

$$f^{\leftarrow} = \{(x, y) : (y, x) \in f\}.$$

$f^{\leftarrow}$, of course, is a function only in the event f is an injection. The symbol f^{-1} is usually used for this but we prefer to reserve the latter symbol for denoting an algebraic inverse.

2. $\mathfrak{S}$-semigroups

Let X and Y denote two non-empty sets and let $\mathscr{S}$ denote a set of functions with domains contained in X and ranges contained in Y. For any function f, we denote the domain of f by $\mathscr{D}(f)$ and the range of f by $\mathscr{R}(f)$. If $\mathscr{D}(f) = \emptyset$, (or equivalently, $\mathscr{R}(f) = \emptyset$) then $f = \emptyset$. We shall use the letter e to denote the empty set when it is to be regarded as a function. Let A be any subset of X and y a point of Y, then A_y denotes the function whose domain is A and which is defined by

$$A_y(x) = y \text{ for each } x \text{ in } A.$$

We refer to such functions as constant functions. Now let $\mathfrak{f}$ be a function with the properties: $\mathscr{D}(\mathfrak{f}) = Y$, $\mathscr{R}(\mathfrak{f}) \subseteqq X$ and $f \circ \mathfrak{f} \circ g$ belongs to $\mathscr{S}$ when both f and g belong to $\mathscr{S}$. Then $\mathscr{S}$ is a semigroup if the product fg of f and g is defined by

$$fg = f \circ \mathfrak{f} \circ g \text{ for all } f \text{ and } g \text{ in } \mathscr{S}.$$

Definition (2.1). $\mathscr{S}$ is referred to as an $\mathfrak{S}$-semigroup and is denoted by $\mathfrak{S}(X, Y, \mathfrak{f})$ if the following two conditions are satisfied.

(2.1.1) $\mathscr{S}$ is point-separating, i.e., for each pair x_1 and x_2 of distinct

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points of X there exists a function f in $\mathscr{S}$ whose domain contains both x_1 and x_2 with the property that $f(x_1) \neq f(x_2)$.

(2.1.2) For each x in X and y in Y, there is a subset A of X containing x such that $A_y \in \mathscr{S}$.

Let y_1 and y_2 be two points of Y. Then (2.1.2) implies that there is a nonempty subset A of X containing $\mathfrak{f}(y_2)$ such that $A_{y_1} \in \mathscr{S}$ and a nonempty subset B of X such that $B_{y_2} \in \mathscr{S}$. Thus $A_{y_1} B_{y_2} = A_{y_1} \circ \mathfrak{f} \circ B_{y_2} = B_{y_1} \in \mathscr{S}$ and we see that (2.1.2) implies that for any two points y_1 and y_2 of Y there exists a nonempty subset B of X such that both B_{y_1} and B_{y_2} belong to $\mathscr{S}$. This fact will be used several times in the proof of Theorem (2.3).

In the following definition, $K(\mathfrak{S}(X, Y, \mathfrak{f}))$ denotes the set of all constant functions which belong to the $\mathfrak{S}$-semigroup, $\mathfrak{S}(X, Y, \mathfrak{f})$.

Definition (2.2). A homomorphism φ from $\mathfrak{S}(X, Y, \mathfrak{f})$ into $\mathfrak{S}(U, V, \mathfrak{g})$ is a K-homomorphism if the following conditions are satisfied.

(2.2.1). φ maps $K(\mathfrak{S}(X, Y, \mathfrak{f}))-\{e\}$ into $K(\mathfrak{S}(U, V, \mathfrak{g}))-\{e\}$ and if e belongs to $\mathfrak{S}(X, Y, \mathfrak{f})$, then e also belongs to $\mathfrak{S}(U, V, \mathfrak{g})$ and $\varphi(e) = e$.

(2.2.2) The image of $\mathfrak{S}(X, Y, \mathfrak{f})$ is point-separating.

(2.2.3) If $\varphi(A_y) = B_z$ and $\varphi(C_y) = D_v$ $(A \neq \emptyset \neq C)$, then $v = z$.

(2.2.4) $\mathscr{D}(\varphi(A_y)) = \mathscr{D}(\varphi(B_z))$ if $A = B$.

Concerning K-homomorphisms, we have the following

Theorem (2.3). *Let φ be a K-homomorphism from $\mathfrak{S}(X, Y, \mathfrak{f})$ into $\mathfrak{S}(U, V, \mathfrak{g})$. Then there exists a mapping $\mathfrak{h}$ from $\mathscr{R}(\mathfrak{f})$ into $\mathscr{R}(\mathfrak{g})$ and a mapping $\mathfrak{t}$ from Y into V such that for each f in $\mathfrak{S}(X, Y, \mathfrak{f})$, the following diagram commutes.*

$$\begin{array}{ccccc} \mathscr{D}(f) \cap \mathscr{R}(\mathfrak{f}) & \xrightarrow{f} & Y & \xrightarrow{\mathfrak{f}} & \mathscr{R}(\mathfrak{f}) \\ \Big\downarrow{\scriptstyle \mathfrak{h}} & & \Big\downarrow{\scriptstyle \mathfrak{t}} & & \Big\downarrow{\scriptstyle \mathfrak{h}} \\ \mathscr{D}(\varphi(f)) \cap \mathscr{R}(\mathfrak{g}) & \xrightarrow{\varphi(f)} & V & \xrightarrow{\mathfrak{g}} & \mathscr{R}(\mathfrak{g}) \end{array}$$

Moreover, if φ maps $K(\mathfrak{S}(X, Y, \mathfrak{f}))$ injectively into $K(\mathfrak{S}(U, V, \mathfrak{g}))$, then both $\mathfrak{h}$ and $\mathfrak{t}$ are injections and if φ maps $K(\mathfrak{S}(X, Y, \mathfrak{f}))$ onto $K(\mathfrak{S}(U, V, \mathfrak{g}))$, then $\mathfrak{t}$ is a surjection onto V and $\mathfrak{h}$ is a surjection onto $\mathscr{R}(\mathfrak{g})$ which maps $\mathscr{D}(f) \cap \mathscr{R}(\mathfrak{f})$ onto $\mathscr{D}(\varphi(f)) \cap \mathscr{R}(\mathfrak{g})$ for each f in $\mathfrak{S}(X, Y, \mathfrak{f})$. Finally, the pair $\mathfrak{h}$ and $\mathfrak{t}$ is unique in the sense that if $\mathfrak{h}^*$ and $\mathfrak{t}^*$ are two mappings from $\mathscr{R}(\mathfrak{f})$ into $\mathscr{R}(\mathfrak{g})$ and Y into V respectively with the property that the resulting diagram commutes when $\mathfrak{h}$ is replaced by $\mathfrak{h}^*$ and $\mathfrak{t}$ by $\mathfrak{t}^*$, then $\mathfrak{h} = \mathfrak{h}^*$ and $\mathfrak{t} = \mathfrak{t}^*$.

Proof. We will first define the function $\mathfrak{t}$. Let y in Y be given. Then by (2.1.2), there exists a nonempty subset A of X such that $A_y \in \mathfrak{S}(X, Y, \mathfrak{f})$.

