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# SMOOTH, VERY SMOOTH AND STRONGLY SMOOTH POINTS IN MUSIELAK-ORLICZ SEQUENCE SPACES

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Criteria for smooth points, very smooth points and strongly smooth points in Musielak-Orlicz sequence spaces equipped with the Luxemburg norm are given.

## 0. INTRODUCTION

Let us denote by $X$ a real Banach space and by $S(X)$ the unit sphere of $X$. For any $x \in S(X)$ we denote by $\mathrm{Grad}(x)$ the set of all support functionals at $x$, that is, $\mathrm{Grad}(x) = \{f \in S(X^*)\colon\ f(x) = \|x\|\}$, where $X^*$ denotes the dual space of $X$. A point $x \in S(X)$ is said to be a smooth point if $\mathrm{Grad}(x)$ is a singleton. A point $x \in S(X)$ is said to be a very smooth (strongly smooth or equivalently Frechet differentiable) point if it is a smooth point and for any sequence $(f_n)$ in $S(X^*)$ such that $f_n(x) \to 1$ we have $f_n - f \to 0$ weakly (respectively $\|f_n - f\| \to 0$), where $\{f\} = \mathrm{Grad}(x)$.

It is obvious that strong smoothness implies very smoothness and this implies smoothness. For these definitions and their applications we refer to [**5**].

A mapping $\Phi\colon \mathbb{R} \to [0,\infty]$ is said to be an Orlicz function if it is even, convex left-continuous on $[0,\infty)$, $\Phi(0) = 0$ and $\Phi(u) < \infty$ for some $u > 0$ (see [**1, 10, 11, 12, 13, 15**]). A sequence $M = (M_i)$ of Orlicz functions is called a Musielak-Orlicz function (see [**14**]). We associate with this function two sequences $(e_i)$ and $(b_i)$, where

$$e_i = \sup\{u \geqslant 0\colon\ M_i(u) = 0\}, \quad b_i = \sup\{u \geqslant 0\colon\ M_i(u) < \infty\}$$

for each $i \in \mathbb{N}$. Moreover, $p_i^-(u)$ and $p_i(u)$ denote the left and the right derivative of $M_i$ at $u \in \mathbb{R}$ with $|u| \leqslant b_i$. Of course we assume $p_i(b_i) = \infty$ and $p_i^-(u) = p(u) = \infty$ for $u > b_i$.

If $N = (N_i)$ is the Musielak-Orlicz function complementary to $M = (M_i)$ in the sense of Young, that is, $N_i(v) = \sup_{u\geqslant 0}\{u|v| - M_i(u)\}$ for each $i \in \mathbb{N}$ and $v \in \mathbb{R}$, then we have Young's inequality

$$|uv| \leqslant M_i(u) + N_i(v)$$

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for every $u, v \in \mathbb{R}$, and for any $u \in \mathbb{R}$ the equality

$$|uv| = M_i(u) + N_i(v)$$

holds if and only if $p_i^-(u) \leqslant v \leqslant p_i(u)$.

We say a Musielak-Orlicz function $M = (M_i)$ satisfies the $\delta_2^0$-condition ($M \in \delta_2^0$ for short) if there are positive constants $a$ and $k$, a natural number $i_0$ and a sequence $(c_i)_{i=i_0}^{\infty}$ with $c_i \geqslant 0$ such that $\sum_{i=i_0}^{\infty} c_i < \infty$ and the inequality

$$M_i(2u) \leqslant kM_i(u) + c_i$$

holds for all $i \geqslant i_0$ and $u$ satisfying $M_i(u) \leqslant a$ (see [**14**]).

Let $l^0$ denote the space of all real sequences $x = \big(x(i)\big)$. As usual, for $x \in l^0$, we denote $\operatorname{supp} x = \big\{i \in \mathbb{N}\colon\ x(i) \neq 0\big\}$. With any Musielak-Orlicz function $M = (M_i)$ we associate the convex modular function $\rho_M\colon\ l^0 \to [0, \infty]$ defined by

$$\rho_M(x) = \sum_{i=1}^{\infty} M_i\big(x(i)\big) \quad (\forall x \in l^0)$$

and the Musielak-Orlicz sequence space

$$l_M = \big\{x \in l^0\colon\ \rho(x/\lambda) < \infty \quad \text{for some} \quad \lambda > 0\big\}.$$

In the space $l_M$ we define two norms; the Luxemburg norm

$$\|x\|_M = \inf\big\{\lambda > 0\colon\ \rho_M(x/\lambda) \leqslant 1\big\}$$

and the Orlicz norm

$$\|x\|_M^0 = \sup\left\{\sum_{i=1}^{\infty} x(i)y(i)\colon\ \rho_N(y) \leqslant 1\right\}.$$

By $h_M$ we denote the subspace of $l_M$ which is defined to be the closure in $l_M$ of the space of all sequences in $l^0$ with finite number of coordinates different from 0 (the closure is taken in the norm topology). It is easy to see that

$$h_M = \left\{x \in l^0\colon\ \text{for any } \lambda > 0 \text{ there is } i_\lambda \in \mathbb{N} \text{ such that} \sum_{i \geqslant i_\lambda} M_i\big(\lambda x(i)\big) < \infty\right\}.$$

The spaces $l_M$ and $h_M$ are Banach spaces under either of these two norms (see [**1, 11, 14**]). In [**11**] these spaces are called modular sequence spaces. The function $d_M\colon\ l_M \to [0, \infty)$ defined by

$$d_M(x) = \inf\left\{\lambda > 0\colon\ \sum_{i \geqslant i_0} M_i\Big(\frac{x(i)}{\lambda}\Big) < \infty \text{ for some } i_0 \in \mathbb{N}\right\}$$

is nothing but the distance of $x \in l_M$ from the subspace $h_M$ (see [1]).

Every $f \in \operatorname{Grad}(x)$ for $x \in l_M \setminus \{0\}$ is of the from $f = y + \varphi$, where $y \in l_N^0$ and $\varphi$ is a singular functional, that is, $\varphi(z) = 0$ for any $z \in h_M$, and $y$ is identified with the functional

$$\langle w, y\rangle = \sum_{i=1}^{\infty} w(i)y(i) \quad \Big(\forall\, w = \big(w(i)\big) \in l_M\Big).$$

If $\varphi = 0$, we say that $f$ is a regular support functional at $x$. The set of all regular support functionals at $x$ is denoted by $R\operatorname{Grad}(x)$. It is well known that (see [1, 15])

$$\|f\| = \|y\|_N^0 + \|\varphi\|.$$

Smooth points and smoothness in Orlicz spaces and Musielak-Orlicz spaces has been discussed for both the Luxemburg and the Orlicz norm as well as for a non-atomic measure and for the counting measure in the papers [2, 3, 4, 6, 7, 8, 9, 10, 17].

