Forum of Mathematics, Sigma (2019), Vol. 7, e40, 27 pages
doi:10.1017/fms.2019.39

BOREL DENSITY FOR APPROXIMATE LATTICES

MICHAEL BJÖRKLUND[1],
TOBIAS HARTNICK[2] and THIERRY STULEMEIJER[3]

[1] Department of Mathematics, Chalmers, Gothenburg, Sweden;
email: micbjo@chalmers.se

[2] Institut für Algebra und Geometrie, Karlsruher Institut für Technologie, Germany;
email: tobias.hartnick@kit.edu

[3] Mathematisches Institut, Justus-Liebig-Universität Gießen, Germany;
email: thierry.stulemeijer@math.uni-giessen.de

Received 9 July 2019; accepted 7 August 2019

Abstract

We extend classical density theorems of Borel and Dani–Shalom on lattices in semisimple, respectively solvable algebraic groups over local fields to approximate lattices. Our proofs are based on the observation that Zariski closures of approximate subgroups are close to algebraic subgroups. Our main tools are stationary joinings between the hull dynamical systems of discrete approximate subgroups and their Zariski closures.

2010 Mathematics Subject Classification: 22D40 (primary); 20P05, 20G25 (secondary)

1. Introduction

Borel's density theorem [6] is a cornerstone of the theory of lattices in semisimple algebraic groups over local fields, and can be stated as follows.

THEOREM 1 (Borel Density Theorem). *Let k be a local field and let $\mathbf{G}$ be a connected semisimple algebraic group over k. If $\mathbf{G}(k)$ does not have any compact factors, then every lattice $\Gamma < \mathbf{G}(k)$ is Zariski-dense.*

Here and in the sequel, $\mathbf{G}(k)$ is considered as a topological group with respect to its natural Hausdorff group topology, which turns $\mathbf{G}(k)$ into a locally compact second countable (lcsc) topological group. A similar density theorem for lattices in solvable algebraic groups was established by Dani [8] (for $k = \mathbb{R}$) and

Shalom [18] (for general local fields). Recall that a solvable algebraic group $\mathbf{G}$ over a field k is called *k-split* if every composition factor is isomorphic over k to the additive or multiplicative group of k. For example, unipotent algebraic groups over fields of characteristic 0 are k-split, but they need not be k-split in positive characteristic (see Section 5.2).

THEOREM 2 (Dani–Shalom Density Theorem). *Let k be a local field and let $\mathbf{G}$ be a connected solvable algebraic group over k. If $\mathbf{G}$ is k-split, then every lattice $\Gamma < \mathbf{G}(k)$ is Zariski-dense.*

In this article, we generalize these density theorems to (certain) approximate lattices. Approximate lattices are certain discrete approximate subgroups (in the sense of Tao [20]) of locally compact groups. They were introduced in [2] as generalizations of Meyer sets in abelian lcsc groups [14] and further studied in [3–5, 12]. By definition, a *uniform approximate lattice* $\Lambda \subset G$ is a discrete approximate subgroup which is cocompact in the sense that $G = \Lambda K$ for a compact subset $K \subset G$. More generally, *approximate lattices* are defined by the existence of nontrivial stationary measures on an associated hull dynamical system, and *strong approximate lattices* are those approximate lattices, for which the hull even admits an invariant measure (see Section 2 for precise definitions). With this terminology understood, our main result can be stated as follows:

THEOREM 3 (Main theorem). *Let k be a local field and let $\mathbf{G}$ be a connected algebraic group over k. Assume that either $\mathbf{G}$ is semisimple and $\mathbf{G}(k)$ does not have any compact factors, or that $\mathbf{G}$ is solvable and k-split. Then every strong approximate lattice $\Lambda \subset \mathbf{G}(k)$ and every uniform approximate lattice $\Lambda \subset \mathbf{G}(k)$ is Zariski-dense.*

By definition, a subgroup $\Gamma < \mathbf{G}(k)$ is a uniform approximate lattice if and only if it is a uniform lattice. We show in Section 2.4 below that a subgroup $\Gamma < \mathbf{G}(k)$ is an approximate lattice if and only if it is a strong approximate lattice if and only if it is a lattice. Thus our main theorem is indeed a proper generalization of the classical density theorems.

As we discuss in Section 2.1, a symmetric subset Λ of a locally compact group G containing the identity is a uniform approximate lattice if and only if it a Delone set (that is, cocompact and uniformly discrete) and if moreover $\Lambda \cdot \Lambda \cdot \Lambda$ is uniformly discrete; this in turn implies that Λ^n is uniformly discrete for all $n \in \mathbb{N}$. In the abelian case, the latter property is known as *long-range order* (since it is a nonlocal property) and is often used as the defining property of quasicrystals. We emphasize that our Main Theorem does not hold for arbitrary Delone sets without the assumption of long-range order (see Example 1 below).

https://doi.org/10.1017/fms.2019.39 Published online by Cambridge University Press

Our proof of the main theorem is inspired by Furstenberg's proof of Borel density [9], which can be sketched as follows: if Γ is a lattice in $G = \mathbf{G}(k)$ and H denotes the Zariski closure of Γ in G, then the invariant probability measure on G/Γ pushes forward to an invariant probability measure on G/H, which by Chevalley's theorem can be realized as a quasiprojective variety. Using recurrence properties of unipotents on projective space with respect to the invariant measure at hand one then deduces that G/H must be a point.

This approach does not apply directly to our more general setting for several reasons: Firstly, the Zariski closure of an approximate lattice $\Lambda \subset G$ is not a group. It is however, in a sense made precise in Theorem 17 below, close to an algebraic subgroup H of G. We would thus like to connect a stationary measure on the hull of Λ (which serves as a natural replacement for the homogeneous space G/Γ in the group case) to a measure on G/H. Unlike the group case, we cannot embed the hull of Λ into G/H, but we can use a stationary joining between the hull and G/H to obtain a measure on G/H. A crucial difference to the group case will be that the measure obtained on G/H will in general not be invariant, but only stationary. To obtain the desired conclusion, we thus need to investigate further properties of the measure in question. In this final step we also need information concerning maximal algebraic subgroups of semisimple groups over local fields as provided by Stuck [19].

This article is organized as follows. In Section 2, we recall the precise definitions of strong and uniform approximate lattices. We use this opportunity to establish a number of basic results concerning hull dynamical systems, which will be used throughout the article. In Section 3, we show that Zariski closures of approximate subgroups are again approximate subgroups, and that such 'algebraic' approximate subgroups are close to algebraic subgroups. This statement is made precise in Theorem 17, which is the main result of this section. In Section 4, we use this result to deduce Borel density, first in the uniform case, and then in the strong case. In Section 5, we derive Dani–Shalom density and discuss various variants and refinements.

Appendix A contains some background concerning the existence of stationary joinings. Appendix B generalizes the unimodularity theorem from [2] to the case of groups which are not compactly generated; this is used in the proof of the main theorem in the uniform case.

Throughout this article, we use the following convention: if k is a local field and $\mathbf{G}$ is a linear algebraic group over k, then all topological terms (for example, closure, compactness) concerning subsets of $G := \mathbf{G}(k)$ refer to the Hausdorff topology on G and not to the Zariski topology, unless explicitly mentioned otherwise.

https://doi.org/10.1017/fms.2019.39 Published online by Cambridge University Press

2. Approximate lattices and their hulls

2.1. Uniform approximate lattices. Let G be a group. Given subsets $A, B \subset G$ we denote by

$$AB := \{ab \in G \mid a \in A, b \in B\} \quad \text{and} \quad A^{-1} := \{a^{-1} \in G \mid a \in A\}$$

their product set, respectively set of inverses. We also define $A^k := A^{k-1}A$ for $k \geqslant 2$. (For distinction we write $A^{\times k}$ for the k-fold Cartesian product.)

We recall that if G is a group, then a subset $\Lambda \subset G$ is called an *approximate subgroup* if it is symmetric, contains the identity and satisfies $\Lambda^2 \subset F\Lambda$ for a finite subset $F \subset G$. Since Λ is symmetric, this implies $\Lambda^2 \subset \Lambda F^{-1}$, and hence we may choose a finite set F_Λ such that $\Lambda^2 \subset F_\Lambda \Lambda \cap \Lambda F_\Lambda$, and hence

$$\Lambda^k \subset F_\Lambda^{k-1}\Lambda \cap \Lambda F_\Lambda^{k-1} \quad \text{for all } k \geqslant 2. \tag{1}$$

If G is an lcsc group, then a subset $P \subset G$ is called *(left-)relatively dense* if there exists a compact subset $K \subset G$ such that $G = PK$. It is called *uniformly discrete* if e is not an accumulation point of $P^{-1}P$. An approximate subgroup $\Lambda \subset G$ is called a *uniform approximate lattice*, if it is relatively dense and discrete.

We refer the reader to [2] for a thorough discussion of uniform approximate lattices. To give an alternative characterization of uniform approximate lattices, recall that a (left-)relatively dense and uniformly discrete subset of an lcsc group G is also called a *(left-)Delone set*, and it is said to have *long-range order* if Λ^n is uniformly discrete for all $n \in \mathbb{N}$. As pointed out in [2, Proposition 2.9], a symmetric subset of G containing the identity is a uniform approximate lattice if and only if it is Delone and has long-range order; equivalently, it is Delone and Λ^3 is closed and discrete. The following example shows that the conclusion of our main theorem may fail for Delone sets without long-range order.

EXAMPLE 1. It follows from [7, Proposition 3.C.3] and the metric characterization of Delone sets in lcsc groups [2, Proposition 2.2] that every lcsc group contains a (left-) Delone set. Moreover, if G is an lcsc group and $P < G$ is a cocompact closed subgroup, then every Delone set in P is also a Delone set in G.