By (2.2.1), $\varphi(A_y) = B_v$ where $v \in V$ and B is a nonempty subset of U. Define $\mathfrak{t}(y) = v$. Note that t is single-valued because of (2.2.3) and also observe that

$$\varphi(A_y) = B_{\mathfrak{t}(y)}. \tag{2.3.1}$$

Now let y_1 and y_2 be two elements of Y with the property that $\mathfrak{f}(y_1) = \mathfrak{f}(y_2)$. We will show that $\mathfrak{g}(\mathfrak{t}(y_1)) = \mathfrak{g}(\mathfrak{t}(y_2))$. Suppose g is any function in the image of $\mathfrak{S}(X, Y, \mathfrak{f})$ whose domain contains both $\mathfrak{g}(\mathfrak{t}(y_1))$ and $\mathfrak{g}(\mathfrak{t}(y_2))$. Then $\varphi(f) = g$ for some f in $\mathfrak{S}(X, Y, \mathfrak{f})$ and if $\mathfrak{f}(y_1) \in \mathscr{D}(f)$, we have

$$f(\mathfrak{f}(y_1)) = f(\mathfrak{f}(y_2)). \tag{2.3.2}$$

By (2.1.2), there exists a nonempty subset A of X such that A_{y_1} and A_{y_2} both belong to $\mathfrak{S}(X, Y, \mathfrak{f})$. Then (2.3.2) implies

$$f(\mathfrak{f}(A_{y_1}(x))) = f(\mathfrak{f}(A_{y_2}(x))) \tag{2.3.3}$$

for each x in $\mathscr{D}(fA_{y_1}) = \mathscr{D}(fA_{y_2})$. Thus

$$fA_{y_1} = fA_{y_2} \tag{2.3.4}$$

which implies $\varphi(f)\varphi(A_{y_1}) = \varphi(f)\varphi(A_{y_2})$, i.e.,

$$gB_{\mathfrak{t}(y_1)} = gB_{\mathfrak{t}(y_2)} \tag{2.3.5}$$

where B is a nonempty subset of U. Since $\mathfrak{g}(\mathfrak{t}(y_1))$ and $\mathfrak{g}(\mathfrak{t}(y_2))$ both belong to the domain of g, $\mathscr{D}(gB_{\mathfrak{t}(y_1)}) = \mathscr{D}(gB_{\mathfrak{t}(y_2)}) = B$. Then for any point u in B, we have

$$\begin{aligned} g(\mathfrak{g}(\mathfrak{t}(y_1))) = g(\mathfrak{g}(B_{\mathfrak{t}(y_1)}(u))) = (gB_{\mathfrak{t}(y_1)})(u) = \\ (gB_{\mathfrak{t}(y_2)})(u) = g(\mathfrak{g}(B_{\mathfrak{t}(y_2)}(u))) = g(\mathfrak{g}(\mathfrak{t}(y_2))). \end{aligned}$$

This implies $\mathfrak{g}(\mathfrak{t}(y_1)) = \mathfrak{g}(\mathfrak{t}(y_2))$ since the image of $\mathfrak{S}(X, Y, \mathfrak{f})$ is point-separating. Thus we have shown that

(2.3.6) *if* $\mathfrak{f}(y_1) = \mathfrak{f}(y_2)$ *for two points of* Y, *then* $\mathfrak{g}(\mathfrak{t}(y_1)) = \mathfrak{g}(\mathfrak{t}(y_2))$.

Now we define a function $\mathfrak{h}$ from $\mathscr{R}(\mathfrak{f})$ into $\mathscr{R}(\mathfrak{g})$. Let x in $\mathscr{R}(\mathfrak{f})$ be given. Choose y in Y such that $\mathfrak{f}(y) = x$ and let

$$\mathfrak{h}(x) = \mathfrak{g}(\mathfrak{t}(y)). \tag{2.3.7}$$

Because of (2.3.6), $\mathfrak{h}$ is a single-valued function. Observe that for any point y in Y, (2.3.7) implies

$$\mathfrak{h}(\mathfrak{f}(y)) = \mathfrak{g}(\mathfrak{t}(y)). \tag{2.3.8}$$

Now let x be any point of $\mathscr{D}(f) \cap \mathscr{R}(\mathfrak{f})$. We will show that

$$\mathfrak{t}(f(x)) = (\varphi(f))(\mathfrak{h}(x)). \tag{2.3.9}$$

Since $x \in \mathscr{R}(\mathfrak{f})$, there is an element y in Y such that $x = \mathfrak{f}(y)$. By (2.1.2), there is a nonempty subset A of X such that $A_{f(x)}$ belongs to $\mathfrak{S}(X, Y, \mathfrak{f})$. It follows that

$$A_{f(x)} = A_{f(\mathfrak{f}(y))} = f \circ \mathfrak{f} \circ A_y = fA_y.$$

This, in conjunction with (2.2.1) and (2.3.1) implies that for some nonempty subset B of U,

$$B_{\mathfrak{t}(f(x))} = \varphi(A_{f(x)}) = \varphi(fA_y) = \varphi(f)\varphi(A_y) = \varphi(f) \circ \mathfrak{g} \circ C_{\mathfrak{t}(y)}$$

for some (necessarily nonempty) subset C of U. This, together with (2.3.8) implies that for any point p in B,

$$\begin{aligned} \mathfrak{t}(f(x)) &= B_{\mathfrak{t}(f(x))}(p) = \varphi(f)(\mathfrak{g}(C_{\mathfrak{t}(y)}(p))) = \varphi(f)(\mathfrak{g}(\mathfrak{t}(y))) \\ &= \varphi(f)(\mathfrak{h}(\mathfrak{f}(y))) = \varphi(f)(\mathfrak{h}(x)). \end{aligned}$$

Therefore, (2.3.9) is verified and it, together with (2.3.8) implies that the diagram is commutative.

Now suppose φ maps $K(\mathfrak{S}(X, Y, \mathfrak{f}))$ injectively into $K(\mathfrak{S}(U, V, \mathfrak{g}))$. Let y_1 and y_2 be two distinct elements of Y. Then, by (2.1.2) there exists a nonempty subset A of X such that A_{y_1} and A_{y_2} both belong to $\mathfrak{S}(X, Y, \mathfrak{f})$. Since $A_{y_1} \neq A_{y_2}$, $\varphi(A_{y_1}) \neq \varphi(A_{y_2})$. Now $\varphi(A_{y_1}) = B_{v_1}$ and $\varphi(A_{y_2}) = C_{v_2}$ for two nonempty subsets B and C of U and two points v_1 and v_2 of V. Since, by (2.2.4), $B = C$, we must have $v_1 \neq v_2$. But this implies $\mathfrak{t}$ is an injection since $\mathfrak{t}(y_1) = v_1$ and $\mathfrak{t}(y_2) = v_2$.