In [8] the following theorem was presented.

**Theorem 0.1.** *Let $x \in S(l_M)$.*

I *If $|x(i)| < b_i$ for $i = 1, 2, \ldots$, then $x$ is smooth if and only if*

(a) $d_M(x) < 1$,

(b) $\operatorname{Card}\Big\{i \in \mathbb{N}\colon\ p_i\big(|x(i)|\big) \neq 0\Big\} = 1$ *or* $p_i^-\big(|x(i)|\big) = p_i\big(|x(i)|\big)$ $(i = 1, 2, \ldots)$.

II *If $|x(i_0)| = b_{i_0}$ for some $i_0 \in \mathbb{N}$, $\rho_M(x) = \sup\big\{\rho_M(y)\colon\ \|y\|_M = 1,\ \operatorname{supp} y \subset \operatorname{supp} x\big\}$, then $x$ is smooth if and only if*

(a) $|x(i)| < b_i$ *for any* $i \neq i_0$,

(b) $p_{i_0}^-(b_{i_0}) = \infty$ *or* $p_i\big(|x(i)|\big) = 0$ *for* $i \neq i_0$ *or* $v \notin l_N$, *where* $v = \big\{v(i)\big\}$, $v(i) \in \Big[p_i^-\big(|x(i)|\big), p_i\big(|x(i)|\big)\Big]$ *for* $i = 1, 2, \ldots$,

(c) $d_M(x) < 1$.

III *If $|x(i_0)| = b_{i_0}$ for some $i_0 \in \mathbb{N}$, $\rho_M(x) < \sup\big\{\rho_M(y)\colon\ \|y\|_M = 1,\ \operatorname{supp} y \subset \operatorname{supp} x\big\}$, then $x$ is smooth if and only if*

(a) $|x(i)| < b_i,\ i \neq i_0$,

(b) $d_M(x) < 1$.

The formulation of this theorem is too complicated and, as we shall see below, its part II is not true (the assumptions are not necessary in general). Indeed, if $M = (M_i)$, where

$$M_i(u) = \begin{cases} |u| & \text{if } |u| \leqslant \dfrac{1}{2} \\ \infty & \text{if } |u| > \dfrac{1}{2} \end{cases}$$

for each $i \in \mathbb{N}$, define $x = (1/2, 0, 0, \ldots)$. Obviously $\|x\|_M = 1$. Since $x(1) = 1/2 = b_1$, $\rho_M(x) = 1/2 = \sup\big\{\rho_M(y)\colon\ \|y\|_M = 1,\ \operatorname{supp} y \subset \operatorname{supp} x = \{1\}\big\}$, $x$ belongs to case II. Since $p_1^-(b_1) = p_1^-(1/2) = 1 < \infty$, $p_2(x(2)) = p_2(0) = 1 \neq 0$, we have $\sum_{i=1}^{\infty} N_i\Big(p_i^-(x(i))\Big) = \sum_{i=1}^{\infty} N_i(1) = \sum_{i=1}^{\infty} 0 = 0$, whence $\Big(p_i^-(x(i))\Big) \in l_N^0$. So, condition (b) of case II is not satisfied, whence it follows that if case II of Theorem 0.1 is true, $x$ should be not a smooth point. However, we shall prove that $x$ is smooth. Since $d_M(x) = 0 < 1$, $\mathrm{Grad}(x)$ contains only regular (that is, order continuous) functionals (see [**1**]). We show that if $y \in S(l_N^0)$ belongs to $\mathrm{Grad}(x)$, then $y(i) = 0$ for any $i \neq 1$. Indeed, if $y(2) > 0$, then for $\overline{x} = (1/2, 1/2, 0, 0, \ldots) \in S(l_M)$, we have

$$1 \geqslant \langle \overline{x}, y\rangle = \langle x, y\rangle + \frac{1}{2}y(2) = 1 + \frac{y(2)}{2},$$

which means that $y \notin \mathrm{Grad}(x)$, a contradiction. Therefore $y(i) = 0$ for any $i \neq 1$ if $y \in \mathrm{Grad}(x)$, which means that $\mathrm{Grad}(x)$ is a singleton, that is, $x$ is a smooth point.

We shall establish a new criterion for smooth points in $S(l_M)$ and we shall also give criteria for very smooth points and strongly smooth points of $S(l_M)$.

Before proving new results let us recall some results concerning $l_M$ that will be used in this paper.

**Lemma 0.1.** *For each $x \in l_M$, $d(x, h_M) = d_M(x)$ (see [**1**, Theorem 1.4.3]).*

**Lemma 0.2.** *If $x \in l_M$ and $d_M(x) < 1$, then $\mathrm{Grad}(x) = R\,\mathrm{Grad}(x)$ (see [**7**, Lemma 1.7]).*

**Lemma 0.3.** *If $x \in S(l_M)$ and $d_M(x) = 1$, then there exist $y, z \in S(l_M)$ with $\mathrm{supp}\, x \cap \mathrm{supp}\, y = \emptyset$ and $y + z = x$ (see [**8**, Proposition 1]).*

**Lemma 0.4.** *If $x \in S(l_M)$ and $|x(i)| = b_i$, $|x(j)| = b_j$ for $i \neq j$, then there exist $y, z \in S(l_M)$ such that $\mathrm{supp}\, y \cap \mathrm{supp}\, z = \emptyset$ and $y + z = x$ (see [**8**, Proposition 2]).*

**Lemma 0.5.** *Assume that $y \in l_N^0 \setminus \{0\}$. Then*

(i) $\|y\|_N^0 = \big(1 + \rho_N(ky)\big)/k$ *for some $k > 0$ whenever* $\sum_{i \in \mathrm{supp}\, y} M_i(b_i) > 1$,

(ii) $\|y\|_N^0 = \sum_{i=1}^{\infty} b_i|y(i)|$ *whenever* $\sum_{i \in \mathrm{supp}\, y} M_i(b_i) \leqslant 1$ *(see [**16**]).*

## 1. Results

First we prove some auxiliary lemmas.