Now if $\mathbf{G}$ is a connected semisimple algebraic group over a local field k, and $\mathbf{P}$ is any proper parabolic subgroup of $\mathbf{G}$, then $P := \mathbf{P}(k)$ is a cocompact algebraic subgroup of $G := \mathbf{G}(k)$. By the previous remark there exists a Delone set $\Lambda \subset P$; any such Delone set Λ is also a Delone set in G, but its Zariski closure will be contained in P. This shows that general Delone sets in connected semisimple algebraic groups over local fields need not be Zariski-dense, not even in the absence of compact factors.

https://doi.org/10.1017/fms.2019.39 Published online by Cambridge University Press

One of the main difficulties in the proof of our main theorem in the semisimple case is indeed to show that a Delone set with long-range order cannot be contained in a parabolic subgroup.

Note that a subgroup of G is a uniform approximate lattice if and only if it is a uniform lattice. We now proceed towards the definition of nonuniform approximate lattices, which generalize nonuniform lattices in a similar way.

2.2. The Chabauty–Fell topology. Given an lcsc space X we denote by $\mathcal{C}(X)$ the collection of closed subsets of X with the Chabauty–Fell topology, that is, the topology on $\mathcal{C}(X)$ generated by the basic open sets

$$U_K = \{A \in \mathcal{C}(X) \mid A \cap K = \emptyset\} \quad \text{and} \quad U^V = \{A \in \mathcal{C}(X) \mid A \cap V \neq \emptyset\},$$

where K runs over all compact subsets of X and V runs over all open subsets of X.

Under the present assumptions on X, the space $\mathcal{C}(X)$ is a compact metrizable space (see for example, [17, Propositions 1.7 and 1.8]), and in particular its topology is characterized by convergence of sequences in $\mathcal{C}(X)$. A sequence (F_i) in $\mathcal{C}(X)$ converges if and only if the two following conditions are satisfied:

(CF1) For all $x \in F$ there exist $x_i \in F_i$ such that (x_i) converges to x.

(CF2) If $x_i \in F_i$ for all $i \in \mathbb{N}$ then every accumulation point of the sequence (x_i) is contained in F.

We derive two consequences: Firstly, if a G acts jointly continuously on X, then it acts jointly continuously on $\mathcal{C}(X)$ by $g \,.\, A := \{ga \mid a \in A\}$. Secondly, taking finite unions is continuous in the Chabauty–Fell topology:

COROLLARY 4. *For every lcsc space X the map $\pi\colon \mathcal{C}(X)^{\times k} \to \mathcal{C}(X)$, $(F_1, \ldots, F_k) \mapsto F_1 \cup \cdots \cup F_k$ is continuous.*

Proof. Let $((F_{1,i}, \ldots, F_{k,i}))_{i \geqslant 1}$ be a sequence in $\mathcal{C}(G)^{\times k}$ converging to $(E_1, \ldots, E_k)$, and abbreviate $F_i := F_{1,i} \cup \cdots \cup F_{k,i}$ and $E := E_1 \cup \cdots \cup E_k$. We have to show that $F_i \to E$; for this we check Conditions (CF1) and (CF2):

(CF1) If $x \in E$, then $x \in E_j$ for some $j \in \{1, \ldots, k\}$. Since $F_{j,i} \to E_j$ there thus exist $x_i \in F_{j,i} \subseteq F_i$ such that $x_i \to x$.

(CF2) Let $x_i \in F_i$ and let $x \in X$ be an accumulation point of (x_i), say $x_{n_i} \to X$. Passing to a further subsequence we may assume by the pigeon hole principle that $x_{n_i} \in F_{j,n_i}$ for some $j \in \{1, \ldots, k\}$. Since $F_{j,i} \to E_j$ it then follows that $x \in E_j \subset E$. □

https://doi.org/10.1017/fms.2019.39 Published online by Cambridge University Press

2.3. Hulls of closed subsets. Let G be an lcsc group. We refer to a jointly continuous action of G on a compact space Ω as a *topological dynamical system (TDS)* and to a continuous G-equivariant map between TDSs as a factor map. If $G \curvearrowright \Omega$ is a TDS, then so is the orbit closure of every element of Ω, and every factor map maps an orbit closure of an element onto the orbit closure of its image.

By the results recalled in the previous subsection, the left action of an lcsc group G on itself induces a TDS $G \curvearrowright \mathcal{C}(G)$, $(g, A) \mapsto gA$, and more generally the diagonal action of G on $G^{\times n}$ induces a TDS $G \curvearrowright \mathcal{C}(G)^{\times n}$, $(g, (A_1, \dots, A_n)) \mapsto (gA_1, \dots, gA_n)$. We are going to consider orbit closures in these TDSs.

DEFINITION 5. Let G be an lcsc group and let $P, P_1, \dots, P_n \in \mathcal{C}(G)$.

(i) The *(left-)hull* of a closed subset $P \subset G$ is defined as the orbit closure

$$\Omega_P := \overline{\{gP \mid g \in G\}} \subset \mathcal{C}(G).$$

(ii) The *simultaneous (left-)hull* of $P_1, \dots, P_n$ is

$$\Omega_{P_1,\dots,P_n} := \overline{\{(gP_1, \dots, gP_n) \mid g \in G\}} \subset \mathcal{C}(G)^{\times n}.$$

In the case of closed subgroup $H < G$, the hull is a compactification of G/H, but it turns out to be the trivial compactification:

LEMMA 6 (Hulls of closed subgroups). *If $H < G$ is a closed subgroup, then $\Omega_H \setminus \{\emptyset\} = G/H$.*

Proof. Let (g_n) be a sequence in G such that $g_n H$ converges to some $H' \in \mathcal{C}(G)$ and assume $H' \neq \emptyset$. Then there exists $x \in H'$ and by (CF1) there exist $h_n \in H$ such that $g_n h_n \to x$. In Particular, there exists a compact set K such that $k_n := g_n h_n \in K$ and $k_n H = g_n H \to H'$. Passing to a subsequence we may assume that k_n converges to some $k \in K$, and by continuity of the G-action we deduce that $k_n H \to kH$. Thus $H' = kH \in G/H$. □

Note that if $P_1, \dots, P_n \in \mathcal{C}(G)$, then the projection onto the ith factor yields a continuous surjective G-factor map $\pi_i : \Omega_{P_1,\dots,P_n} \to \Omega_{P_i}$. If we set $P := P_1 \cup \dots \cup P_N$, then by Lemma 4 we also have a continuous G-factor map $\pi : \Omega_{P_1,\dots,P_n} \to \Omega_P$ given by $(Q_1, \dots, Q_n) \mapsto Q_1 \cup \dots \cup Q_n$.

We apply these factor maps to study relatively dense subsets of lcsc groups. This is made possible by the observation that if $P \in \mathcal{C}(G)$, then $\emptyset \in \Omega_P$ if and only if P is not relatively dense [2, Proposition 4.4]. We use the fact that every TDS contains a *minimal* subset, that is, a subset which is the orbit closure of each of its elements. Note that if a minimal system Z contains a fixpoint p, then $Z = \{p\}$.

https://doi.org/10.1017/fms.2019.39 Published online by Cambridge University Press

LEMMA 7. *Let $P_1, \dots, P_k \in \mathcal{C}(G)$. If $P := P_1 \cup \dots \cup P_n$ is relatively dense then there exists $i \in \{1, \dots, n\}$ such that $P_i^{-1} P_i$ is relatively dense.*

Proof. We choose a minimal subset Z of the joint hull $\Omega_{P_1,\dots,P_n}$ and an element $z \in Z$. We then set $Q := \pi(z)$ and $Q_i := \pi_i(z)$, where $\pi_i : \Omega_{P_1,\dots,P_n} \to \Omega_{P_i}$ and $\pi : \Omega_{P_1,\dots,P_n} \to \Omega_P$ are the factor maps defined above.

We first observe that $z \neq (\emptyset, \dots, \emptyset)$, since otherwise $Q = \emptyset$, contradicting the assumption that P be relatively dense. There thus exists $i \in \{1, \dots, n\}$ such that $Q_i \neq \emptyset$. Note that $\emptyset$ is a G-fixpoint in $\mathcal{C}(G)$. Since $\Omega_{Q_i} = \pi_i(Z)$ is minimal and $Q_i \neq \emptyset$, we thus deduce that $\emptyset \notin \Omega_{Q_i}$, hence Q_i is relatively dense in G. Let $K_1 \subset G$ be a compact subset such that $G = Q_i K_1$.

Since $Q_i \in \Omega_{P_i}$, we deduce from [2, Lemma 4.6] that $Q_i^{-1} Q_i \subset \overline{P_i^{-1} P_i}$, and hence $Q_i^{-1} Q_i \subset P_i^{-1} P_i K_0$, where K_0 is any compact identity neighbourhood in G. We thus obtain

$$G \subset Q_i^{-1} Q_i K_1 \subset P_i^{-1} P_i K_0 K_1$$

which shows that $P_i^{-1} P_i$ is relatively dense in G. □

2.4. Nonuniform approximate lattices. We now turn to the definition of nonuniform approximate lattices. Let G be an lcsc group and let $\Gamma < G$ be a discrete subgroup. By Lemma 6, we have $\Omega_\Gamma \setminus \{\emptyset\} = G/\Gamma$. Thus Γ is a lattice in G if and only if there exists a G-invariant probability measure on $\Omega_\Gamma \setminus \{\emptyset\}$. Equivalently, if μ is any admissible probability measure on μ (that is, absolutely continuous with respect to Haar measure and with support generating G as a semigroup), then there exists a μ-stationary probability measure on $\Omega_\Gamma \setminus \{\emptyset\}$.

DEFINITION 8. Let G be an lcsc group, let $P \in \mathcal{C}(G)$ and let $\Lambda \subset G$ be a closed and discrete approximate subgroup.

(i) A probability measure ν on Ω_P is called *nontrivial* if $\nu(\{\emptyset\}) = 0$.

(ii) Λ is called a *strong approximate lattice* if there exists a nontrivial G-invariant probability measure ν on Ω_Λ.

(iii) Λ is called an *approximate lattice* if for every admissible probability measure μ on G there exists a nontrivial μ-stationary probability measure ν on Ω_Λ.