Now we show $\mathfrak{h}$ is an injection. Let x_1 and x_2 be two distinct points of $\mathscr{R}(\mathfrak{f})$. Then there exists a function f in $\mathfrak{S}(X, Y, \mathfrak{f})$ such that both x_1 and x_2 belong to $\mathscr{D}(f)$ and $f(x_1) \neq f(x_2)$. Choose y_1 and y_2 such that $\mathfrak{f}(y_1) = x_1$ and $\mathfrak{f}(y_2) = x_2$. By (2.1.2), there exists a nonempty subset A of X such that both A_{y_1} and A_{y_2} belong to $\mathfrak{S}(X, Y, \mathfrak{f})$. Then $f \circ \mathfrak{f} \circ A_{y_1} = fA_{y_1}$ and $f \circ \mathfrak{f} \circ A_{y_2} = fA_{y_2}$ are elements of $K(\mathfrak{S}(X, Y, \mathfrak{f}))$ such that $\mathscr{D}(fA_{y_1}) = \mathscr{D}(fA_{y_2}) = A$ and $fA_{y_1} \neq fA_{y_2}$. It follows from (2.2.4) and (2.3.1) that

$$\begin{aligned} &\varphi(f) \circ \mathfrak{g} \circ B_{\mathfrak{t}(y_1)} = \varphi(f)B_{\mathfrak{t}(y_1)} = \\ &\varphi(f)\varphi(A_{y_1}) = \varphi(fA_{y_1}) \neq \varphi(fA_{y_2}) = \\ &\varphi(f)\varphi(A_{y_2}) = \varphi(f) \circ \mathfrak{g} \circ B_{\mathfrak{t}(y_2)} \end{aligned}$$

where B is some nonempty subset of U. Moreover, it follows from (2.2.4) and the fact that both fA_{y_1} and fA_{y_2} belong to $K(\mathfrak{S}(X, Y, \mathfrak{f}))$ that $\mathscr{D}(\varphi(fA_{y_1})) = \mathscr{D}(\varphi(fA_{y_2}))$. Thus, there must exist a point p in $\mathscr{D}(\varphi(fA_{y_1}))$ such that

$$\begin{aligned} &\varphi(f)(\mathfrak{g}(\mathfrak{t}(y_1))) = (\varphi(f) \circ \mathfrak{g} \circ B_{\mathfrak{t}(y_1)})(p) \neq \\ &(\varphi(f) \circ \mathfrak{g} \circ B_{\mathfrak{t}(y_2)})(p) = \varphi(f)(\mathfrak{g}(\mathfrak{t}(y_2))). \end{aligned}$$

Therefore, $\mathfrak{g}(\mathfrak{t}(y_1)) \neq \mathfrak{g}(\mathfrak{t}(y_2))$. Recall from (2.3.7) that $\mathfrak{h}(x_1) = \mathfrak{g}(\mathfrak{t}(y_1))$

and $\mathfrak{h}(x_2) = \mathfrak{g}(\mathfrak{t}(y_2))$. Thus $\mathfrak{h}(x_1) \neq \mathfrak{h}(x_2)$ and we conclude that $\mathfrak{h}$ is an injection.

Now suppose φ maps $K(\mathfrak{S}(X, Y, \mathfrak{f}))$ onto $K(\mathfrak{S}(U, V, \mathfrak{g}))$. For any v in V, there exists a nonempty subset B of U such that B_v belongs to $K(\mathfrak{S}(U, V, \mathfrak{g}))$. Then there exists a constant function A_y such that $\varphi(A_y) = B_v$. Thus, by (2.3.1), $\mathfrak{t}(y) = v$ and we conclude $\mathfrak{t}$ is a surjection onto V. Now let any u in $\mathscr{R}(\mathfrak{g})$ be given. Then $\mathfrak{g}(v) = u$ for some v in V and since $\mathfrak{t}$ is a surjection, $\mathfrak{t}(y) = v$ for some y in Y. Using (2.3.8), we see that

$$\mathfrak{h}(\mathfrak{f}(y)) = \mathfrak{g}(\mathfrak{t}(y)) = \mathfrak{g}(v) = u.$$

Therefore, $\mathfrak{h}$ is a surjection onto $\mathscr{R}(\mathfrak{g})$.

Suppose now f is any element of $\mathfrak{S}(X, Y, \mathfrak{f})$ and u belongs to $\mathscr{D}(\varphi(f)) \cap \mathscr{R}(\mathfrak{g})$. Again $\mathfrak{g}(v) = u$ for some v in V and once more we use the fact that $\mathfrak{t}$ is a surjection to conclude that $\mathfrak{t}(y) = v$ for some y in Y. Then $\mathfrak{f}(y) \in \mathscr{R}(\mathfrak{f})$. Suppose, however,

$$\mathfrak{f}(y) \notin \mathscr{D}(f). \tag{2.3.10}$$

Then there is a nonempty subset A of X such that A_y belongs to $\mathfrak{S}(X, Y, \mathfrak{f})$. Now, $fA_y = e$ (the empty function) and thus, for some nonempty subset B of U, (2.2.1) implies

$$\varphi(f)B_{\mathfrak{t}(y)} = \varphi(f)\varphi(A_y) = \varphi(fA_y) = \varphi(e) = e. \tag{2.3.11}$$

This implies $\mathfrak{g}(\mathfrak{t}(y)) \notin \mathscr{D}(\varphi(f))$ which is a contradiction since $\mathfrak{g}(\mathfrak{t}(y)) = \mathfrak{g}(v) = u$ which belongs to $\mathscr{D}(\varphi(f))$. Therefore, statement (2.3.10) is not valid. That is, $\mathfrak{f}(y) \in \mathscr{D}(f)$ and since, appealing to (2.3.8), we have $\mathfrak{h}(\mathfrak{f}(y)) = \mathfrak{g}(\mathfrak{t}(y)) = u$, we conclude that $\mathfrak{h}$ maps $\mathscr{D}(f) \cap \mathscr{R}(\mathfrak{f})$ onto $\mathscr{D}(\varphi(f)) \cap \mathscr{R}(\mathfrak{g})$.

And now let us show that the mappings $\mathfrak{h}$ and $\mathfrak{t}$ are unique. Let $\mathfrak{h}^*$ and $\mathfrak{t}^*$ be two mappings from $\mathscr{R}(\mathfrak{f})$ into $\mathscr{R}(\mathfrak{g})$ and Y into V respectively such that the resulting diagram commutes when $\mathfrak{h}$ is replaced by $\mathfrak{h}^*$ and $\mathfrak{t}$ by $\mathfrak{t}^*$. Choose any point y in Y and any point x in $\mathscr{R}(\mathfrak{f})$. Then there exists a subset A of X containing x such that A_y belongs to $\mathfrak{S}(X, Y, \mathfrak{f})$. Then from the diagram,

$$\mathfrak{t}(y) = \mathfrak{t}(A_y(x)) = \varphi(A_y)(\mathfrak{h}(x))$$

and from the diagram which results from replacing $\mathfrak{h}$ by $\mathfrak{h}^*$ and $\mathfrak{t}$ by $\mathfrak{t}^*$, we obtain

$$\mathfrak{t}^*(y) = \mathfrak{t}^*(A_y(x)) = \varphi(A_y)(\mathfrak{h}^*(x)).$$

But $\varphi(A_y)$ is a constant function and thus

$$\varphi(A_y)(\mathfrak{h}(x)) = \varphi(A_y)(\mathfrak{h}^*(x)).$$

Therefore, $\mathfrak{t}(y) = \mathfrak{t}^*(y)$.

Now, concerning the point x in $\mathscr{R}(\mathfrak{f})$, we have $\mathfrak{f}(z) = x$ for some z in Y. Therefore,

$$\mathfrak{h}(x) = \mathfrak{h}(\mathfrak{f}(z)) = \mathfrak{g}(\mathfrak{t}(z))$$

and

$$\mathfrak{h}^*(x) = \mathfrak{h}^*(\mathfrak{f}(z)) = \mathfrak{g}(\mathfrak{t}^*(z)).$$

Thus $\mathfrak{h}(x) = \mathfrak{h}^*(x)$ since $\mathfrak{t}(z) = \mathfrak{t}^*(z)$. This completes the proof of the theorem.