**Lemma 1.1.** *If $M = (M_i)$ is a Musielak-Orlicz function such that $N \in \delta_2^0$ and $(y_n)$ is a sequence in $l_N^0$ such that $y_n(i) \to 0$ for each $i \in \mathbb{N}$ and*

$$\lim_{i_0 \to \infty} \sup_n \sum_{i > i_0} N_i(y_n(i)) = 0, \tag{1}$$

*then $\|y_n\|_N^0 \to 0$.*

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PROOF: Choose any $\varepsilon > 0$ and take $h > 0$ such that $4/h < \varepsilon$. By $N \in \delta_2^0$, there exist $k > 0$, $a > 0$, $i_0 \in \mathbb{N}$ and $c_i \geqslant 0$ $(i > i_0)$ with $\sum_{i>i_0} c_i < \infty$ such that

$$N_i(hv) \leqslant kN_i(v) + c_i \quad \text{if } i > i_0,\ N_i(v) \leqslant a.$$

Without loss of generality we assume that $\sum_{i>i_0} c_i \leqslant 1$. By (1), there is $i_0' > i_0$ such that

$$\sup_n \sum_{i>i_0'} N_i\big(y_n(i)\big) \leqslant \min\Big\{a, \frac{1}{k}\Big\}.$$

By $y_n(i) \to 0$ for each $i \in \mathbb{N}$, we get $\sum_{i=1}^{i_0'} N_i\big(hy_n(i)\big) \leqslant 1$ for $n$ large enough. Hence, we get

$$\begin{aligned}\|y_n\|_N^0 &\leqslant \frac{1}{h}\big(1 + \rho_N(hy_n)\big) = \frac{1}{h}\Big(1 + \sum_{i=1}^{i_0'} N_i\big(hy_n(i)\big) + \sum_{i>i_0'} N_i\big(hy_n(i)\big)\Big)\\ &\leqslant \frac{1}{h}\Big(2 + \sum_{i>i_0'}\big(kN_i\big(y_n(i)\big) + c_i\big)\Big) \leqslant \frac{1}{h}\Big(2 + k\cdot\frac{1}{k} + 1\Big) < \varepsilon\end{aligned}$$

for $n$ large enough, which means that $\|y_n\|_N^0 \to 0$ as $n \to \infty$. ∎

LEMMA 1.2. *If* $\|x\|_M = 1$, $d_M(x) < 1 - \theta < 1$, $y_n \in S(l_N^0)$ *for any* $n \in \mathbb{N}$ *and* $\langle x, y_n\rangle \to 1$ *as* $n \to \infty$, *then condition* (1) *from Lemma 1.1 holds.*

PROOF: If (1) is not true, by passing to a subsequence if necessary, we may assume that there are a sequence $(i_n) \subset \mathbb{N}$ with $i_n \nearrow \infty$ and $\varepsilon_0 > 0$ such that

$$\sum_{i>i_n} N_i\big(y_n(i)\big) \geqslant \varepsilon_0 \quad (n = 1, 2, \ldots).$$

We consider two cases.

I. $\|y_n\|_N^0 = 1/k_n\big(1 + \rho_N(k_ny_n)\big)$ for an infinite number of $n$. Noticing that $k_n \geqslant 1$, we get a contradiction:

$$\begin{aligned}1 &\leftarrow \langle x, y_n\rangle\\ &= \frac{1}{k_n}\langle x, k_ny_n\rangle\\ &= \frac{1}{k_n}\Big(\sum_{i=1}^{i_n} k_nx(i)y_n(i) + (1-\theta)\sum_{i>i_n}\frac{x(i)}{1-\theta}y_n(i)\Big)\\ &\leqslant \frac{1}{k_n}\Big(\sum_{i=1}^{i_n}\Big(M_i\big(x(i)\big)\Big) + N_i\big(k_ny_n(i)\big) + (1-\theta)\sum_{i>i_n}\Big(M_i\Big(\frac{x(i)}{1-\theta}\Big) + N_i\big(k_ny_n(i)\big)\Big)\Big)\\ &\leqslant \frac{1}{k_n}\Big(\rho_M(x) + \rho_N(k_ny_n) - \theta\sum_{i>i_n} N_i\big(k_ny_n(i)\big) + (1-\theta)\sum_{i>i_n} M_i\Big(\frac{x(i)}{1-\theta}\Big)\Big)\end{aligned}$$

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$$\leqslant \frac{1}{k_n}\Big(1+\rho_N(k_n y_n)-\theta\varepsilon_0+\sum_{i>i_n} M_i\Big(\frac{x(i)}{1-\theta}\Big)\Big)$$
$$\leqslant \|y_n\|_N^0-\theta\varepsilon_0+\sum_{i>i_n} M_i\Big(\frac{x(i)}{1-\theta}\Big)\to 1-\theta\varepsilon_0.$$

II. $\|y_n\|_N^0=\sum\limits_{i=1}^{\infty} b_i|y_n(i)|$ for an infinite number of $n$.

By $d_M(x)<1-\theta$, there is $i_0\in\mathbb{N}$ such that $\sum\limits_{i>i_0} M_i\big(x(i)/(1-\theta)\big)<\infty$. So, $|x(i)|/(1-\theta)\leqslant b_i$ for $i>i_0$. From

$$N_i\big(y_n(i)\big)=\int_0^{|y_n(i)|} q_i(s)\,ds\leqslant\int_0^{|y_n(i)|} b_i\,ds=b_i|y_n(i)|,$$

where $q_i(s)=\sup\{t\geqslant 0\colon\ p_i(t)\leqslant s\}$, we have $\sum\limits_{i>i_n} b_i|y_n(i)|\geqslant\varepsilon_0$. Hence for $i_n>i_0$, we get a contradiction:

$$\begin{aligned}1\leftarrow\langle x,y_n\rangle&=\sum_{i=1}^{i_n}x(i)y_n(i)+(1-\theta)\sum_{i>i_n}\frac{x(i)}{1-\theta}y_n(i)\\&\leqslant\sum_{i=1}^{i_n}b_i|y_n(i)|+(1-\theta)\sum_{i>i_n}b_i|y_n(i)|=\sum_{i=1}^{\infty}b_i|y_n(i)|-\theta\sum_{i>i_n}b_i|y_n(i)|\\&\leqslant\|y_n\|_N^0-\theta\varepsilon_0=1-\theta\varepsilon_0.\end{aligned}$$

□

The following lemma is very important in the remaining considerations.