Concerning the relations between these definitions we remark: Every uniform approximate lattice and every strong approximate lattice is an approximate lattice. We do not currently know whether every approximate lattice is strong. If G is

https://doi.org/10.1017/fms.2019.39 Published online by Cambridge University Press

amenable, then every uniform approximate lattice is strong, and if G is nilpotent, then every approximate lattice is both strong and uniform [2, Theorem 4.25]. A discrete subgroup $\Gamma < G$ is a strong approximate lattice if and only if it is an approximate lattice if and only if it is a lattice by the remark preceding Definition 8.

2.5. Quasimonotone joining of hulls. Throughout this subsection let G be an lcsc group and let $P, Q \in \mathcal{C}(G)$. We have surjective factor maps $\pi_1 : \Omega_{P,Q} \to \Omega_P$ and $\pi_2 : \Omega_{P,Q} \to \Omega_Q$. A triple $(\nu_{P,Q}, \nu_P, \nu_Q)$ of probability measures on $\Omega_{P,Q}$, Ω_P and Ω_Q respectively is called a *hull joining* if $(\pi_1)_*\nu_{P,Q} = \nu_P$ and $(\pi_2)_*\nu_{P,Q} = \nu_Q$:

$$\begin{array}{ccc} & (\Omega_{P,Q}, \nu_{P,Q}) & \\ {\scriptstyle \pi_1}\swarrow & & \searrow{\scriptstyle \pi_2} \\ (\Omega_P, \nu_P) & & (\Omega_Q, \nu_Q) \end{array}$$

The hull joining is called *invariant* if $\nu_{P,Q}$ (and hence ν_P and ν_Q) is G-invariant; it is called *μ-stationary* for an admissible probability measure μ on G if $\nu_{P,Q}$ is μ-stationary, that is, $\mu * \nu_{P,Q} = \nu_{P,Q}$. An invariant or μ-stationary hull joining is called *ergodic* if $\nu_{P,Q}$ (and hence ν_P and ν_Q) is G-ergodic.

If μ is an admissible probability measure on G, then by Lemma 31 in Appendix A, every (ergodic) μ-stationary probability measure ν_P on Ω_P lifts to an (ergodic) μ-stationary probability measure on $\Omega_{P,Q}$, hence is part of an (ergodic) μ-stationary hull joining $(\nu_{P,Q}, \nu_P, \nu_Q)$. This construction works for arbitrary P, $Q \in \mathcal{C}(G)$, but there are two caveats:

- It may happen that $\nu_Q = \delta_\emptyset$, even if ν_P is nontrivial.
- Even if ν_P is G-invariant, it will in general not be part of an *invariant* joining (unless G is amenable).

To deal with the first point, we need to add some assumption on the relation between P and Q. For example, nontriviality of ν_Q is guaranteed in the case of a *monotone* joining, that is, if $P \subset Q$ and hence $P' \subset Q'$ for all $(P', Q') \in \Omega_{P,Q}$. We need a slightly more general version of this result.

LEMMA 9 (Quasimonotone joinings). *Let $P, Q \in \mathcal{C}(G)$ and assume that $P \subset QF$ for some finite set $F \subset G$. Then $P' \subset Q'F$ for all $(P', Q') \in \Omega_{P,Q}$. In particular, if $Q' = \emptyset$, then $P' = \emptyset$.*

Proof. If $(P', Q') \in \Omega_{P,Q}$, then there exist $g_n \in G$ such that $g_n P \to P'$ and $g_n Q \to Q'$. It thus follows from (CF1) that every $p \in P'$ is the limit of a sequence

https://doi.org/10.1017/fms.2019.39 Published online by Cambridge University Press

of the form $(g_n p_n)$ with $p_n \in P$. Since $P \subset QF$ we can write $p_n = q_n f_n$ with $q_n \in Q$ and $f_n \in F$. Passing to a subsequence we may assume that $(f_n = f)$ is constant. It then follows from (CF2) that $g_n p_n f^{-1}$ converges to an element $q' \in Q'$, and hence $p' = q' f \in Q'F$. □

COROLLARY 10. *Let $P, Q \in \mathcal{C}(G)$ and let $(\nu_{P,Q}, \nu_P, \nu_Q)$ be a μ-stationary hull joining. If $P \subset QF$ for some finite set $F \subset G$ and if ν_P is nontrivial, then also ν_Q is nontrivial.*

Proof. Otherwise, $\mathrm{supp}(\nu_{P,Q}) \subset (\Omega_P \times \{\emptyset\}) \cap \Omega_{P,Q}$, and then Lemma 9 would imply that ν_P is the Dirac mass at the empty set. □

To deal with the second point, we observe that if $(\nu_{P,Q}, \nu_P, \nu_Q)$ is a μ-stationary hull joining and ν_P is nontrivial and G-invariant, then while ν_Q need not be invariant, it can at least not satisfy certain strong negations of invariance. To make this precise we recall the following definition.

DEFINITION 11. Let $H \curvearrowright \Omega$ be a TDS, μ an admissible probability measure on H and let ν be a μ-stationary Borel probability measure on Ω. We denote by $\mathbf{P} := \mu^{\times\mathbb{N}}$ the product measure on $G^{\times\mathbb{N}}$.

(i) Given $\xi = (\xi_n) \in G^{\times\mathbb{N}}$ we say that ν has *conditional measure* ν_ξ with respect to ξ if
$$(\xi_1 \cdots \xi_n)_* \nu \longrightarrow \nu_\xi$$
in the weak-$*$-topology as $n \to \infty$.

(ii) ν is called *μ-proximal* if for $\mathbf{P}$-almost every $\xi \in G^{\times\mathbb{N}}$ the conditional measure μ_ξ exists and is a point measure. In this case, (Ω, ν) is called a *μ-boundary*.

In fact, it follows from the martingale convergence theorem that conditional measures exist for $\mathbf{P}$-almost every $\xi \in G^{\times\mathbb{N}}$. Typical examples of μ-boundaries are given by generalized flag varieties: Let k be a local field and $\mathbf{H}$ be a semisimple algebraic group over k such that $\mathbf{H}(k)$ has no compact factors. Then for every parabolic subgroup $\mathbf{P}$ of $\mathbf{H}$, and every admissible probability measure μ on $\mathbf{H}(k)$ there exists a unique μ-stationary probability measure ν on $(\mathbf{H}/\mathbf{P})(k)$, and then $((\mathbf{H}/\mathbf{P})(k), \nu)$ is a μ-boundary [**13**, Theorem VI.3.7].

PROPOSITION 12. *Assume that $(\nu_{P,Q}, \nu_P, \nu_Q)$ is an ergodic μ-stationary hull joining and that $P \subset QF$ for some finite set $F \subset G$. If ν_P is nontrivial and G-invariant and ν_Q is μ-proximal, then there exists $Q' \in \Omega_Q$ such that $G = Q'F$.*

https://doi.org/10.1017/fms.2019.39 Published online by Cambridge University Press

Proof. Since ν_Q is μ-proximal, it follows from [10, Proposition 3.1] that (in the terminology of *loc. cit.*) the only μ-joining of (Ω_P, ν_P) and (Ω_Q, ν_Q) is the product joining. In our terminology this means that $(\Omega_{P\times Q}, \nu_{P,Q}) \cong (\Omega_P \times \Omega_Q, \nu_P \times \nu_Q)$ as measurable G-spaces, hence in particular, the support of $\nu_{P,Q}$ would be $G \times G$ invariant.

Now since ν_P is nontrivial, we can find $P' \in \mathrm{supp}(\nu_P)$ such that $P' \neq \emptyset$. There then exists $Q' \in \mathrm{supp}(\nu_Q)$ such that $(P', Q') \in \mathrm{supp}(\nu_{P,Q})$ and hence $G \times G$ invariance of the latter set implies that $(gP', Q') \in \mathrm{supp}(\nu_{P,Q})$ for all $g \in G$. By Lemma 9 we thus have $gP' \subset Q'F$ for all $g \in G$ and thus $Q'F = G$. □

2.6. Commensurability invariance of approximate lattices. Given a group G we say that two subsets $A, B \subset G$ are *commensurable* if there exist finite subset $F_1, F_2 \subset G$ such that $A \subset BF_1$ and $B \subset AF_2$. Commensurability defines an equivalence relation on subsets of G, and as a first application of stationary hull joinings we show that the class of approximate lattices is invariant under commensurability.

LEMMA 13. *Let Λ be an approximate lattice in an lcsc group G. If $\Lambda' \subset G$ is a discrete approximate subgroup and $\Lambda \subset \Lambda' F$ for some finite subset $F \subset G$, then also Λ' is an approximate lattice. In particular, this is the case if Λ' is commensurable to Λ.*

Proof. Since Λ is an approximate lattice, there exists a nontrivial μ-stationary probability measure ν_Λ on Ω_Λ for every admissible probability measure μ on G, and hence a stationary hull joining $(\nu_{\Lambda,\Lambda'}, \nu_\Lambda, \nu_{\Lambda'})$ between Ω_Λ and $\Omega_{\Lambda'}$. By Corollary 10, the measure $\nu_{\Lambda'}$ is nontrivial, hence Λ' is an approximate lattice. □

If G is amenable, then invariant hull joining always exist, hence we deduce:

COROLLARY 14. *Let Λ be a strong approximate lattice in an amenable lcsc group G. If $\Lambda' \subset G$ is a discrete approximate subgroup and $\Lambda \subset \Lambda' F$ for some finite subset $F \subset G$, then Λ' is a strong approximate lattice. In particular, this is the case if Λ' is commensurable to Λ.* □

3. Algebraic approximate subgroups

In this section, we are going to study Zariski closures of approximate subgroups of linear algebraic groups. Throughout this section, we fix a field k and a linear algebraic group $\mathbf{G}$ defined over k and denote by $G := \mathbf{G}(k)$ its group of k-points.

https://doi.org/10.1017/fms.2019.39 Published online by Cambridge University Press

We equip G with its Zariski topology and given a subset $A \subset G$ we denote by $\overline{A}^Z$ its Zariski closure in G.