It is not difficult to find examples of homomorphisms which are not K-homomorphisms. Recall that for a topological space X, we are using the symbol $C(X, X)$ to denote the semigroup of all continuous functions mapping X into X under the binary operation of composition. Note that such a semigroup is a $\mathfrak{S}$-semigroup. Now let X be a space with a proper subset Y which is both open and closed. Define a mapping from $C(Y, Y)$ into $C(X, X)$ by

$$\varphi(f)(y) = f(y) \text{ for } y \text{ in } Y \text{ and}$$
$$\varphi(f)(y) = y \quad \text{ for } y \text{ in } X-Y.$$

Then φ is a homomorphism but not a K-homomorphism since (2.2.1) is violated, i.e., the constant functions of $C(Y, Y)$ are not mapped into the constant functions of $C(X, X)$.

Definition (2.3). A $\mathfrak{S}$-semigroup $\mathfrak{S}(X, Y, \mathfrak{f})$ is referred to as an $\mathfrak{S}^*$-semigroup and is denoted by $\mathfrak{S}^*(X, Y, \mathfrak{f})$ if $\mathfrak{f}$ is a surjection onto X.

We intend to consider isomorphisms between $\mathfrak{S}^*$-semigroups but it will be convenient to prove some lemmas first.

Lemma (2.4). *Let g be an element of a $\mathfrak{S}^*$-semigroup $\mathfrak{S}^*(X, Y, \mathfrak{f})$. Then g is a constant function if and only if $ghg = g$ or e for each h in $\mathfrak{S}^*(X, Y, \mathfrak{f})$.*

Proof. First suppose g is a constant function. Then $g = A_y$ for some subset A of X and some point y in Y. Let h be any function in $\mathfrak{S}^*(X, Y, \mathfrak{f})$. Then $A_y h A_y = A_y \circ \mathfrak{f} \circ h \circ \mathfrak{f} \circ A_y = A_y$ if $\mathfrak{f}(h(\mathfrak{f}(y)))$ is an element of A and e otherwise.

Now suppose g is not a constant function and let x be any point in the domain of g. Since $\mathfrak{f}$ is a surjection, there exists a point y in Y such that $\mathfrak{f}(y) = x$. By condition (2.1.2), there exists a subset A of X containing $\mathfrak{f}(g(x))$ such that A_y belongs to $\mathfrak{S}^*(X, Y, \mathfrak{f})$. Then $x \in \mathscr{D}(gA_y g)$ which implies $gA_y g \neq e$. On the other hand, $gA_y g$ is a constant function and therefore cannot be equal to g. This completes the proof.

Lemma (2.5). *Let A_x and B_y be two constant functions ($\neq e$) of an $\mathfrak{S}^*$-semigroup $\mathfrak{S}^*(X, Y, \mathfrak{f})$. Then $x = y$ if and only if there exists a constant function C_z in $\mathfrak{S}^*(X, Y, \mathfrak{f})$ such that $A_x C_z B_y = B_y$.*

Proof. Suppose $x = y$ and choose z such that $\mathfrak{f}(z) \in A$. Then there exists a subset C of X containing $\mathfrak{f}(y)$ such that C_z belongs to $\mathfrak{S}^*(X, Y, \mathfrak{f})$.

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Since $\mathfrak{f}(z) \in A$ and $\mathfrak{f}(y) \in C$, it follows that $\mathscr{D}(A_x C_z B_y) = B$. Moreover, for any p in B,

$$(A_x C_z B_y)(p) = A_x\big(\mathfrak{f}(C_z(\mathfrak{f}(B_y(p))))\big) = x = y.$$

Thus $A_x C_z B_y = B_y$.

On the other hand, suppose $x \neq y$. Then for any C_z, $C_z B_y = B_z$ or e. This implies that $A_x C_z B_y = B_x$ or e which, in either event, is not equal to B_y.

LEMMA (2.6). *For f and g in $\mathfrak{S}^*(X, Y, \mathfrak{f})$, $\mathscr{D}(f) = \mathscr{D}(g)$ if and only if for each constant function A_x in $\mathfrak{S}^*(X, Y, \mathfrak{f})$, the following two statements are equivalent.*

(2.6.1) $\qquad B_y f A_x = A_x$ *for some* B_y *in* $\mathfrak{S}^*(X, Y, \mathfrak{f})$.

(2.6.2) $\qquad C_z g A_x = A_x$ *for some* C_z *in* $\mathfrak{S}^*(X, Y, \mathfrak{f})$.

PROOF. Suppose $\mathscr{D}(f) = \mathscr{D}(g)$ and let A_x be given. Furthermore, suppose $B_y f A_x = A_x$ for some B_y. If $A = \emptyset$, it is evident that $B_y g A_x = A_x$. If $A \neq \emptyset$, then $\mathfrak{f}(x) \in \mathscr{D}(f)$ and thus $\mathfrak{f}(x) \in \mathscr{D}(g)$. Then some subset C of X contains $\mathfrak{f}(g(\mathfrak{f}(x)))$ and C_x belongs to $\mathfrak{S}^*(X, Y, \mathfrak{f})$. It follows that $C_x g A_x = A_x$. Thus, (2.6.1) implies (2.6.2) and one proves the reverse implication in a similar manner.

Now we prove sufficiency. Suppose $\mathscr{D}(f) \neq \mathscr{D}(g)$. There will be no loss in generality in assuming there exists a point x in $\mathscr{D}(f) - \mathscr{D}(g)$. Choose y such that $\mathfrak{f}(y) = x$. Then there exists a subset A of X containing $\mathfrak{f}(f(x))$ such that A_y belongs to $\mathfrak{S}^*(X, Y, \mathfrak{f})$. It follows that $A_y f A_y = A_y$. However, $g A_y = e$ since $\mathfrak{f}(y) = x \notin \mathscr{D}(g)$. Thus, $B_z g A_y = e \neq A_y$ for each B_z in $K(\mathfrak{S}^*(X, Y, \mathfrak{f}))$. This completes the proof.

Now we are in a position to characterize isomorphisms between $\mathfrak{S}^*$-semigroups.

THEOREM (2.7). *A bijection φ from a $\mathfrak{S}^*$-semigroup $\mathfrak{S}^*(X, Y, \mathfrak{f})$ onto a $\mathfrak{S}^*$-semigroup $\mathfrak{S}^*(U, V, \mathfrak{g})$ is an isomorphism if and only if there exist bijections $\mathfrak{h}$ and $\mathfrak{t}$ from X onto U and Y onto V respectively such that for each f in $\mathfrak{S}^*(X, Y, \mathfrak{f})$, $\mathfrak{h}$ maps $\mathscr{D}(f)$ bijectively onto $\mathscr{D}(\varphi(f))$ and the following diagram commutes.*

$$\begin{tikzcd}
\mathscr{D}(f) \arrow[r, "f"] \arrow[d, "\mathfrak{h}"'] & Y \arrow[r, "\mathfrak{f}"] \arrow[d, "\mathfrak{t}"'] & X \arrow[d, "\mathfrak{h}"'] \\
\mathscr{D}(\varphi(f)) \arrow[r, "\varphi(f)"] & V \arrow[r, "\mathfrak{g}"] & U
\end{tikzcd}$$

Moreover, the functions $\mathfrak{h}$ and $\mathfrak{t}$ are unique in the sense that if $\mathfrak{h}^*$ and $\mathfrak{t}^*$ are two mappings from X into U and Y into V respectively with the property

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that the resulting diagram commutes when $\mathfrak{h}$ *is replaced by* $\mathfrak{h}^*$ *and* $\mathfrak{t}$ *by* $\mathfrak{t}^*$ *then* $\mathfrak{h} = \mathfrak{h}^*$ *and* $\mathfrak{t} = \mathfrak{t}^*$.