**Lemma 1.3.** *If $\|x\|_M=1$, $\|y_n\|_N^0=1$ for each $n\in\mathbb{N}$ and $\langle x,y_n\rangle\to 1$ as $n\to\infty$, then:*

(i) *$y_n(j)\to 0$ as $n\to\infty$ whenever $|x(j)|<e_j$,*

(ii) *$\liminf\limits_{n\to\infty}\Big(y_n(i)p_j\big(x(j)\big)-y_n(j)p_i^-\big(x(i)\big)\Big)\geqslant 0$ whenever $|x(j)|\geqslant e_j$ and $|x(i)|>0$.*

Proof: We may assume without loss of generality that $x(i)\geqslant 0$ for any $i\in\mathbb{N}$. If (i) is not true, we may assume that there is $j\in\mathbb{N}$ such that $x(j)<e_j$ and $y_n(j)\geqslant c>0$ for each $n\in\mathbb{N}$. Let us define $\overline{x}$ with $\overline{x}(j)=e_j$ and $\overline{x}(i)=x(i)$ for $i\neq j$. It is easy to see that $\rho_M(\overline{x})=\rho_M(x)\leqslant 1$, whence $\|\overline{x}\|_M=1$. Hence

$$\begin{aligned}1\geqslant\|\overline{x}\|_M\|y_n\|_N^0&\geqslant\langle\overline{x},y_n\rangle=\langle x,y_n\rangle+y_n(j)\big(e_j-x(j)\big)\\&\geqslant\langle x,y_n\rangle+c\big(e_j-x(j)\big)\to 1+c\big(e_j-x(j)\big),\end{aligned}$$

which is a contradiction. So (i) is proved.

If (ii) is not true, there are $x(j)\geqslant e_j$, $x(i)>0$ and $a>0$ satisfying

$$y_n(i)p_j\big(x(j)\big)<y_n(j)p_i^-\big(x(i)\big)-2a\quad(n=1,2,\ldots).$$

Since $\rho_N(y_n) \leqslant \|y_n\|_N \leqslant \|y_n\|_N^0 = 1$, we get $|y_n(j)| \leqslant A_j := \sup\{v \geqslant 0\colon\ N_j(v) \leqslant 1\}$ for any $j \in \mathbb{N}$. (Note that $A_j = N_j^{-1}(1)$ if $N_j(A_j) = 1$ and $A_j = b_j$ if $N_j(A_j) < 1$.) Since $p_i^-$ is left continuous and $p_i$ is right continuous one can find a number $r > 0$ such that

$$p_i^-\big(x(i)\big) - p_i^-\big(x(i) - r\big) < \frac{a}{2A_j},\quad p_j\big(x(j) + r\big) - p_j\big(x(j)\big) < \frac{1}{2A_i}.$$

Then

$$\Big(p_i^-\big(x(i)\big) - p_i^-\big(x(i) - r\big)\Big)y_n(j) < \frac{a}{2},\quad \Big(p_j\big(x(j) + r\big) - p_j\big(x(j)\big)\Big)y_n(i) < \frac{a}{2}.$$

Thus

$$\begin{aligned} y_n(i)p_j\big(x(j) + r\big) &< y_n(i)p_j\big(x(j)\big) + \frac{a}{2} < y_n(j)p_i^-\big(x(i)\big) - 2a + \frac{a}{2} \\ &< y_n(j)p_i^-\big(x(i) - r\big) + \frac{a}{2} - 2a + \frac{a}{2} = y_n(j)p_i^-\big(x(i) - r\big) - a \end{aligned}$$

for all $n \in \mathbb{N}$. We have

$$\int_{x(j)}^{x(j)+r} p_j(s)\,ds > \int_{x(i)-r}^{x(i)} p_i^-(s)\,ds \quad\text{or}\quad \int_{x(j)}^{x(j)+r} p_j(s)\,ds \leqslant \int_{x(i)-r}^{x(i)} p_i^-(s)\,ds$$

and we may assume that the second inequality holds. Denote $c_j = x(j) + r$. One can find a number $c_i$, $x(i) > c_i \geqslant x(i) - r$ such that

$$\int_{x(j)}^{c_j} p_j(s)\,ds = \int_{c_i}^{x(i)} p_i^-(s)\,ds. \tag{2}$$

Of course

$$y_n(i)p_j(c_j) < y_n(j)p_i^-(c_i) - a \quad (n = 1, 2, \ldots).$$

Since $0 < p_j(c_j),\ p_i^-(c_i) < \infty$, there is $k > 0$ such that

$$\frac{y_n(i)}{p_i^-(c_i)} < k < \frac{y_n(j)}{p_j(c_j)} - \frac{a}{p_i^-(c_i)p_j(c_j)} \quad (n = 1, 2, \ldots). \tag{3}$$

Define $\overline{x}$ with $\overline{x}(i) = c_i$, $\overline{x}(j) = c_j$, $\overline{x}(t) = x(t)$ for $t \neq i$ and $t \neq j$. Then

$$\begin{aligned} \rho_M(\overline{x}) &= \rho_M(x) + M_j(c_j) - M_j\big(x(j)\big) + M_i(c_i) - M_i\big(x(i)\big) \\ &= \rho_M(x) + \int_{x(j)}^{c_j} p_j(s)\,ds - \int_{c_i}^{x(i)} p_i^-(s)\,ds = \rho_M(x) \leqslant 1, \end{aligned}$$

whence $\|\overline{x}\|_M \leqslant 1$. From (2) and (3), we get

$$\begin{aligned} 1 \geqslant \langle \overline{x}, y_n\rangle &= \langle x, y_n\rangle + y_n(j)\big(c_j - x(j)\big) - y_n(i)\big(x(i) - c_i\big) \\ &= \langle x, y_n\rangle + \int_{x(j)}^{c_j} \big(y_n(j) - kp_j(s)\big)\,ds - \int_{c_i}^{x(i)} \big(y_n(i) - kp_i^-(s)\big)\,ds \\ &\geqslant \langle x, y_n\rangle + \int_{x(j)}^{c_j} \frac{a}{p_i^-(c_i)}\,ds = \langle x, y_n\rangle + \frac{ar}{p_i^-(c_i)} \to 1 + \frac{ar}{p_i^-(c_i)}, \end{aligned}$$

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which is a contradiction finishing the proof. ∎

As an immediate consequence of Lemma 1.3, we get

**Lemma 1.4.** *Assume that $\|x\|_M = 1$, $\|y\|_N^0 = 1$ and $\langle x, y\rangle = 1$. Then:*

(i) $y(j) = 0$ *whenever* $|x(j)| < e_j$,

(ii) $y(i)p_j(x(j)) \geqslant y(j)p_i^-(x(i))$ *whenver* $|x(j)| \geqslant e_j$ *and* $|x(i)| > 0$.

Now, we are ready to give criteria for smooth points of $S(l_M)$.