3.1. Zariski closures of approximate subgroups. We first observe that approximate subgroups are preserved under Zariski closure:

LEMMA 15 (Zariski closures of approximate subgroups). *If $\Lambda \subset G$ is an approximate subgroup, then its Zariski closure $\overline{\Lambda}^Z \subset G$ is an approximate subgroup as well.*

Note that if we equip G and $G \times G = (\mathbf{G} \times \mathbf{G})(k)$ with their respective Zariski topologies then inversion and multiplication are continuous. Since the Zariski topology on $G \times G$ is finer than the product topology, G is not a topological group with respect to the Zariski topology. It is however a (non-Hausdorff) *semitopological* group in the sense that multiplication is separately continuous. Thus Lemma 15 is a special case of the following general result:

PROPOSITION 16. *Let G be a semitopological group (not necessarily Hausdorff). Then the closure of an approximate subgroup of G is again an approximate subgroup.*

Proof. Let F_Λ be as in (1) and let H be the closure of Λ in G. Enumerate $F_\Lambda = \{g_1, \dots, g_N\}$ and let $\lambda \in \Lambda$. Then using the fact that left and right multiplication by an element of G is a homeomorphism we obtain

$$\lambda H = \lambda \overline{\Lambda} = \overline{\lambda \Lambda} \subset \overline{\Lambda^2} \subset \overline{\Lambda F_\Lambda} = \overline{\bigcup_{i=1}^{N} \Lambda g_i} \subset \bigcup_{i=1}^{N} \overline{\Lambda g_i} = \bigcup_{i=1}^{N} \overline{\Lambda} g_i = H F_\Lambda$$

and hence $\Lambda H \subset H F_\Lambda$. Since the right-hand side is closed as a finite union of closed set we deduce that for every $h \in H$,

$$Hh = \overline{\Lambda} h = \overline{\Lambda h} \subset \overline{\Lambda H} \subset H F_\Lambda,$$

which shows that $H^2 \subset H F_\Lambda$ and finishes the proof. □

3.2. Algebraic approximate subgroups are almost subgroups. The following is the main result of this section:

THEOREM 17 (Algebraic approximate subgroups are almost subgroups). *Let k be a field and $\mathbf{G}$ be a linear algebraic group over k. If $\Lambda \subset \mathbf{G}(k)$ is an approximate*

https://doi.org/10.1017/fms.2019.39 Published online by Cambridge University Press

subgroup, then there exists a connected k-algebraic subgroup $\mathbf{H}$ of $\mathbf{G}$, an element $g \in N_{\mathbf{G}(k)}(\mathbf{H}(k))$ and a finite subset $F \subset \mathbf{G}(k)$ such that

$$g\mathbf{H}(k) \subset \overline{\Lambda}^Z \subset F\mathbf{H}(k) \cap \mathbf{H}(k)F. \tag{2}$$

For the proof we recall that a topological space is called *irreducible* if it is nonempty and it is not the union of two proper closed subsets. We call a (not necessarily Zariski-closed) subset of $G := \mathbf{G}(k)$ irreducible, if it is irreducible with respect to the restriction of the Zariski topology from G. We need the following lemma, which will be proved in the next subsection.

LEMMA 18. *Let $A, B \subset G$ be irreducible subsets. Then $A^{-1}B$ is irreducible as well.*

Proof of Theorem 17. In view of Lemma 15 we can replace Λ by its Zariski closure and thereby assume that Λ is Zariski-closed. Let $F_\Lambda = \{g_1, \dots, g_N\}$ be as in (1) and let $\Lambda_0, \dots, \Lambda_m$ be the irreducible components of Λ, where Λ_0 is an irreducible component of maximal dimension. We claim that if $\lambda_0 \in \Lambda_0$, then $H := \lambda_0^{-1}\Lambda_0$ is a subgroup of $\mathbf{G}(k)$. Since $e \in H$ we have $H \subset H^{-1}H$, and it remains to show $H^{-1}H \subset H$. We have

$$H^{-1}H = \Lambda_0^{-1}\Lambda_0 \subset \Lambda^2 \subset \bigcup_{i=1}^{N} g_i\Lambda = \bigcup_{i=1}^{N}\bigcup_{j=0}^{m} g_i\Lambda_j.$$

Now H is irreducible, since Λ_0 is irreducible, and hence also $H^{-1}H$ is irreducible by Lemma 18. There thus exists $i \in \{1, \dots, n\}$ and an irreducible component Λ_j of Λ such that

$$g_i\Lambda_j \supset H^{-1}H \supset H, \tag{3}$$

where we have used that $e \in H^{-1}$. Since $g_i\Lambda_j$ is Zariski-closed we also have $\overline{H^{-1}H}^Z \subset g_i\Lambda_j$. Since Λ_0 was an irreducible component of maximal dimension, we have

$$\dim g_i\Lambda_j = \dim \Lambda_j \leqslant \dim \Lambda_0 = \dim H \leqslant \dim \overline{H^{-1}H}^Z \leqslant \dim g_i\Lambda_j.$$

We deduce that $\dim H = \dim \overline{H^{-1}H}^Z$, and since $H \subset H^{-1}H$ and the latter is irreducible we have $H = H^{-1}H$, hence the claim is proved.

We deduce that H is an irreducible Zariski-closed subgroup of G, and hence $H = \mathbf{H}(k)$, where $\mathbf{H}$ is the connected k-algebraic subgroup of $\mathbf{G}$ defined by the vanishing ideal of H. With $g := \lambda_0$ we then have $g\mathbf{H}(k) = gH = \Lambda_0 \subset \Lambda$.

https://doi.org/10.1017/fms.2019.39 Published online by Cambridge University Press

Next we show that finitely many left cosets of H, or equivalently Λ_0, cover Λ. This will then imply that also finitely many right cosets of H cover Λ by symmetry of H and Λ. Assume otherwise; then the set $\{\lambda\Lambda_0 \mid \lambda \in \Lambda\}$ would be infinite, hence we could find a sequence (λ_n) in Λ such that $\lambda_i\Lambda_0 \neq \lambda_j\Lambda_0$ for all $i \neq j$. Since Λ_0 is a coset of a group we thus have a *disjoint* union

$$\bigsqcup_{n=1}^{\infty} \lambda_n \Lambda_0 \subset \Lambda^2 \subset \bigcup_{i=1}^{N} g_i \Lambda \subset \bigcup_{i=1}^{N} \bigcup_{j=0}^{m} g_i \Lambda_j$$

By the pigeonhole principle, one of the irreducible sets $g_i\Lambda_j$ would thus be a disjoint union of infinitely many irreducible subsets of the form $\lambda_n\Lambda_0$. Since $\dim \lambda_n\Lambda_0 \geqslant \dim g_i\Lambda_j$, this is a contradiction. We have thus established (2), and it remains only to show that $g \in N_G(H)$.

Let F_Λ be finite such that $\Lambda^2 \subset \Lambda F_\Lambda$ and let F as in (2). Then

$$gHg^{-1} = gH(gH)^{-1} \subset \Lambda\Lambda^{-1} = \Lambda^2 \subset \Lambda F_\Lambda \subset HFF_\Lambda.$$

Since gHg^{-1} is irreducible and the irreducible components of HFF_Λ are the left-H-cosets in HFF_Λ, we must have $gHg^{-1} \subset Hf$ for some $f \in FF_\Lambda$. Since $e \in gHg^{-1}$ this forces $f \in H^{-1}$ and thus $gHg^{-1} = H$, that is, g normalizes H. □

3.3. Product sets of irreducible sets are irreducible. This subsection is devoted to the proof of Lemma 18. We keep the notation of the previous subsection. In particular, k denotes a field, $\mathbf{G}$ a linear algebraic group defined over k and $G := \mathbf{G}(k)$.

To show Lemma 18 we first observe that the image of an irreducible topological space under a continuous map is irreducible, and that the map $q : G \times G \to G$ given by $q(a, b) := a^{-1}b$ is continuous, if G and $G \times G = (\mathbf{G} \times \mathbf{G})(k)$ are equipped with their respective Zariski topology. Since $A^{-1}B = q(A \times B)$ it thus suffices to show that if A and B are irreducible subsets of G, then $A \times B$ is irreducible in $G \times G$. We can choose a representation $\rho : \mathbf{G} \to \mathbf{GL}_n$ defined over k and thereby consider G as a subset of k^{n^2}. It then suffices to establish the following (see [**11**, Exercise 3.15 (a)]):

LEMMA 19. *Let X be an irreducible subset of k^n, and let Y be an irreducible subset of k^m. Then $X \times Y$ is irreducible in k^{n+m}.*

Proof. Let Z_1 and Z_2 be closed subsets of k^{n+m}, with corresponding vanishing ideals I_{Z_1} and I_{Z_2} in $k[T_1, \ldots, T_{n+m}]$. Assume that $X \times Y$ is contained in $Z_1 \cup Z_2$. We have to show that either $X \times Y \subset Z_1$ or $X \times Y \subset Z_2$. For $i \in \{1, 2\}$ we denote $X_i = \{x \in X \mid \{x\} \times Y \subset Z_i\}$.

https://doi.org/10.1017/fms.2019.39 Published online by Cambridge University Press

We first claim that $X \subset X_1 \cup X_2$. Indeed, $\{x\} \times Y$ is the image of Y under the Zariski continuous map $k^m \to k^{n+m}$ given by $b \mapsto (x, b)$, hence an irreducible subset of k^{n+m}, and thus of $Z_1 \cup Z_2$. We deduce that for all $x \in X$ we have either $\{x\} \times Y \subset Z_1$ or $\{x\} \times Y \subset Z_2$, which proves the first claim.