We point out that if one takes $X = Y$, $U = V$ and $\mathfrak{f}$ and $\mathfrak{g}$ to be identity functions, one obtains Theorem (2.4) of [7].

PROOF. Suppose that for each f in $\mathfrak{S}^*(X, Y, \mathfrak{f})$, $\mathfrak{h}$ maps $\mathscr{D}(f)$ bijectively onto $\mathscr{D}(\varphi(f))$ and that the diagram commutes. Then $\varphi(f) = \mathfrak{t} \circ f \circ \mathfrak{h}^{\leftarrow}$ for each f and for any two functions f and g of $\mathfrak{S}^*(X, Y, \mathfrak{f})$, it follows that

$$\varphi(f)\varphi(g) = \varphi(f) \circ \mathfrak{g} \circ \varphi(g) = (\mathfrak{t} \circ f \circ \mathfrak{h}^{\leftarrow}) \circ \mathfrak{g} \circ (\mathfrak{t} \circ g \circ \mathfrak{h}^{\leftarrow}) =$$
$$(\mathfrak{t} \circ f \circ \mathfrak{h}^{\leftarrow}) \circ (\mathfrak{g} \circ \mathfrak{t}) \circ (g \circ \mathfrak{h}^{\leftarrow}) = (\mathfrak{t} \circ f \circ \mathfrak{h}^{\leftarrow}) \circ (\mathfrak{h} \circ \mathfrak{f}) \circ (g \circ \mathfrak{h}^{\leftarrow}) =$$
$$\mathfrak{t} \circ f \circ \mathfrak{f} \circ g \circ \mathfrak{h}^{\leftarrow} = \mathfrak{t} \circ (fg) \circ \mathfrak{h}^{\leftarrow} = \varphi(fg).$$

Thus φ is an isomorphism.

Now suppose φ is any isomorphism from $\mathfrak{S}^*(X, Y, \mathfrak{f})$ onto $\mathfrak{S}^*(U, V, \mathfrak{g})$. We will show that if e belongs to $\mathfrak{S}^*(X, Y, \mathfrak{f})$, then e also belongs to $\mathfrak{S}^*(U, V, \mathfrak{g})$ and $\varphi(e) = e$. This, together with Lemma (2.4), will imply that φ maps $K(\mathfrak{S}^*(X, Y, \mathfrak{f}))$ bijectively onto $K(\mathfrak{S}^*(U, V, \mathfrak{g}))$ and hence that condition (2.2.1) is satisfied. Suppose, then, $e \in \mathfrak{S}^*(X, Y, \mathfrak{f})$. Then $\mathfrak{S}^*(X, Y, \mathfrak{f})$ contains more than one element since otherwise it would not be a $\mathfrak{S}^*$-semigroup. Thus $\mathfrak{S}^*(U, V, \mathfrak{g})$ contains more than one element and, in addition, has a zero $\varphi(e)$. Suppose $\varphi(e) \neq e$. Then some point u belongs to $\mathscr{D}(\varphi(e))$. Choose v in V different from $\varphi(e)(u)$. This can be done since otherwise, (2.1.1) implies that U also has only one point which in turn implies $\mathfrak{S}^*(U, V, \mathfrak{f})$ consists of two elements. This results in a contradiction since $\varphi(e)$ would not be the zero. Now there is a subset A of U containing $\mathfrak{g}(\varphi(e)(u))$ such that A_v belongs to $\mathfrak{S}^*(U, V, \mathfrak{g})$. Then $(A_v(\varphi(e))(u) = A_v(\mathfrak{g}(\varphi(e)(u))) = v \neq \varphi(e)(u)$. Thus $A_v\varphi(e) \neq \varphi(e)$ which is a contradiction since $\varphi(e)$ is the zero of $\mathfrak{S}^*(U, V, \mathfrak{g})$. Thus $\varphi(e) = e$ and (2.2.1) is satisfied. (2.2.2) is satisfied since the image of $\mathfrak{S}^*(X, Y, \mathfrak{f})$ is a $\mathfrak{S}^*$-semigroup. Lemma (2.5) implies that condition (2.2.3) is satisfied. Finally, condition (2.2.4) follows from Lemma (2.6) and we conclude that φ is a K-homomorphism which maps $K(\mathfrak{S}^*(X, Y, \mathfrak{f}))$ bijectively onto $K(\mathfrak{S}^*(U, V, \mathfrak{g}))$. The proof now follows from Theorem (2.3) and the fact that $\mathscr{R}(\mathfrak{f}) = X$ and $\mathscr{R}(\mathfrak{g}) = U$.

Note that in the first portion of the proof the fact that both $\mathfrak{f}$ and $\mathfrak{g}$ are surjections was not used. The theorem, however, cannot be proven if $\mathfrak{f}$ and $\mathfrak{g}$ are assumed merely to be mappings into Y and V respectively or equivalently, if the two semigroups involved are only assumed to be $\mathfrak{S}$-semigroups. Let us consider the following example: let X be any set with more than two elements and choose x_0 in X. Define a function $\mathfrak{f}$ mapping X into X by

$$\mathfrak{f}(x) = x_0 \text{ for each } x \text{ in } X.$$

Let $\mathfrak{S}(X, X, \mathfrak{f})$ denote the $\mathfrak{S}$-semigroup of all functions f with $\mathscr{D}(f) = X$

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and $\mathscr{R}(f) \subseteqq X$. Let i be the identity function on X and choose two nonconstant functions k and l different from i with the properties $k(x_0) = l(x_0)$ and $k(x_1) \neq l(x_1)$ for some x_1 in X. Define a bijection φ on $\mathfrak{S}(X, X, \mathfrak{f})$ by

$$\begin{aligned} \varphi(f) &= f \text{ for } k \neq f \neq l, \\ \varphi(k) &= l \text{ and } \varphi(l) = k. \end{aligned}$$

Notice that for any function f in $\mathfrak{S}(X, X, \mathfrak{f})$, $\varphi(f)(x_0) = f(x_0)$. Then for any f, g in $\mathfrak{S}(X, X, \mathfrak{f})$,

$$\varphi(fg) = \varphi(X_{f(x_0)}) = X_{f(x_0)} = X_{\varphi(f)(x_0)} = \varphi(f)\varphi(g).$$

Thus, φ is an isomorphism. Now suppose $\mathfrak{h}$ and $\mathfrak{t}$ are bijections on X such that the diagram commutes (in this case, of course, $X = Y = U = V$). Then for any x in X,

$$\mathfrak{t}(x) = \mathfrak{t}(X_x(x)) = \varphi(X_x)(\mathfrak{h}(x)) = X_x(\mathfrak{h}(x)) = x$$

and for the point x_1,

$$x_1 = \mathfrak{t}(i(x_1)) = \varphi(i)(\mathfrak{h}(x_1)) = i(\mathfrak{h}(x_1)) = \mathfrak{h}(x_1).$$