**Theorem 1.1.** *If $x \in S(l_M)$, then $x$ is a smooth point if and only if:*

(i) $d_M(x) < 1$,

(ii) *there is at most one index $i$ satisfying $|x(i)| = b_i$,*

(iii-1) *if $|x(i)| < b_i$ for all $i \in \mathbb{N}$, then $p_i^-(|x(i)|) = p_i(|x(i)|)$ whenever $|x(i)| < M_i^{-1}(1)$,*

(iii-2) *if $|x(i_0)| = b_{i_0}$ for some $i_0 \in \mathbb{N}$ and $|x(i)| < b_i$ for $i \neq i_0$, then $\rho_N(x) < 1$ or $p_{i_0}^-(b_{i_0}) = \infty$ or $p_i(|x(i)|) = 0$ for $i \neq i_0$.*

Proof: Assume without loss of generality that $x(i) \geqslant 0$ for all $i \in \mathbb{N}$.

Necessity. If (i) or (ii) is not true, then by Lemma 0.3 and Lemma 0.4, there are $y, z \in S(l_M)$ such that $\operatorname{supp} y \cap \operatorname{supp} z = \emptyset$ and $x = y + z$. Clearly, $y - z \in S(l_M)$ too. Take $y^* \in \operatorname{Grad}(y), z^* \in \operatorname{Grad}(z)$. Then

$$1 \pm y^*(z) = y^*(y \pm z) \leqslant \|y^*\|\,\|y \pm z\|_M = \|y^*\| = 1.$$

So $y^*(z) = 0$. Similarly $z^*(y) = 0$. Consequently $y^*(y) = z^*(z) = 1$, whence $y^* \neq z^*$. Moreover

$$y^*(x) = y^*(y + z) = y^*(y) = \|y\|_M = 1.$$

So $y^* \in \operatorname{Grad}(x)$. Similarly, $z^* \in \operatorname{Grad}(x)$, which is a contradiction proving the necessity of (i) and (ii).

Now, we shall prove that $d_M(x) < 1$ and $|x(i)| < b_i$ for all $i \in \mathbb{N}$ imply

$$\rho_M(x) = 1 \tag{4}$$

and

$$\rho_N(p^-(x)) \leqslant \sum_{i=1}^{\infty} |x(i)| p_i^-(|x(i)|) < \infty. \tag{5}$$

Indeed, by $d_M(x) < 1$, there exists $\lambda > 1$ and $i_0 \in \mathbb{N}$ such that $\sum_{i>i_0} M_i(\lambda x(i)) < \infty$. By $x(i) < b_i$, there exists $\lambda_0$ with $1 < \lambda_0 \leqslant \lambda$ such that $\sum_{i=1}^{i_0} M_i(\lambda_0 x(i)) < \infty$. Hence $\rho_M(\lambda_0 x) < \infty$. Since $\rho_M(tx)$ is a continuous function of $t$ on the interval $[0, \lambda_0]$, we obtain (4). From

$$M_i(\lambda_0 u) \geqslant \int_u^{\lambda_0 u} p_i(s)\, ds \geqslant (\lambda_0 - 1) u p_i^-(u)$$

and $\rho_M(\lambda_0 x) < \infty$, we get (5).

If (iii-1) does not hold, we may assume that $p_1^-(x(1)) < p_1(x(1))$ and $M_1(x(1)) < 1$. By (4), there is $i \in \mathbb{N}$ (we may assume that $i = 2$) such that $M_2(x(2)) > 0$. Therefore $p_2^-(x(2)) > 0$ and, by (5),

$$\sum_{i=1}^{\infty} x(i)p_i^-(x(i)) < \infty \quad \text{and} \quad \sum_{i=2}^{\infty} x(i)p_i^-(x(i)) + x(1)p_1(x(1)) < \infty.$$

Let $x^*, \overline{x}^*$ be defined as follows:

$$x^*(i) = \frac{p_i^-(x(i))}{\sum_{i=1}^{\infty} x(i)p_i^-(x(i))} \quad (i = 1, 2, \ldots,),$$

$$\overline{x}^*(i) = \begin{cases} \dfrac{p_1(x(1))}{\sum\limits_{i \neq 1} x(i)p_i^-(x(i)) + x(1)p_1(x(1))} & \text{if } i = 1 \\ \dfrac{p_i^-(x(i))}{\sum\limits_{i \neq 1} x(i)p_i^-(x(i)) + x(1)p_1(x(1))} & \text{if } i \neq 1. \end{cases}$$

Since $\langle x, x^* \rangle = 1 = \|x\|_M$, so $\|x^*\|_N^0 \geqslant 1$. Moreover, for any $y \in l_M$ with $\rho_M(y) \leqslant 1$ we have

$$\begin{aligned} \langle y, x^* \rangle &= \sum_{i=1}^{\infty} \frac{y(i)p_i^-(x(i))}{\sum_{i=1}^{\infty} x(i)p_i^-(x(i))} \leqslant \frac{\rho_M(y) + \rho_N(p^-(x))}{\sum_{i=1}^{\infty} x(i)p_i^-(x(i))} \\ &\leqslant \frac{1 + \rho_N(p^-(x))}{\sum_{i=1}^{\infty} x(i)p_i^- x(i))} = \frac{\rho_M(x) + \rho_N\Big(p^-(x(i))\Big)}{\sum_{i=1}^{\infty} x(i)p_i^-(x(i))} = 1, \end{aligned}$$

whence $\|x^*\|_N^0 \leqslant 1$ and consequently $\|x^*\|_N^0 = 1$. This means that $x^* \in \operatorname{Grad}(x)$. Similarly $\overline{x}^* \in \operatorname{Grad}(x)$.

But if $x(1) \neq 0$, then $x^*(2) \neq \overline{x}^*(2)$; if $x(1) = 0$, then $x^*(1) \neq \overline{x}^*(1)$. Therefore $x^* \neq \overline{x}^*$. This contradicts the assumption that $x$ is a smooth point, finishing the proof of the necessity of condition (iii-1).