Secondly, we claim that X_1 is closed in k^n. Given $b \in k^m$ and $f \in k[T_1, \ldots, T_{n+m}]$, we define $f_b \in k[T_1, \ldots, T_n]$ by $f_b(T_1, \ldots, T_n) := f(T_1, \ldots, T_n, b_1, \ldots, b_m)$. Now for every $x \in X$ we have $\{x\} \times Y \subset Z_1$ if and only if $f(x, y) = 0$ for all $f \in I_{Z_1}$ and all $y \in Y$, and hence

$$X_1 = \{x \in X \mid f_y(x) = 0 \text{ for all } y \in Y \text{ and for all } f \in I_{Z_1}\}$$

is closed. This proves the second claim, and the same argument shows that X_2 is closed in k^n.

We have written $X = X_1 \cup X_2$ as the union of two proper closed subsets. Since X is irreducible this implies that either $X \subset X_1$ or $X \subset X_2$. Consequently we have either $X \times Y \subset Z_1$ or $X \times Y \subset Z_2$, which finishes the proof. □

This finishes the proof of Lemma 18.

4. Proof of Borel density

4.1. General setting. Throughout this section k denotes a local field, $\mathbf{G}$ is a connected linear algebraic group defined over k and Λ denotes an approximate subgroup of $G := \mathbf{G}(k)$. By Theorem 17 there exists an algebraic subgroup $\mathbf{H}$ of $\mathbf{G}$, an element $g \in G$ and a finite subset $F \subset G$ such that

$$g\mathbf{H}(k) \subset \overline{\Lambda}^Z \subset F\mathbf{H}(k) \cap \mathbf{H}(k)F.$$

We abbreviate $H := \mathbf{H}(k)$ enumerate $F = \{g_1, \ldots, g_N\}$ so that

$$\Lambda \subset \overline{\Lambda}^Z \subset \bigcup_{j=1}^{n} g_j H \quad \text{and} \quad \Lambda \subset \overline{\Lambda}^Z \subset \bigcup_{j=1}^{n} H g_j. \tag{4}$$

We then have to show that $H = G$. The argument for this will be different in the uniform case (where we use relative denseness of Λ) and in the strong case (where we use a joining argument between the hulls of Λ and H).

4.2. The uniform case. We consider the general setting (and notation) of Section 4.1. In addition we are going to assume that $\Lambda \subset G$ is a *uniform* approximate lattice. From this assumption and (4) one immediately deduces:

https://doi.org/10.1017/fms.2019.39 Published online by Cambridge University Press

LEMMA 20. *If Λ is a uniform approximate lattice, then H is cocompact in G.*

Proof. Since Λ is relatively dense, so is its superset $g_1 H \cup \cdots \cup g_n H$. It thus follows from Lemma 7 that there exists $j \in \{1, \ldots, n\}$ such that $(g_j H)^{-1} g_j H = H^{-1} H = H$ is relatively dense, that is, cocompact. □

There is also a more subtle consequence of (4) and the assumption that Λ be uniform. Namely, let $\Lambda_j := \Lambda \cap g_j H$ so that $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_N$. Since Λ is relatively dense in G, Lemma 7 implies that there exists $j \in \{1, \ldots, N\}$ such that $\Delta_j := \Lambda_j^{-1} \Lambda_j$ is relatively dense in G. Note that

$$\Delta_j = (\Lambda \cap g_j H)^{-1} (\Lambda \cap g H_j) \subset \Lambda^2 \cap H \subset \Lambda^2.$$

In particular, Δ_j is a symmetric subset of the uniform approximate lattice Λ^2 which contains the identity and is relatively dense in G. It thus follows from [2, Corollary 2.10], that Δ_j is a uniform approximate lattice in G itself. Since $\Delta_j \subset H$ we deduce:

LEMMA 21. *It Λ is a uniform approximate lattice, then H contains a uniform approximate lattice.*

In [2, Theorem 5.8], it was established that if a compactly generated lcsc group contains a uniform approximate lattice, then it is unimodular. In Theorem 32 in Appendix B, we show that this also holds without the assumption of compact generation. This then implies that H is unimodular, in addition to being cocompact. We have established:

THEOREM 22. *Let k be a local field and $\mathbf{G}$ be a connected affine algebraic group defined over k. Assume that $\mathbf{G}$ does not contain any proper algebraic subgroup $\mathbf{H}$ such that $\mathbf{H}(k)$ is unimodular and cocompact in $\mathbf{G}(k)$. Then every uniform approximate lattice in $\mathbf{G}(k)$ is Zariski-dense.*

Proof. In view of Lemma 20, Lemma 21 and Theorem 32, the assumption forces $H = G$, hence Λ is Zariski-dense. □

COROLLARY 23 (Borel density, uniform case). *Let k be a local field, $\mathbf{G}$ be a connected semisimple algebraic group and assume that $G := \mathbf{G}(k)$ has no compact factors. Then every uniform approximate lattice in G is Zariski-dense.*

Proof. It only remains to check that the assumptions of Theorem 22 are satisfied in this case. This follows from [19, Corollary 2.3]. □

https://doi.org/10.1017/fms.2019.39 Published online by Cambridge University Press

4.3. The nonuniform case. We consider the general setting (and notation) of Section 4.1, but assume in addition that $\Lambda \subset G$ is a *strong* approximate lattice. We also fix a left-Haar measure m_G and an admissible probability measure $\mu = \rho m_G$ on G. We assume that μ is symmetric and that $\rho \in L^1(G) \cap L^\infty(G)$. From the results of Section 2.5, we then deduce:

LEMMA 24. *If $H \neq G$, then there exists a μ-stationary ergodic probability measure ν on G/H which is not μ-proximal.*

Proof. Since Λ is a strong approximate lattice, we can choose a nontrivial G-invariant ergodic probability measure on Ω_Λ. Applying Lemma 31 to the canonical projection $\Omega_{\Lambda,H} \to \Omega_\Lambda$ we conclude (as in Section 2.5) that this measure lifts to a μ-stationary ergodic probability measure on $\Omega_{\Lambda,H}$, which then in turn pushes forward to a μ-stationary ergodic probability measure ν on Ω_H. Since $\Lambda \subset HF$ we deduce from Corollary 10 that ν is nontrivial, hence it is supported on the orbit $G/H \subset \Omega_H$ by Lemma 6. To see that ν is not μ-proximal it suffices to show by Proposition 12 that if $H' \in \Omega_H$, then $H'F \subsetneq G$, that is, that H' has infinite index in G. Since $\mathbf{G}$ is connected and $\mathbf{H}$ is a proper algebraic subgroup of $\mathbf{G}$ we have $\dim \mathbf{H} < \dim \mathbf{G}$, and hence H has infinite index in G. This implies that every $H' \in \Omega_H$ also has infinite index in G, since $\Omega_H \subset G/H \cup \{\emptyset\}$. □

We thus have to investigate, which homogeneous spaces of the form G/H admit nonproximal stationary probability measures. Note that since G contains a strong approximate lattice, it is automatically unimodular by [**2**, Theorem 5.8].

LEMMA 25. *Assume that G/H admits a μ-stationary probability measure ν. Then ν is actually the unique μ-stationary probability measure on G/H, and if H is unimodular, then ν is G-invariant, and hence H has finite covolume.*

Proof. Every μ-stationary probability measure on G/H is G-quasi-invariant. If there was more than one μ-stationary μ-probability measure on G/H, then there would be two different ergodic such measures, and these would then be mutually singular. This would contradict the fact that the quotient G/H admits a unique G-invariant measure class.

Now assume that H is unimodular, and denote by η the unique G-invariant Radon measure on G/H. We are going to show that $\eta(G/H) < \infty$; this will finish the proof, since ν will then be a multiple of η by the uniqueness statement.

Since ν and η both represent the unique G-invariant measure class on G/H we can write $\nu = u\eta$ for some nonnegative η-integrable Borel function u on G/H. Since ν is μ-stationary and $\mu = \rho m_G$ is symmetric we deduce that

$$\rho * u = u.$$

https://doi.org/10.1017/fms.2019.39 Published online by Cambridge University Press

Moreover, since $\rho \in L^\infty(G)$ we have, by Hölder's inequality,

$$\|u\|_\infty = \|\rho * u\|_\infty \leqslant \|\rho\|_\infty \|u\|_1 < \infty.$$

In particular, $u \in L^1(\eta) \cap L^\infty(\eta) \subset L^2(\eta)$ and u is continuous. Since $\mu * u = \check{\mu} * u = u$ and η is G-invariant we have

$$\begin{aligned}
&\int_{G/H} \int_G \left|u(gx) - u(x)\right|^2 d\mu(g)\, d\eta(x) \\
&\quad = \int_{G/H} \int_G u(gx)^2 d\mu(g)\, d\eta(x) - 2\int_{G/H} \int_G u(gx)\, d\mu(g)\, u(x)\, d\eta(x) + \int_{G/H} u(x)^2 d\eta(x) \\
&\quad = \int_G \int_{G/H} u(gx)^2 d\eta(x)\, d\mu(g) - 2\int_{G/H} (\rho * u)(x) u(x)\, d\eta(x) + \int_{G/H} u(x)^2 d\eta(x) \\
&\quad = \int_G \int_{G/H} u(x)^2 d\eta(x)\, d\mu(g) - 2\int_{G/H} u(x) u(x)\, d\eta(x) + \int_{G/H} u(x)^2 d\eta(x) = 0.
\end{aligned}$$

We conclude that u (as an element in L^2) is invariant under μ-a.e. $g \in G$. Since the support of μ generates G, u is G-invariant and thus constant. We thus have $\nu = \lambda\eta$ for some $\lambda > 0$ and hence $\eta(G/H) = \lambda^{-1}\nu(G/H) = \lambda^{-1} < \infty$. This finishes the proof. □

COROLLARY 26. *Let k be a local field, $\mathbf{G}$ be a connected algebraic group over k and assume that every Zariski-closed proper subgroup of $\mathbf{G}(k)$ is contained in a closed subgroup $M < G$ satisfying one of the following three properties:*

(i) *M is unimodular of infinite covolume in G.*

(ii) *G/M admits a unique μ-stationary measure which is μ-proximal.*

(iii) *G/M does not admit a μ-stationary probability measure.*

Then every strong approximate lattice in $\mathbf{G}(k)$ is Zariski-dense.