But then,

$$k(x_1) = \mathfrak{t}(k(x_1)) = \varphi(k)(\mathfrak{h}(x_1)) = l(x_1)$$

which is a contradiction. Therefore, no such bijections $\mathfrak{h}$ and $\mathfrak{t}$ of X exist. The isomorphism φ is a K-homomorphism which maps $K(\mathfrak{S}(X, X, \mathfrak{f}))$ bijectively onto itself, however, and Theorem (2.3) implies that there does exist a bijection $\mathfrak{h}$ from $\mathscr{R}(\mathfrak{f})$ onto $\mathscr{R}(\mathfrak{f})$ and a bijection $\mathfrak{t}$ from X onto X such that the following diagram commutes.

$$\begin{tikzcd} \mathscr{R}(\mathfrak{f}) \arrow[r, "f"] \arrow[d, "\mathfrak{h}"'] & X \arrow[r, "\mathfrak{f}"] \arrow[d, "\mathfrak{t}"'] & \mathscr{R}(\mathfrak{f}) \arrow[d, "\mathfrak{h}"'] \\ \mathscr{R}(\mathfrak{f}) \arrow[r, "\varphi(f)"] & X \arrow[r, "\mathfrak{f}"] & \mathscr{R}(\mathfrak{f}) \end{tikzcd}$$

In this particular case, $\mathfrak{h}(x_0) = x_0$ and $\mathfrak{t}(x) = x$ for each x in X. To show that commutativity of the diagram above is not sufficient to insure that φ be an isomorphism, we give an example of a bijection φ of $\mathfrak{S}(X, X, \mathfrak{f})$ which is not an isomorphism but for which the diagram commutes (where $\mathfrak{h}$ and $\mathfrak{t}$ are defined as above). Choose a nonconstant function k and let $y = k(x_0)$. Define a bijection φ on $\mathfrak{S}(X, X, \mathfrak{f})$ by

$$\begin{aligned} \varphi(f) &= f \quad \text{for} \quad k \neq f \neq X_y, \\ \varphi(k) &= X_y \text{ and } \varphi(X_y) = k. \end{aligned}$$

Note that for any f in $\mathfrak{S}(X, X, \mathfrak{f})$, $\varphi(f)(x_0) = f(x_0)$. It follows from this that the diagram commutes. However, φ is not an isomorphism since

$$\varphi(X_y k) = \varphi(X_y) = k$$

while

$$\varphi(X_y)\varphi(k) = kX_y = X_y.$$

In concluding this section, we make one more observation. A $\mathfrak{S}$-semigroup $\mathfrak{S}(X, Y, \mathfrak{f})$ can be partially ordered in a very natural way. We define $f \leqq g$ if g is an extension of f, i.e., if $f \subseteqq g$. With this ordering, $\mathfrak{S}(X, Y, \mathfrak{f})$ is a partially ordered semigroup ([2]), page 153) since $f \leqq g$ implies $fh \leqq gh$ and $hf \leqq hg$ for every h in $\mathfrak{S}(X, Y, \mathfrak{f})$. It follows from Theorem (2.7) that every semigroup isomorphism from $\mathfrak{S}^*(X, Y, \mathfrak{f})$ onto $\mathfrak{S}^*(U, V, \mathfrak{g})$ is also an order isomorphism.

3. Applications of theorem (2.7) to some special $\mathfrak{S}^*$-semigroups

Suppose X is a nonempty set, Y is a set with more than one element and $\mathfrak{f}$ is a surjection from Y onto X. Then the semigroup of all functions f with $\mathscr{D}(f) = X$ and $\mathscr{D}(f) \subseteqq Y$ is a $\mathfrak{S}^*$-semigroup and will be denoted by $\mathfrak{S}^*_T(X, Y, \mathfrak{f})$. Therefore, with each function $\mathfrak{f}$ from Y into X, it is possible to associate a $\mathfrak{S}^*$-semigroup, namely, $\mathfrak{S}^*_T(\mathscr{R}(\mathfrak{f}), Y, \mathfrak{f})$. We first direct our efforts toward answering the question, "Precisely when do two functions $\mathfrak{f}$ and $\mathfrak{g}$ give rise to isomorphic semigroups?" For each y in Y, we let $y_\mathfrak{f} = \{z \in Y : \mathfrak{f}(z) = \mathfrak{f}(y)\}$ and $\mathfrak{D}_\mathfrak{f} = \{y_\mathfrak{f} : y \in Y\}$. Then $\mathfrak{D}_\mathfrak{f}$ is a decomposition of Y into a family of nonempty, mutually disjoint subsets and will be referred to as the decomposition of Y which is induced by $\mathfrak{f}$. We find that one can tell if the semigroups associated with $\mathfrak{f}$ and $\mathfrak{g}$ are isomorphic merely by looking at the decompositions of Y which are induced by $\mathfrak{f}$ and $\mathfrak{g}$. We state this more precisely as

THEOREM (3.1). *The semigroups $\mathfrak{S}^*_T(\mathscr{R}(\mathfrak{f}), Y, \mathfrak{f})$ and $\mathfrak{S}^*_T(\mathscr{R}(\mathfrak{g}), Y, \mathfrak{g})$ are isomorphic if and only if there is a one-to-one correspondence between the sets of $\mathfrak{D}_\mathfrak{f}$ and $\mathfrak{D}_\mathfrak{g}$ such that corresponding sets have the same cardinality.*

Before proving the theorem, it will be convenient to first prove a lemma.

LEMMA (3.2). *Suppose $\mathfrak{f}$ maps Y onto X, $\mathfrak{g}$ maps Y onto Z and $\mathfrak{t}$ is a bijection from Y onto Y. Then there exists a bijection $\mathfrak{h}$ from X onto Z such that $\mathfrak{h} \circ \mathfrak{f} = \mathfrak{g} \circ \mathfrak{t}$ if and only if $\mathfrak{t}[y_\mathfrak{f}] \in \mathfrak{D}_\mathfrak{g}$ for each $y_\mathfrak{f}$ in $\mathfrak{D}_\mathfrak{f}$.*

PROOF. Suppose first that $\mathfrak{t}[y_\mathfrak{f}] \in \mathfrak{D}_\mathfrak{g}$ for each $y_\mathfrak{f}$ in $\mathfrak{D}_\mathfrak{f}$. We define a function $\mathfrak{h}$ from X onto Z by

$$\mathfrak{h}(\mathfrak{f}(y)) = \mathfrak{g}(\mathfrak{t}(y)) \text{ for each } y \text{ in } Y.$$

Suppose $\mathfrak{f}(v) = \mathfrak{f}(y)$. Then $v \in y_\mathfrak{f}$ and $\mathfrak{t}(v) \in \mathfrak{t}[y_\mathfrak{f}]$. This implies $\mathfrak{g}(\mathfrak{t}(v)) = \mathfrak{g}(\mathfrak{t}(y))$ since $\mathfrak{t}[y_\mathfrak{f}] \in \mathfrak{D}_\mathfrak{g}$. Thus, $\mathfrak{h}(\mathfrak{f}(v)) = \mathfrak{h}(\mathfrak{f}(y))$, that is, $\mathfrak{h}$ is a (single-valued) function. Since $\mathfrak{f}$ and $\mathfrak{g}$ are surjections onto X and Z respectively and $\mathfrak{t}$ is a bijection from Y onto Y, it follows that $\mathscr{D}(\mathfrak{h}) = X$ and $\mathscr{R}(\mathfrak{h}) = Z$.