If (iii-2) does not hold, we may assume that $x(1) = b_1$, $x(i) < b_i$ for $i \neq 1$, $\rho_M(x) = 1$, $p_1^-(b_1) < \infty$ and $p_2(x(2)) > 0$. Notice that

$$\sum_{i=1}^{\infty} x(i)p_i^-(x(i)) = x(1)p_1^-(x(1)) + \sum_{i \neq 1} x(i)p_i^-(x(i)) < \infty.$$

Let us define

$$x^*(i) = \begin{cases} \dfrac{p_i^-(x(i))}{\sum\limits_{i \neq 2} x(i)p_i^-(x(i)) + x(2)p_2(x(2))} & \text{for } i \neq 2 \\ \dfrac{p_2(x(2))}{\sum\limits_{i \neq 2} x(i)p_i^-(x(i)) + x(2)p_2(x(2))} & \text{for } i = 2. \end{cases}$$

Similarly as in case (iii-1) we can prove that $x^* \in \operatorname{Grad}(x)$. Consider also $\overline{x}^*$ defined by

$$\overline{x}^*(i) = \begin{cases} \dfrac{1}{b_1} & \text{for } i = 1 \\ 0 & \text{for } i \neq 1. \end{cases}$$

Then $\overline{x}^*(x) = 1$ and for any $y \in l_M$ with $\rho_M(y) \leqslant 1$ we have $\langle y, \overline{x}_1^* \rangle = y(1)/b_1 \leqslant b_1/b_1 = 1$. So $\|\overline{x}^*\|_N^0 = 1$, whence it follows that $\overline{x}^* \in \operatorname{Grad}(x)$. Note that $\overline{x}^* \neq x^*$ because $\overline{x}^*(2) \neq x^*(2)$. Therefore, $x$ cannot be smooth if (iii-2) is not satisfied.

SUFFICIENCY. By condition (i) and Lemma 0.2, $\operatorname{Grad}(x) = R\operatorname{Grad}(x)$. Assume also that conditions (ii), (iii) are satisfied. We consider separately five cases.

I. $x(i) < b_i$ for all $i \in \mathbb{N}$ and $x(i) < M_i^{-1}(1)$ for all $i \in \mathbb{N}$.

Take an arbitrary $x^* \in R\operatorname{Grad}(x)$. By condition (iii-1) and Lemma 1.4, we have

$$x^*(i)p_j^-\big(x(j)\big) = x^*(i)p_j\big(x(j)\big) \geqslant x^*(j)p_i^-\big(x(i)\big) = x^*(j)p_i\big(x(i)\big) \geqslant x^*(i)p_j^-\big(x(j)\big).$$

So, $x^*(i)p_j\big(x(j)\big) = x^*(j)p_i\big(x(i)\big)$ for every $i, j$ with $p_i\big(x(i)\big) > 0$, $p_j\big(x(i)\big) > 0$, whence it follows that there is $d > 0$ such that

$$\frac{x^*(j)}{p_j\big(x(j)\big)} = d \quad \text{for any } j \text{ with } p_j\big(x(j)\big) > 0.$$

If $x(j) < e_j$, we have $p_j\big(x(j)\big) = 0$ and Lemma 1.4 (i) yields $x^*(j) = 0$. Assume that $p_j\big(x(j)\big) = 0$ and $x(j) = e_j$. By $\rho_M(x) = 1$ there is $i_0 \in \mathbb{N}$ such that $M_{i_0}\big(x(i_0)\big) > 0$, so $p_{i_0}^-\big(x(i_0)\big) > 0$. By Lemma 1.4(i),

$$0 = x^*(i_0)p_j\big(x(j)\big) \geqslant x^*(j)p_{i_0}^-\big(x(i_0)\big).$$

So, we still have $x^*(j) = 0$. This means that $x^*(j) = dp_j\big(x(j)\big)$ for all $i \in \mathbb{N}$. From $1 = \langle x, x^* \rangle = d\sum_{i=1}^{\infty} x(i)p_i\big(x(i)\big)$, we obtain $d = 1 \Big/ \sum_{i=1}^{\infty} x(i)p_i\big(x(i)\big)$. Hence $x^*(i) = p_i\big(x(i)\big) \Big/ \sum_{i=1}^{\infty} x(i)p_i\big(x(i)\big)$ for all $i \in \mathbb{N}$, and so $x^*$ is unique, that is, $x$ is smooth.

II. $x(i) < b_i$ for all $i \in \mathbb{N}$ and $M_1\big(x(1)\big) = 1$.

Obviously, $M_i\big(x(i)\big) = 0$ for all $i \neq 1$. So, by (iii-1) $p_i\big(x(i)\big) = p_i^-\big(x(i)\big) = 0$ for all $i \neq 1$. We can prove similarly to case I that for any $x^* \in R\operatorname{Grad}(x)$, we have $x^*(i) = 0$ for all $i \neq 1$. Therefore, $\operatorname{supp} x^* = \{1\}$, whence

$$x^* = \Big(\frac{1}{x(1)}, 0, 0, \dots\Big)$$

is the only element of $\operatorname{Grad}(x)$, that is, $x$ is a smooth point.

III. $x(1) = b_1$, $x(i) < b_i$ for $i \neq 1$ and $p_1^-(b_1) = \infty$.

For any $x^* \in R\operatorname{Grad}(x)$ and any $i \neq 1$, by Lemma 1.4 we have $x^*(i) = 0$ if $x(i) \leqslant e_i$ and

$$\infty > x^*(1)p_i^-\big(x(i)\big) \geqslant x^*(i)p_1^-\big(x(1)\big) \geqslant x^*(i)p_1^-(b_1)$$

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if $x(i) \geqslant e_i$. (We apply Lemma 1.4 (ii) with $j = 1$ if $e_i = 0$ and Lemma 1.4 (i) if $e_i > 0$). So, we have $x^*(i) = 0$ for $i \neq 1$. This shows that $\operatorname{supp} x^* = \{1\}$, that is, $x^*$ is unique, namely $x^* = (b_1^{-1}, 0, 0, \ldots)$.

IV. $x(1) = b_1$, $x(i) < b_i$ for $i \neq 1$ and $\rho_M(x) < 1$.

For any $x^* \in R\operatorname{Grad}(x)$, if $x^*(i) > 0$ for $i \neq 1$, then by $x(i) < b_i$, one can find $c$ with $x(i) < c < b_i$ satisfying $\sum_{j \neq i} M_j(x(j)) + M_i(c) \leqslant 1$. Define $\bar{x}$ with $\bar{x}(i) = c$ and $\bar{x}(j) = x(j)$ for $j \neq i$. Then $\rho_M(\bar{x}) \leqslant 1$ and $\|\bar{x}\|_M \leqslant 1$. Hence

$$1 \geqslant \langle \bar{x}, x^* \rangle = \langle x, x^* \rangle + x^*(i)\big(c - x(i)\big) = 1 + x^*(i)\big(c - x(i)\big),$$

which is a contradiction. So, $x^*(i) = 0$ for $i \neq 1$, which shows that $x^*$ is unique (in fact $x^* = (1/b_1, 0, 0, \ldots)$).