Proof. In the notation of Section 4.1, we have to show that $H = G$. Assume otherwise, and let $M < G$ be a subgroup containing H as in the corollary. By Lemma 24 there exists a μ-stationary measure ν on G/H, and we denote by ν_1 its push-forward to G/M. Since ν is μ-stationary, but not μ-proximal the same holds for ν_1, and hence (ii) and (iii) cannot hold. This forces M to be unimodular of infinite covolume, which contradicts Lemma 25. □

We conclude:

THEOREM 27 (Borel density for strong approximate lattices). *Let k be a local field and let $\mathbf{G}$ be a connected semisimple algebraic group over k. If $G := \mathbf{G}(k)$*

https://doi.org/10.1017/fms.2019.39 Published online by Cambridge University Press

does not have any compact factors, then every strong approximate lattice $\Lambda \subset G$ is Zariski-dense.

Proof. We have to check that the conditions of Corollary 26 are satisfied. By [19, Lemma 2.2] every Zariski-closed subgroup of G is contained in either a parabolic subgroup P of G or in an algebraic subgroup M whose identity component M^0 is reductive with anisotropic centre over k. In the first case, G/P admits a unique μ-stationary measure which is μ-proximal [13, Theorem VI.3.7]. In the second case, M is unimodular, since M^0 is reductive and $[M : M^0]$ is finite. In this case, it then has infinite covolume in G by [19, Corollary 2.3]. Now the theorem follows from Corollary 26. □

5. Variants and refinements

5.1. Dani–Shalom density. Let $\mathbf{G}$ be a connected affine algebraic group defined over a local field k. In this subsection, we assume that $G = \mathbf{G}(k)$ is amenable. In particular, this is the case if $\mathbf{G}$ is solvable.

THEOREM 28 (Dani–Shalom density for strong approximate lattices). *Assume that G does not contain any proper normal cocompact algebraic subgroup. Then every strong approximate lattice and every uniform approximate lattice in G is Zariski-dense.*

Proof. Since G is amenable, every uniform approximate lattice in G is strong, hence we assume that $\Lambda \subset G$ is a strong approximate lattice. We then define H as in Section 4.1. There then exists a finite set F such that $\Lambda \subset HF$. Since Λ is a strong approximate lattice, there exists a G-invariant ergodic probability measure ν_Λ on Ω_Λ. Since G is amenable, this measure is part of a G-invariant ergodic hull joining $(\nu_{\Lambda,H}, \nu_\Lambda, \nu_H)$ between Ω_Λ and Ω_H, and the invariant probability measure ν_H is nontrivial by Lemma 10, hence supported on the orbit $G/H \subset \Omega_H$ by Lemma 6. In particular, ν_H has full support on G/H.

Now $\mathrm{supp}(\nu_H) = G/H$ is a subset of $(\mathbf{G}/\mathbf{H})(k)$, and since $\mathbf{G}$ acts algebraically on $\mathbf{G}/\mathbf{H}$, it follows from [18, Theorem 1.1] (which generalizes [8, Corollary 2.6]) that this support consists entirely of G-fixpoints. This forces $H = G$ and finishes the proof. □

REMARK 29. Solvable algebraic groups over local fields k whose k points do not admit any proper normal algebraic cocompact subgroups are called *k-discompact* and have been characterized by Shalom in [18, Theorem 3.6]. They are precisely the k-algebraic solvable group which are *k-split* in the sense that

https://doi.org/10.1017/fms.2019.39 Published online by Cambridge University Press

every composition factor is isomorphic to either the additive or multiplicative group over k.

Taken together, Theorems 22, 27, 28 and Remark 29 now yield our Main Theorem from the introduction.

5.2. The case of unipotent groups. Theorem 28 applies in particular to unipotent algebraic groups. Over local fields k of positive characteristic there do exist nonsplit unipotent groups.

EXAMPLE 2 (Rosenlicht). Let p be a prime, $k := \mathbb{F}_p((t))$ and consider the algebraic subgroup of the additive group of k^2 whose k-points are given by

$$\{(x, y) \in k^2 \mid y^p = tx^p - x\}.$$

One checks that its group of k-points is actually infinite and contained in $(F_p[[t]])^2$, hence it is an example of a *compact* unipotent group. Every finite subset of this group is thus a uniform approximate lattice, which is not Zariski-dense.

The natural context of this example is that of k-wound unipotent groups. A unipotent algebraic group $\mathbf{G}$ defined over a field k is called *k-wound* if every k-morphism from the additive group of k to $\mathbf{G}$ is constant. Over a local field this is equivalent to compactness of $\mathbf{G}(k)$ [**16**, Section VI, 1, Théorème]. If k is of characteristic 0, then every k-wound unipotent groups is trivial [**15**, Corollary 14.55], but Rosenlicht's example shows that nontrivial examples exist in positive characteristic. In general, if $\mathbf{G}$ is any unipotent algebraic group over an arbitrary field k, then there exists a unique k-split unipotent normal subgroup $\mathbf{G}_s$ of $\mathbf{G}$, such that the quotient group $\mathbf{G}/\mathbf{G}_s$ is k-wound [**21**, page 733, 4.2, Theorem]. In characteristic 0 we thus have $\mathbf{G} = \mathbf{G}_s$, hence we deduce:

EXAMPLE 3. Let k be a local field of characteristic 0 and let $\mathbf{G}$ be a unipotent algebraic group over k. Then every strong approximate lattice and every uniform approximate lattice in $\mathbf{G}(k)$ is Zariski-dense.

We do not know whether Theorem 28 holds for approximate lattices which are not strong. However, the following example shows that it does not hold for *weak approximate lattices*. Here, a discrete approximate subgroup Λ of an lcsc group G is called a *weak approximate lattice* if its hull admits a nontrivial μ-stationary probability measure for *some* admissible probability measure μ on G.

https://doi.org/10.1017/fms.2019.39 Published online by Cambridge University Press

EXAMPLE 4. Consider the algebraic group $\mathbf{G} := \mathbf{GL}_1 \ltimes \mathbb{A}^1$ over $\mathbb{R}$ so that $G = \mathbf{G}(\mathbb{R}) = \mathbb{R}^\times \ltimes \mathbb{R}$ is the $(ax+b)$-group. By [2, Section 5.4] the subgroup $\Lambda := \{1\} \ltimes \mathbb{Z}$ is a weak approximate lattice in G, but its Zariski closure is given by $\{1\} \ltimes \mathbb{R}$. Thus weak approximate lattices in the $(ax+b)$-group need not be Zariski-dense, despite the fact that G is solvable without cocompact algebraic subgroups.

5.3. Thin approximate subgroups of abelian groups. If $\mathbf{G}$ is an algebraic group over a local field k, then an approximate subgroup $\Lambda \subset G$ is called *thin* if it is Zariski-dense, but not an approximate lattice. In the case where Λ is an actual subgroup one recovers the notion of a thin subgroup. It is well known that nilpotent algebraic groups over $\mathbb{R}$ do not admit thin subgroups. On the contrary we show:

PROPOSITION 30. *The additive group $\mathbb{R}^2$ admits thin approximate subgroups.*

Proof. We set

$$\Gamma = \{(m + n\sqrt{2}, m - n\sqrt{2}) \mid m, n \in \mathbb{Z}\} \quad \text{and} \quad S = \mathbb{R} \times [-1, 1].$$

It is easy to check that $\Lambda := \Gamma \cap S$ is an infinite approximate subgroup, but not a uniform approximate lattice, hence not an approximate lattice at all by [2, Corollary 4.19]. It remains to show that Λ is Zariski-dense. Otherwise, by Theorem 17, Λ would be contained in a finite union of translates of a fixed proper algebraic subgroup of G. Since Λ is infinite, H would have to be nontrivial, hence a line. This implies that either $H \subset S$ or that $H \cap S$ is compact. In the second case, $S \cap \Lambda$ would have to be contained in a compact subset of G; since Γ is discrete, this implies that Λ is finite, a contradiction. Thus $H \subset S$, and thus all points of Λ lie on a finite union of lines which are parallel to the line $\mathbb{R} \times \{0\}$. Then there exist $\alpha_1, \ldots, \alpha_N \in [-1, 1]$ such that

$$\Lambda \subset \bigcup_{k=1}^{N} \{(m + n\sqrt{2}, m - n\sqrt{2}) \mid m - n\sqrt{2} = \alpha_k\}.$$

Thus the second coordinate of elements of Λ can take only finitely many values, but since the first coordinate is just the Galois conjugate of the second coordinate we deduce that Λ is actually finite, which is a contradiction. □

Appendix A. Lifting stationary measures

The purpose of this appendix is to record a proof of the following fact from measure theory, to be used in Section 2.5. Given an lcsc group G and a

https://doi.org/10.1017/fms.2019.39 Published online by Cambridge University Press

compact G-space Ω and an admissible probability measure μ on G, we denote by $\mathrm{Prob}_\mu(\Omega) \subset \mathrm{Prob}(\Omega)$ the compact convex sets of μ-stationary, respectively arbitrary probability measures on Ω.