Now suppose x_1 and x_2 are two distinct points of X. Choose y_1 and y_2 in Y such that $\mathfrak{f}(y_1) = x_1$ and $\mathfrak{f}(y_2) = x_2$. Evidently, $y_1 \notin (y_2)_{\mathfrak{f}}$. Due to the fact that $\mathfrak{t}$ is a bijection and that $\mathfrak{t}[y_{\mathfrak{f}}] \in \mathfrak{D}_{\mathfrak{g}}$ for each $y_{\mathfrak{f}}$ in $\mathfrak{D}_{\mathfrak{f}}$, it follows that $\mathfrak{t}(y_1) \notin (\mathfrak{t}(y_2))_{\mathfrak{g}}$. Thus $\mathfrak{g}(\mathfrak{t}(y_1)) \neq \mathfrak{g}(\mathfrak{t}(y_2))$ which implies $\mathfrak{h}(x_1) \neq \mathfrak{h}(x_2)$. Thus $\mathfrak{h}$ is a bijection from X onto Z.

Now suppose there exists a bijection $\mathfrak{h}$ from X onto Z such that $\mathfrak{h} \circ \mathfrak{f} = \mathfrak{g} \circ \mathfrak{t}$. We will show that $\mathfrak{t}[y_{\mathfrak{f}}] = (\mathfrak{t}(y))_{\mathfrak{g}}$ for each $y_{\mathfrak{f}} \in \mathfrak{D}_{\mathfrak{f}}$. Suppose $p \in y_{\mathfrak{f}}$. Then $\mathfrak{f}(p) = \mathfrak{f}(y)$ and thus $\mathfrak{h}(\mathfrak{f}(p)) = \mathfrak{h}(\mathfrak{f}(y))$. Therefore, $\mathfrak{g}(\mathfrak{t}(p)) = \mathfrak{g}(\mathfrak{t}(y))$. That is to say, $\mathfrak{t}(p) \in (\mathfrak{t}(y))_{\mathfrak{g}}$. Now let q be any element of $(\mathfrak{t}(y))_{\mathfrak{g}}$. Then $\mathfrak{t}(v) = q$ for some v in Y. Suppose $v \notin y_{\mathfrak{f}}$. Then $\mathfrak{f}(v) \neq \mathfrak{f}(y)$ and therefore $\mathfrak{h}(\mathfrak{f}(v)) \neq \mathfrak{h}(\mathfrak{f}(y))$. This implies $\mathfrak{g}(\mathfrak{t}(v)) \neq \mathfrak{g}(\mathfrak{t}(y))$, i.e., $q \notin (\mathfrak{t}(y))_{\mathfrak{g}}$ which is a contradiction. Thus $v \in y_{\mathfrak{f}}$ and we conclude $\mathfrak{t}[y_{\mathfrak{f}}] = (\mathfrak{t}(y))_{\mathfrak{g}}$.

Now we are in a position to prove Theorem (3.1). First suppose there exists a one-to-one correspondence between the sets of $\mathfrak{D}_{\mathfrak{f}}$ and $\mathfrak{D}_{\mathfrak{g}}$ such that corresponding sets have the same cardinality. It follows that there exists a bijection $\mathfrak{t}$ from Y onto Y such that $\mathfrak{t}[y_{\mathfrak{f}}] \in \mathfrak{D}_{\mathfrak{g}}$ for each $y_{\mathfrak{f}} \in \mathfrak{D}_{\mathfrak{f}}$. Then according to Lemma (3.2) there exists a bijection $\mathfrak{h}$ from $\mathscr{R}(\mathfrak{f})$ onto $\mathscr{R}(\mathfrak{g})$ such that $\mathfrak{h} \circ \mathfrak{f} = \mathfrak{g} \circ \mathfrak{t}$. It follows from Theorem (2.7) that the mapping φ from $\mathfrak{S}^*_T(\mathscr{R}(\mathfrak{f}), Y, \mathfrak{f})$ onto $\mathfrak{S}^*_T(\mathscr{R}(\mathfrak{g}), Y, \mathfrak{g})$ defined by $\varphi(f) = \mathfrak{t} \circ \mathfrak{f} \circ \mathfrak{h}^{\leftarrow}$ is an isomorphism.

On the other hand, if φ is any isomorphism from $\mathfrak{S}^*_T(\mathscr{R}(\mathfrak{f}), Y, \mathfrak{f})$ onto $\mathfrak{S}^*_T(\mathscr{R}(\mathfrak{g}), Y, \mathfrak{g})$, Theorem (2.7) implies that there exists a bijection $\mathfrak{h}$ from $\mathscr{R}(\mathfrak{f})$ onto $\mathscr{R}(\mathfrak{g})$ and a bijection $\mathfrak{t}$ from Y onto Y such that $\mathfrak{h} \circ \mathfrak{f} = \mathfrak{g} \circ \mathfrak{t}$. Lemma (3.2) implies $\mathfrak{t}[y_{\mathfrak{f}}] \in \mathfrak{D}_{\mathfrak{g}}$ for each $y_{\mathfrak{f}}$ in $\mathfrak{D}_{\mathfrak{f}}$ and therefore $y_{\mathfrak{f}} \to (\mathfrak{t}(y))_{\mathfrak{g}} = \mathfrak{t}[y_{\mathfrak{f}}]$ is a one-to-one correspondence between the sets of $\mathfrak{D}_{\mathfrak{f}}$ and those of $\mathfrak{D}_{\mathfrak{g}}$ with the property that corresponding sets have the same cardinality.

Next, we use Theorem (2.7) to determine the automorphism group of $\mathfrak{S}^*_T(\mathscr{R}(\mathfrak{f}), Y, \mathfrak{f})$. For any decomposition $\mathfrak{D}$ of a set Y, we let $G(\mathfrak{D})$ denote the group of all bijections f from Y onto Y with the property $f[A] \in \mathfrak{D}$ for each A in $\mathfrak{D}$ where the binary operation is that of composition.

Theorem (3.3). *The automorphism group of* $\mathfrak{S}^*_T(\mathscr{R}(\mathfrak{f}), Y, \mathfrak{f})$ *is isomorphic to* $G(\mathfrak{D}_{\mathfrak{f}})$.