V. $x(1) = b_1$, $x(i) < b_i$ for $i \neq 1$, $p_i\big(x(i)\big) = 0$ for $i \neq 1$, $\rho_M(x) = 1$.

Then for any $x^* \in R\operatorname{Grad}(x)$, if $x^*(i) > 0$ and $x(i) \geqslant e_i$ for some $i \neq 1$, by Lemma 1.4,

$$0 = x^*(1)p_i\big(x(i)\big) \geqslant x^*(i)p_1(b_1) = x^*(i)p_1^-\big(M_1^{-1}(1)\big) > 0,$$

because $M\big(x(1)\big) = 1$ by $\rho_M(x) = 1$. This is a contradiction, which shows that $x^*(i) = 0$ for $i \neq 1$ whenever $x(i) \geqslant e_i$. If $0 \leqslant x(i) < e_i$, then $x^*(i) = 0$ by Lemma 1.4 (i). Therefore $\operatorname{supp} x^* = \{1\}$, whence $x^*$ is unique. This finishes the proof. □

**Theorem 1.2.** *For $x \in S(l_M)$ the following assertions are equivalent:*

(A) *$x$ is a strongly smooth point,*

(B) *$x$ is a very smooth point,*

(C)
- (i) *$d_M(x) < 1$,*
- (ii) *$|x(i)| = b_i$ for at most one $i \in \mathbb{N}$,*
- (iii-1) *if $|x(i)| < b_i$ for all $i \in \mathbb{N}$, then $N \in \delta_2^0$ and $|x(i)| < M_i^{-1}(1)$ implies $p_i^-\big(|x(i)|\big) = p_i\big(|x(i)|\big)$,*
- (iii-2) *if $|x(i_o)| = b_{i_0}$ for some $i_0 \in \mathbb{N}$ and $|x(i)| < b_i$ for $i \neq i_o$, then $p_{i_0}^-(b_{i_0}) = \infty$ or $\rho_M(x) < 1$ or $p_i\big(|x(i)|\big) = 0$ for $i \neq i_0$ and $N \in \delta_2^0$.*

Proof: We still assume without loss of generality that $x(i) \geqslant 0$ for all $i \in \mathbb{N}$. The implications (A) $\Rightarrow$ (B) $\Rightarrow$(C)(i),(ii) are trivial. Since very smooth points are smooth points, if (iii-1) is not true, then $N \notin \delta_2^0$ and

$$|x(i)| < b_i \quad \text{for all} \quad i \in \mathbb{N}.$$

From Theorem 1.1, we get in this case that $\operatorname{supp} y = \operatorname{supp} x$, where $y$ determines the unique support functional of $x$. Combining this with $\rho_M(x) = 1$, we get

$$\sum_{i \in \operatorname{supp} y} M_i(b_i) = \sum_{i \in \operatorname{supp} x} M_i(b_i) > \sum_{i \in \operatorname{supp} x} M_i\big(x(i)\big) = \rho_M(x) = 1.$$

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By Lemma 0.5, there is $k \geqslant 1$ such that $\|y\|_N^0 = \big(1+\rho_N(ky)\big)/k$. Since $N \notin \delta_2^0$, there exists $z \in l_N^0$ satisfying

$$\rho_N(z) \leqslant 1 \quad \text{and} \quad d_N\Big(\frac{z}{k} - y\Big) \neq 0.$$

(If $d_N(y) = 0$ we choose $z \in l_N^0$ with $d_N(z) > 0$; if $d_N(y) > 0$ we choose $z \in l_N^0$ with $d_N(z) = 0$.) Let

$$y_n = \Big(y(1), y(2), \ldots, y(n), \frac{z(n+1)}{k}, \frac{z(n+2)}{k}, \ldots\Big).$$

Then

$$\begin{aligned}\|y_n\|_N^0 &\leqslant \frac{1}{k}\big(1+\rho_N(ky_n)\big) = \frac{1}{k}\Big(1 + \sum_{i\leqslant n} N_i\big(ky(i)\big) + \sum_{i>n} N_i\big(z(i)\big)\Big)\\ &\leqslant \frac{1}{k}\big(1+\rho_N(ky)\big) + \sum_{i>n} N_i\big(z(i)\big) \to \|y\|_N^0 = 1.\end{aligned}$$

So $\limsup_{n\to\infty} \|y_n\|_N^0 \leqslant 1$. On the other hand

$$\langle x, y_n\rangle = \sum_{i=1}^{n} x(i)y_n(i) + \frac{1}{k}\sum_{i=n+1}^{\infty} x(i)z(i)$$

and by

$$\frac{1}{k}\sum_{i>n} x(i)z(i) \leqslant \frac{1}{k}\sum_{i=n+1}^{\infty}\Big(M_i\big(x(i)\big) + N_i\big(z(i)\big)\Big) \to 0,$$

we have $\langle x, y_n\rangle \to \langle x, y\rangle = 1$. So, $\liminf_{n\to\infty} \|y_n\|_N^0 \geqslant 1$. Hence $\lim_{n\to\infty} \|y_n\|_N^0 = 1$. But since $d_N\big((z/k) - y\big) \neq 0$, there is a singular functional $\varphi$ such that $\varphi\big((z/k) - z\big) \neq 0$. Thus

$$\varphi(y_n - y) = \varphi\Big(\frac{z}{k} - y\Big) \neq 0, \quad \text{that is,} \quad y_n \not\xrightarrow{w} y \quad \text{as} \quad n \to \infty.$$

This contradicts the fact that $x$ is a very smooth point.

If (iii-2) does not hold, then we may assume that $x(1) = b_1$, $x(i) < b_i$ for all $i \neq 1$, $\rho_M(x) = 1$, $p_1^-(b_1) < \infty$ and $N \notin \delta_2^0$. Since $x$ is a smooth point, by condition (iii-2) from Theorem 1.1, we have $p_i\big(x(i)\big) = 0$ for $i \neq 1$. So, $M_i\big(x(i)\big) = 0$ for $i \neq 1$ and $M_1(x(1) = \rho_M(x) = 1$. In this case, $y = (b_1^-, 0, 0, \ldots)$ is the only support functional at $x$. Take $k > 0$ satisfying $k/b_1 = p_1^-(b_1)$. Then

$$\frac{1}{k}\big(1+\rho_N(ky)\big) = \frac{1}{k}\Big(M_1(b_1) + N_1\Big(\frac{k}{b_1}\Big)\Big) = \frac{1}{k}\Big(b_1 \cdot \frac{k}{b_1}\Big) = 1 = \|y\|_N^0.$$

Now, we can deduce a contradiction in the same way as above, proving the necessity of condition (iii-2).