LEMMA 31. *Let X and Y be compact G-spaces, and suppose that there exists a continuous surjective G-map $\pi : X \to Y$. Then the induced map $\pi_* : \mathrm{Prob}_\mu(X) \to \mathrm{Prob}_\mu(Y)$ is surjective as well, and maps ergodic measures surjectively onto ergodic measures.*

Proof. We first show surjectivity of the map $\pi_* : \mathrm{Prob}(X) \to \mathrm{Prob}(Y)$. Thus let $\eta \in \mathrm{Prob}(Y)$ and define $\eta' : \pi^*(C(Y)) \to \mathbb{R}$ by $\eta'(\pi^*(f)) := \eta(f)$, which is well defined since π is surjective and thus $\pi^* : C(Y) \to C(X)$ is injective. We have $\|\eta'\| \leqslant \|\eta\| = 1$ and since $\eta'(1) = 1$ we deduce that $\|\eta'\| = 1$. By Hahn–Banach we can thus extend η' to a continuous linear functional η'' of norm 1 on all of $C(X)$, and by construction $\pi_*\eta'' = \eta$. It thus remains to show only that η'' is a positive linear functional on $C(X)$. Thus let $f \in C(X)$ be nonnegative so that $\|f\|_\infty \geqslant \|f\|_\infty - f \geqslant 0$. Since $\|\eta''\| = 1$ and $\eta''(1) = 1$ we have

$$\|f\|_\infty \geqslant \|f\|_\infty - f \geqslant |\eta''(\|f\|_\infty - f)| \geqslant \eta''(\|f\|_\infty - f) = \|f\|_\infty - \eta''(f),$$

and hence $\eta''(f) \geqslant 0$. This shows that η'' is positive, and hence $\pi_* : \mathrm{Prob}(X) \to \mathrm{Prob}(Y)$ is surjective. Given $\eta \in \mathrm{Prob}(Y)$ we now defined weak-$*$-compact convex sets by

$$F(\eta) := \{\nu \in \mathrm{Prob}(X) \mid \pi_*\nu = \eta\} \quad \text{and} \quad F_\mu(\eta) := \{\nu \in \mathrm{Prob}_\mu(X) \mid \pi_*\nu = \eta\}.$$

We have just seen that $F(\eta)$ is nonempty for every $\eta \in \mathrm{Prob}(Y)$, and if η is moreover μ-stationary, then it is invariant under convolution by μ, since π is G-equivariant and thus for all $\nu \in F(\eta)$ we have

$$\pi_*(\mu * \nu) = \mu * (\pi_*\nu) = \mu * \eta = \eta.$$

It then follows from the Markov–Kakutani fixpoint theorem that $F_\mu(\eta)$ is also nonempty. This shows that $\pi_* : \mathrm{Prob}_\mu(X) \to \mathrm{Prob}_\mu(Y)$ is surjective.

For the second statement we first recall from [1, Corollary 2.7] that the ergodic μ-stationary probability measures are precisely the extremal points of the convex compact set of all μ-stationary probability measures. Assume now that $\eta \in \mathrm{Prob}_\mu(Y)$ is ergodic and let ν be an extremal point of $F_\mu(\eta)$, which exists by the Krein–Milman theorem since $F_\mu(\eta) \neq \emptyset$. We claim that ν is ergodic, that is, an extremal point of $\mathrm{Prob}_\mu(X)$. Otherwise we could write ν as $\nu = \alpha_1\nu_1 + \alpha_2\nu_2$ for some $\alpha_1, \alpha_2 \in (0, 1)$ with $\alpha_1 + \alpha_2 = 1$ and $\nu_1, \nu_2 \in \mathrm{Prob}_\mu(X)$. But then

$$\eta = \pi_*\nu = \alpha_1\pi_*\nu_1 + \alpha_2\pi_*\nu_2,$$

https://doi.org/10.1017/fms.2019.39 Published online by Cambridge University Press

and hence ergodicity of η forces $\pi_*\nu_1 = \pi_*\nu_2 = \eta$ and hence $\nu_1, \nu_2 \in F_\mu(\eta)$. This contradicts extremality of ν in $F_\mu(\eta)$, and hence ν must have been ergodic. □

Appendix B. The unimodularity theorem revisited

The following theorem is used in the proof of Borel density in the uniform case:

THEOREM 32 (Unimodularity theorem, refined version). *Let G be an lcsc group which contains a uniform approximate lattice Λ. Then G is unimodular.*

Under the additional assumption that G be *compactly generated*, this theorem was established in [2, Theorem 5.8]. We revisit the proof to establish the above stronger version; we use this opportunity to correct a few inequalities in the original proof. The following lemma replaces [2, Lemma 5.10]. Here, m_G denotes a fixed choice of left-Haar measure on G and Δ_G denotes the modular function of G.

LEMMA 33. *Assume that G is a nonunimodular lcsc group. Then there exists $\rho \in C(G)$ with the following properties:*

(i) $\rho(g) > 0$ *for all* $g \in G$ *(hence in particular* $\operatorname{supp}(\rho) = G$*).*

(ii) $\int_G \rho(t)\, dm_G(t) = 1$.

(iii) $\int_G \rho(t)\Delta_G(t)\, dm_G(t) > 1$.

We use following simple observation, which we leave as an exercise:

LEMMA 34. *Let S be a countable set. Then for every function $b : S \to [0, \infty)$ there exists $a : S \to (0, \infty)$ such that*

$$\sum_{s\in S} a(s) = 1 \quad \text{and} \quad \sum_{s\in S} a(s)b(s) < \infty. \qquad \square$$

Proof of Lemma 33. We first construct a function $\rho_0 \in C(G)$ which satisfies (i), (ii) and

(iii′) $\gamma := \int_G \rho_0(s)\Delta_G(s)\, dm_G(s) < \infty$.

To construct ρ_0 we pick a countable dense subset $S \subset G$ (which exists since G is second countable) and define $b : S \to [0, \infty)$ by $b(s) := \Delta_G(s)^{-1}$. Using Lemma 34 we then choose a function $a : S \to [0, \infty)$ such that

$$\sum_{s\in S} a(s) = 1 \quad \text{and} \quad \sum_{s\in S} a(s)\Delta_G(s)^{-1} < \infty. \tag{B.1}$$

https://doi.org/10.1017/fms.2019.39 Published online by Cambridge University Press

Next we pick a compactly supported function $\varphi \in C_c(G)$ with $\varphi \geqslant 0$ and $\|\varphi\|_1 = 1$ and set

$$\rho_0(t) := \sum_{s \in S} a(s)\varphi(st).$$

We check that $\rho_0(t)$ satisfies (i), (ii) and (iii′): Property (i) is immediate, since S is dense in G and φ is positive on a nonempty open set, and (ii) follows from

$$\begin{aligned}\int_G \rho_0(t)\, dm_G(t) &= \sum_{s \in S} a(s) \int_G \varphi(st)\, dm_G(t) \\ &= \sum_{s \in S} a(s) \int_G \varphi(t)\, dm_G(t) = \sum_{s \in S} a(s) = 1.\end{aligned}$$

Finally, (iii′) follows from the fact that

$$\begin{aligned}\int_G \rho(t)\Delta_G(t)\, dm_G(t) &= \sum_{s \in S} a(s) \int_G \varphi(st)\Delta_G(t)\, dm_G(t) \\ &= \sum_{s \in S} a(s)\Delta(s)^{-1} \int_G \varphi(t)\Delta_G(t)\, dm_G(t).\end{aligned}$$

Since φ is compactly supported, the integral $\int_G \varphi(t)\Delta_G(t)\, dm_G(t)$ converges, and thus the sum is finite by (B.1). We have thus constructed ρ_0 satisfying (i), (ii) and (iii′).

Now we choose $a > 0$ such that $a\gamma > 1/2$; since Δ_G is unbounded we then find $s \in G$ such that $(1-a)\gamma\Delta(s)^{-1} > 1/2$. We claim that

$$\rho(t) := a\rho_0(t) + (1-a)\rho_0(st)$$

satisfies (i)–(iii). Here, (i) and (ii) are immediate from the corresponding properties of ρ_0 and left invariance of m_G. Concerning (iii) we observe that, using left invariance of m_G and the fact that Δ_G is a homomorphism,

$$\begin{aligned}\int_G \rho(t)\delta(t)\, dm_G(t) &= a\gamma + (1-a) \int_G \rho_0(t)\Delta_G(s^{-1}t)\, dm_G(t) \\ &= a\gamma + (1-a)\gamma\Delta_G(s)^{-1} \quad > \quad 1/2 + 1/2 \quad = \quad 1.\end{aligned}$$

This establishes (iii) and finishes the proof. □

Towards the proof of Theorem 32 we now assume for contradiction that G contains a uniform approximate lattice Λ, but is nonunimodular. We then choose ρ as in Lemma 33 and define an admissible probability measure μ on G by

$$\mu := \rho m_G.$$

https://doi.org/10.1017/fms.2019.39 Published online by Cambridge University Press

By the Markov–Kakutani fixpoint theorem there then exists a μ-stationary probability measure ν on Ω_Λ. Since Λ is relatively dense we have $\emptyset \notin \Omega_\Lambda$, and since Λ is uniformly discrete, every element of Ω_Λ is uniformly discrete and hence $G \notin \Omega_\Lambda$. This shows that ν is nontrivial.

As explained in [2, Section 5.1] we have a well-defined continuous periodization map

$$\mathcal{P} : C_c(G) \to C(\Omega_\Lambda), \quad \mathcal{P}f(\Lambda') = \sum_{x \in \Lambda'} f(x).$$

In particular we can define a Radon measure on G by $\eta(f) := \nu(\mathcal{P}f)$.

LEMMA 35. *There exists $u \in L^1_{\mathrm{loc}}(G, m_G)$ such that $\eta = u\, m_G$.*

Proof. Since $[m_G]$ is the unique G-invariant measure class on G, it suffices to show that η is G-quasi-invariant. This will follows from the fact that ν is μ-stationary and hence G-quasi-invariant. To see this, let $K \subset G$ be a compact subset of positive measure with characteristic function 1_K. There then exists a compact set $L \supset K$ and functions $f_n \in C_c(G)$ supported in L such that $f_n \geqslant 1_K$ and $\liminf f_n = 1_K$. Then $\lim \eta(f_n) = \eta(K) > 0$, and hence for all sufficiently large n we have

$$\nu(\mathcal{P}(f_n)) = \eta(f_n) > 0.$$

Since $\mathcal{P}$ is G-equivariant, ν is G-quasi-invariant and $\nu(\mathcal{P}(f_n)) > 0$ for all sufficiently large n, we have for all $g \in G$,

$$\eta(gK) = \lim \eta(g \,.\, f_n) = \lim \nu(\mathcal{P}(g \,.\, f_n)) = \lim \nu(g \,.\, \mathcal{P}(f_n)) = \lim g_* \nu(\mathcal{P}(f_n)) > 0.$$

Since G is σ-compact, this proves that η is G-quasi-invariant. □

It turns out that the density u is μ-stationary in the following sense; this statement includes in particular the fact that the convolution of μ with u converges.