Proof. Let $\mathfrak{A}$ denote the automorphism group of $\mathfrak{S}^*_T(\mathscr{R}(\mathfrak{f}), Y, \mathfrak{f})$ and let φ be an element of $\mathfrak{A}$. According to Theorem (2.7) there exists a bijection $\mathfrak{h}$ on $\mathscr{R}(\mathfrak{f})$ and a bijection $\mathfrak{t}$ on Y such that $\varphi(f) = \mathfrak{t} \circ f \circ \mathfrak{h}^{\leftarrow}$ and $\mathfrak{h} \circ \mathfrak{f} = \mathfrak{f} \circ \mathfrak{t}$. By Lemma (3.2), $\mathfrak{t}$ is an element of $G(\mathfrak{D}_{\mathfrak{f}})$. Since $\mathfrak{t}$ is uniquely determined by φ, we can define a mapping Φ from $\mathfrak{A}$ into $G(\mathfrak{D}_{\mathfrak{f}})$ by $\Phi(\varphi) = \mathfrak{t}$. One verifies in a straightforward manner that Φ is a homomorphism. Furthermore, if $\mathfrak{t}$ is any element of $G(\mathfrak{D}_{\mathfrak{f}})$, Lemma (3.2) implies the existence of a bijection $\mathfrak{h}$ such that $\mathfrak{h} \circ \mathfrak{f} = \mathfrak{f} \circ \mathfrak{t}$. Then according to Theorem (2.7), the mapping from $\mathfrak{S}^*_T(\mathscr{R}(\mathfrak{f}), Y, \mathfrak{f})$ onto itself which is defined by $\varphi(f) = \mathfrak{t} \circ f \circ \mathfrak{h}^{\leftarrow}$

is an automorphism. Thus $\Phi(\varphi) = \mathfrak{t}$ and we see that Φ is an epimorphism. To conclude the proof, we need only show that the kernel of Φ is the identity automorphism. Suppose, then, $\Phi(\varphi) = i$, the identity mapping of $G(\mathfrak{D}_{\mathfrak{f}})$. Again, we use Theorem (2.7) to conclude that there exists a bijection $\mathfrak{h}$ from $\mathscr{R}(\mathfrak{f})$ onto $\mathscr{R}(\mathfrak{f})$ such that $\mathfrak{h} \circ \mathfrak{f} = \mathfrak{f} \circ i$ and $\varphi(f) = i \circ f \circ \mathfrak{h}^{\leftarrow}$ for each f in $\mathfrak{S}_T^*(\mathscr{R}(\mathfrak{f}), Y, \mathfrak{f})$. Hence, $\mathfrak{h} \circ \mathfrak{f} = \mathfrak{f}$. For any x in $\mathscr{R}(\mathfrak{f})$, there exists a y in Y such that $\mathfrak{f}(y) = x$. Then $\mathfrak{h}(x) = \mathfrak{h}(\mathfrak{f}(y)) = \mathfrak{f}(y) = x$ and $\mathfrak{h}$ is the identity mapping on $\mathscr{R}(\mathfrak{f})$. Therefore, φ is the identity automorphism and Φ is an isomorphism.

Now suppose $X = Y$ and $\mathfrak{f}$ is equal to the identity mapping i on Y. Then $\mathfrak{S}_T^*(\mathscr{R}(\mathfrak{f}), Y, \mathfrak{f}) = \mathfrak{S}_T^*(Y, Y, i)$ is the semigroup of all functions mapping Y into Y under the binary operation of composition. This semigroup is discussed in some detail in both [1] and [4], though, as we mentioned previously, composition is defined by $f \circ g = \{(x, y) : (x, z) \in f$ and $(z, y) \in g$ for some $z\}$ in the former.

It follows from Theorem (2.7) that for any automorphism φ from $\mathfrak{S}_T^*(Y, Y, i)$ onto $\mathfrak{S}_T^*(Y, Y, i)$, there exists a bijection $\mathfrak{h}$ from Y onto Y such that $\varphi(f) = \mathfrak{h} \circ f \circ \mathfrak{h}^{\leftarrow}$ for each f in $\mathfrak{S}_T^*(Y, Y, i)$. Moreover, $\mathfrak{h}$ and $\mathfrak{h}^{\leftarrow}$ are elements of $\mathfrak{S}_T^*(Y, Y, i)$ and, in fact, $\mathfrak{h}^{\leftarrow} = \mathfrak{h}^{-1}$, i.e., $\mathfrak{h}^{\leftarrow}$ is the algebraic inverse of $\mathfrak{h}$. Therefore we have the following well-known result (I. Schreier [10], A. I. Malcev [9] and E. S. Ljapin [5] have all given proofs of this result).

COROLLARY (3.4). *Every automorphism of $\mathfrak{S}_T^*(Y, Y, i)$ is an inner automorphism.*

Now suppose we apply Theorem (3.3) to $\mathfrak{S}_T^*(Y, Y, i)$. In the case of the mapping i, $G(\mathfrak{D}_i)$ is simply the group of all bijections on Y. Therefore, we have

COROLLARY (3.5). *The automorphism group of $\mathfrak{S}_T^*(Y, Y, i)$ is isomorphic to the group of all bijections on Y.*

Let us consider one more family of semigroups. Again, $\mathfrak{f}$ denotes a function from Y into X. The family of all functions f with $\mathscr{D}(f) \subseteqq \mathscr{R}(\mathfrak{f})$ and $\mathscr{R}(f) \subseteqq Y$ is a $\mathfrak{S}^*$-semigroup and is denoted by $\mathfrak{S}_P^*(\mathscr{R}(\mathfrak{f}), Y, \mathfrak{f})$. The proofs of both Theorem (3.1) and Theorem (3.3) carry over completely intact to give the following two results.

THEOREM (3.6). *For two functions $\mathfrak{f}$ and $\mathfrak{g}$ from Y into X, $\mathfrak{S}_P^*(\mathscr{R}(\mathfrak{f}), Y, \mathfrak{f})$ and $\mathfrak{S}_P^*(\mathscr{R}(\mathfrak{g}), Y, \mathfrak{g})$ are isomorphic if and only if there is a one-to-one correspondence between the sets of $\mathfrak{D}_{\mathfrak{f}}$ and $\mathfrak{D}_{\mathfrak{g}}$ such that corresponding sets have the same cardinality.*

THEOREM (3.7). *The automorphism group of $\mathfrak{S}_P^*(\mathscr{R}(\mathfrak{f}), Y, \mathfrak{f})$ is isomorphic to $G(\mathscr{D}_{\mathfrak{f}})$.*

Finally, let us note that if we take $X = Y$ and $\mathfrak{f} = i$, $\mathfrak{S}_P^*(Y, Y, i)$ is the semigroup of all functions f such that $\mathscr{D}(f) \subseteqq Y$ and $\mathscr{R}(f) \subseteqq Y$. This semigroup is discussed in [4]. We invoke Theorems (2.7) and (3.7) to obtain the following analogues of Corollaries (3.4) and (3.5).

COROLLARY (3.8). *Every automorphism of* $\mathfrak{S}_P^*(Y, Y, i)$ *is an inner automorphism.*

COROLLARY (3.9). *The automorphism group of* $\mathfrak{S}_P^*(Y, Y, i)$ *is isomorphic to the group of all bijections on* Y.

Note added in proof. The proof of (2.3.6) was based on the fact that $\mathfrak{f}(y_1) \in \mathscr{D}(f)$. We wish to thank G. B. Preston for pointing out that a verification of this fact was omitted and we take this opportunity to correct the situation. By (2.1.2), there exists a nonempty subset A of X such that $A_{y_1} \in \mathfrak{S}(X, Y, \mathfrak{f})$. Then $\varphi(fA_{y_1}) = \varphi(f)\varphi(A_{y_1}) = gB_{\mathfrak{t}(y_1)}$ for some nonempty subset B of U. Now, $gB_{\mathfrak{t}(y_1)} \neq e$ since $\mathfrak{g}(\mathfrak{t}(y_1)) \in \mathscr{D}(g)$ and it follows from (2.2.1) that $fA_{y_1} \neq e$. This implies $\mathfrak{f}(y_1) \in \mathscr{D}(f)$.

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