LEMMA 36. *For m_G-almost every $x \in G$ we have*

$$u(x) = \int_G \rho(s) u(s^{-1}x)\, dm_G(s) = \int_G u(s^{-1}x)\, d\mu(s) < \infty.$$

Proof. Since ν is μ-stationary and $\mathcal{P}$ is G-equivariant we have for every $f \in C_c(G)$.

$$\eta(f) = \nu(\mathcal{P}f) = \mu * \nu(\mathcal{P}f) = \int_G \nu(g^{-1} \,.\, \mathcal{P}f)\, d\mu(g)$$

https://doi.org/10.1017/fms.2019.39 Published online by Cambridge University Press

$$= \int_G \nu(\mathcal{P}(g^{-1}\,.\,f))\,d\mu(g) = \int_G \eta(g^{-1}\,.\,f)\,d\mu(g).$$

Since $\eta = u\,m_G$ we deduce from left invariant of m_G that

$$m_G(f\cdot u) = \eta(f) = \int_G m_G((g^{-1}\,.\,f)\cdot u)\,d\mu(g) = \int_G m_G(f\cdot(g\,.\,u))\,d\mu(g).$$

If $f \geqslant 0$, then we can apply Fubini to obtain

$$\begin{aligned}\int_G f(x)u(x)\,dm_G(x) &= \int_G m_G(f\cdot(g\,.\,u))d\mu(g)\\ &= \int_G \left(\int_G f(x)u(g^{-1}x)\,dm_G(x)\right)d\mu(g)\\ &= \int_G f(x)\left(\int_G u(g^{-1}x)\,d\mu(g)\right)dm_G(x).\end{aligned}$$

If $f \in C_c(G)$ is arbitrary, then we can write $f = f_+ - f_-$ with $f_+, f_- \geqslant 0$ and apply this formula to f_+ and f_-. The lemma follows. □

COROLLARY 37. *There exists a lower-semicontinuous positive function $v : G \to (0,\infty)$ such that $u(x) = v(x)$ for m_G-almost all $x \in G$.*

Proof. Let ρ_n be an increasing sequence in $C_c(G)$ with $\rho_n \nearrow \rho$. By monotone convergence and Lemma 36 we then have for almost all $x \in G$,

$$u(x) = v(x) := \int_G \rho(s)u(s^{-1}x)\,dm_G(s) = \sup \int_G \rho_n(s)u(s^{-1}x)\,dm_G(s).$$

The integrals on the right-hand side define continuous functions (as convolutions with $\rho_n \in C_c(G)$), hence v is lower-semicontinuous as the supremum of continuous functions. We claim that the function v is strictly positive. Indeed, assume for contradiction that for some $x_0 \in G$ we would have

$$v(x_0) = \int_G \rho(s)u(s^{-1}x_0)\,dm_G(s) = 0$$

Since the integrand is nonnegative and $\rho > 0$ this would imply that $u(x) = 0$ for m_G-almost every x, but then $\eta = u\,m_G = 0$, which is a contradiction. □

In view of the corollary we assume from now on that u has been chosen to be positive and semicontinuous.

https://doi.org/10.1017/fms.2019.39 Published online by Cambridge University Press

Proof of Theorem 32. We fix a compact set $K \subset G$ such that $G = \Lambda K = K\Lambda$. Since u is lower-semicontinuous and strictly positive we then have

$$\delta := \inf_{k \in K} u(k)\delta(k) > 0.$$

It follows from [2, (5.2), page 2957] that there exists a finite set F such that for all $\lambda \in \Lambda$ and m_G-almost all $g \in G$ we have

$$u(g\lambda^{-1})\Delta_G(\lambda^{-1}) \leqslant \sum_{c \in F} u(gc^{-1})\Delta_G(c^{-1}).$$

Note that if g satisfies this inequality and if we write $g = k\lambda$ with $k \in K$ and $\lambda \in \Lambda$, then

$$\sum_{c \in F} u(gc^{-1})\Delta_G(c^{-1}) \geqslant u(g\lambda^{-1})\Delta_G(\lambda^{-1}) = u(k)\Delta_G(k)\Delta_G(g^{-1}) \geqslant \delta \cdot \Delta_G(g)^{-1}.$$

Since this holds for m_G-almost every $g \in G$, we deduce that for all $g \in G$ we have

$$\int_G \left(\sum_{c \in F} u(s^{-1}gc^{-1})\Delta_G(c^{-1}) \right) \rho^{*n}(s)\, dm_G(s) \geqslant \delta \cdot \int_G \Delta_G(s^{-1}g)^{-1}\rho^{*n}(s)\, dm_G(s).$$

By Lemma 36 the left-hand side equals

$$\sum_{c \in F} \rho^{*n} * u(gc^{-1}) = \sum_{c \in C} u(gc^{-1}),$$

whereas the right-hand side equals to

$$\delta \cdot \Delta_G(g)^{-1} \cdot \left(\int_G \rho^* n(s)\delta(s)\, dm_G(s) \right) = \delta \cdot \Delta_G(g)^{-1} \cdot \left(\int_G \rho(s)\delta_G(s)\, dm_G(s) \right)^n,$$

which diverges to ∞ by Property (iii) of Lemma 33. We thus have established for every $g \in G$ that $\sum_{c \in C} u(gc^{-1}) = \infty$, contradicting the fact that c is finite. □

Acknowledgements

We are indebted to Oliver Sargent for challenging us to establish Borel density for approximate lattices. We thank JLU Gießen for providing excellent working conditions during several visits of the first-named author. We also thank two anonymous referees for helpful comments and remarks. We acknowledge support by the KIT-Publication Fund of the Karlsruhe Institute of Technology.

https://doi.org/10.1017/fms.2019.39 Published online by Cambridge University Press

Conflict of Interest: None

References

[1] U. Bader, Y. Shalom and Yehuda, 'Factor and normal subgroup theorems for lattices in products of groups', *Invent. Math.* **163**(2) (2006), 415–454.

[2] M. Björklund and T. Hartnick, 'Approximate lattices', *Duke Math. J.* **167**(15) (2018), 2903–2964.

[3] M. Björklund and T. Hartnick, 'Analytic properties of approximate lattices'. *Ann. Inst. Fourier (Grenoble)*, to appear, Preprint, 2017, arXiv:1709.09942.

[4] M. Björklund and T. Hartnick, 'Spectral theory of approximate lattices in nilpotent Lie groups', Preprint, 2018, arXiv:1811.06563.

[5] M. Björklund, T. Hartnick and F. Pogorzelski, 'Aperiodic order and spherical diffraction, I: auto-correlation of regular model sets', *Proc. Lond. Math. Soc.* **116**(4) (2018), 957–996.

[6] A. Borel, 'Density properties for certain subgroups of semi-simple groups without compact components', *Ann. of Math. (2)* **72**(1) (1960), 179–188.

[7] Y. Cornulier and P. de la Harpe, *Metric Geometry of Locally Compact Groups*, Tracts in Mathematics, 25 (EMS Publishing House, Zürich, 2016).

[8] S. G. Dani, 'On ergodic quasi-invariant measures of group automorphism', *Israel J. Math* **43**(1) (1982), 62–74.

[9] H. Furstenberg, 'A note on Borel's density theorem', *Proc. Amer. Math. Soc.* **55**(1) (1976), 209–212.

[10] H. Furstenberg and E. Glasner, 'Stationary dynamical systems', in *Dynamical Numbers—Interplay Between Dynamical Systems and Number Theory*, Contemporary Mathematics, 532 (American Mathematical Society, Providence, RI) 1–28.

[11] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, 52 (Springer, New York - Heidelberg, 1977).

[12] S. Machado, 'Approximate lattices and Meyer sets in nilpotent Lie groups', Preprint, 2018, arXiv:1810.10870.

[13] G. A. Margulis, *Discrete subgroups of semisimple Lie groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 17 (Springer, Berlin, 1991).

[14] Y. Meyer, *Algebraic Numbers and Harmonic Analysis*, North-Holland Mathematical Library, 2 (North-Holland Publishing Co., New York, 1972).

[15] J. S. Milne, *Algebraic Groups*, Cambridge Studies in Advanced Mathematics 170 (Cambridge University Press, Cambridge, 2017).

[16] J. Oesterlé, 'Nombres de Tamagawa et groupes unipotents en caractéristique p', *Invent. Math.* **78**(1) (1984), 13–88.

[17] F. Paulin, 'De la geometrie et de la dynamique de $\mathrm{SL}_n(\mathbb{R})$ et $\mathrm{SL}_n(\mathbb{Z})$', in *Sur la dynamique des groupes de matrices et applications arithmetiques, Editions de l'Ecole Polytechnique* (eds. N. Berline, A. Plagne and C. Sabbah) (Editions Ellipses, Paris, 2007), 47–110.

[18] Y. Shalom, 'Invariant measures for algebraic actions, Zariski dense subgroups and Kazhdan's property (T)', *Trans. Amer. Math. Soc.* **351**(8) (1999), 3387–3412.

[19] G. Stuck, 'Growth of homogeneous spaces, density of discrete subgroups and Kazhdan's property (T)', *Invent. Math.* **109**(3) (1992), 505–517.

[20] T. Tao, 'Product set estimates for non-commutative groups', *Combinatorica* **28**(5) (2008), 547–594.

[21] J. Tits, *Oeuvres/Collected Works, Vol. IV*, (European Mathematical Society, Zürich, 2013).

https://doi.org/10.1017/fms.2019.39 Published online by Cambridge University Press