(C) $\Rightarrow$ (A). It follows from Theorem 1.1 that $x$ is a smooth point, so $x$ has a unique support functional $y$. Suppose that $f_n \in S(l_M^*)$, $f_n(x) \to 1$ and $f_n = y_n + \varphi_n$, where $y_n \in l_N^0$ and $\varphi_n$ are singular functionals for all $n \in \mathbb{N}$. We need to prove that $\|f_n - y\| \to 0$ as $n \to \infty$. By $d_M(x) < 1$ and Lemma 0.1, there is $z \in h_M$ such that $\|x - z\| < 1 - \theta < 1$. Thus

$$\begin{aligned} 1 \leftarrow f_n(x) &= \langle x, y_n \rangle + \varphi_n(x) \leqslant \|x\|_M \|y_n\|_N^0 + \varphi_n(x - z) \\ &\leqslant \|y_n\|_N^0 + \|\varphi_n\| \, \|x - z\|_M < \|y_n\|_N^0 + \|\varphi_n\|(1 - \theta) \\ &= \|f_n\| - \theta\|\varphi_n\| = 1 - \theta\|\varphi_n\|, \end{aligned}$$

whence $\|\varphi_n\| \to 0$ as $n \to \infty$. Therefore, we can assume in the rest of the proof of this implication that $\|y_n\|_N^0 = 1$ and $\langle x, y_n \rangle \to 1$ as $n \to \infty$. By $d_M(x) < 1$ and Lemma 1.2, we have

$$\lim_{i_0 \to \infty} \sup_n \sum_{i > i_0} N_i\big(y_n(i)\big) = 0. \tag{6}$$

By (6) and Young's equality, we get

$$\lim_{i_0 \to \infty} \sup_n \sum_{i > i_0} x(i) y_n(i) = 0. \tag{7}$$

To show that $\|y_n - y\| \to 0$ as $n \to \infty$, we consider the following five cases.

I. $x(i) < b_i$ and $x(i) < M_i^{-1}(1)$ for all $i \in \mathbb{N}$.

From (C) (iii-1), we get $p_i^-\big(x(i)\big) = p_i\big(x(i)\big)$ for all $i \in \mathbb{N}$ and $N \in \delta_2^0$. By Theorem 1.1, the unique support functional $y$ at $x$ is given by

$$y = \frac{1}{\sum_{i=1}^\infty x(i) p_i\big(x(i)\big)} \big(p_1(x_1), p_2(x_2), \ldots\big).$$

From Lemma 1.3(ii), in the case when $p_i\big(x(i)\big) > 0$ and $p_j\big(x(j)\big) > 0$, we have

$$\lim_{n \to \infty} p_i\big(x(i)\big) y_n(j) = \lim_{n \to \infty} p_j\big(x(j)\big) y_n(i).$$

So, there exists $d > 0$ such that

$$\lim_{n \to \infty} y_n(j) = d p_j\big(x(j)\big)$$

for every $j \in \mathbb{N}$ with $p_j\big(x(j)\big) > 0$ (namely $d$ is the common value of $\lim_{n \to \infty} \Big(y_n(i)/p\big(x(i)\big)\Big)$ which does not depend on $i$).

Assume now that $p_j\big(x(j)\big) = 0$. If $x(j) < e_j$, then $y_n(j) \to 0$ by Lemma 1.3(i). If $x(j) = e_j$ and $y_n(j) \nrightarrow 0$ as $n \to \infty$, we can assume that $y_n(j) \geqslant c > 0$ for all $n \in \mathbb{N}$. There exists $i_0 \in \mathbb{N}$ such that $p_{i_0}^-\big(x(i_0)\big) > 0$ (otherwise $p_i^-\big(x(i)\big) = 0$ and $M_i\big(x(i)\big) = 0$

for all $i \in \mathbb{N}$, whence $\rho_M(x) = 0 \neq 1$, a contradiction to (4)). Therefore $p_{i_0}^-(x(i_0)) > 0$ for some $i_0 \in \mathbb{N}$ and consequently

$$\lim_{n\to\infty}\Big(y_n(i_0)p_j\big(x(j)\big) - y_n(j)p_{i_0}^-\big(x(i_0)\big)\Big) < 0,$$

which contradicts Lemma 1.4(i). So, $y_n(j) \to 0$ as $n \to \infty$. This means that

$$y_n(j) \to dp_j\big(x(j)\big)$$

for any $j \in \mathbb{N}$. Combining this with (7), we get

$$1 \leftarrow \langle x, y_n\rangle = \sum_{i=1}^{\infty} x(i)y_n(i) \to d\sum_{i=1}^{\infty} x(i)p_i\big(x(i)\big),$$

whence

$$d = 1 \Big/ \sum_{i=1}^{\infty} x(i)p_i\big(x(i)\big).$$

This implies that $y_n(j) - y(j) \to 0$ as $n \to \infty$ for all $j \in \mathbb{N}$. Combining this with (6), we get

$$\lim_{i_0\to\infty}\sup_n \sum_{i>i_0} N_i\Big(\frac{y_n(i) - y(i)}{2}\Big) = 0.$$

By Lemma 1.1, $\big\|(y_n - y)/2\big\|_N^0 \to 0$, that is, $\|y_n - y\|_N^0 \to 0$ as $n \to \infty$.

II. $x(i) < b_i$ for all $i \in \mathbb{N}$ and $x(1) = M_1^{-1}(1)$.

In this case it follows from condition (C) (iii-1) that $N \in \delta_2^0$ and $p_i^-\big(x(i)\big) = p_i\big(x(i)\big) = 0$ for $i \neq 1$ (since $M_i\big(x(i)\big) = 0$ and $p_i$ is continuous at $x(i)$ for $i \neq 1$). It follows from the proof of Theorem 1.1 that the unique support functional at $x$ is represented by the sequence

$$y = \Big(\frac{1}{x(1)}, 0, 0, \dots\Big).$$

In a similar way to case I, we can prove that $y_n(i) \to 0$ for $i \neq 1$. By (7), we get

$$1 = \lim_{n\to\infty}\sum_{i=1}^{\infty} x(i)y_n(i) = \lim_{n\to\infty} x(1)y_n(1).$$

Therefore, $y_n(1) \to 1/x(1) = y(1)$. From (6) and Lemma 1.1, we get $\|y_n - y\|_N^0 \to 0$ as $n \to \infty$.

III. $x(1) = b_1$, $x(i) < b_i$ for $i \neq 1$, $p_1^-(b_1) < \infty$ and $\rho_M(x) = 1$.

By (C)(iii-2), we get $p_i\big(x(i)\big) = 0$ for $i \neq 1$ and $N \in \delta_2^0$. From the proof of Theorem 1.1 it follows that the support functional at $x$ is given by $y = \big((1/b_1), 0, 0, \dots\big)$. Similarly to case I we can prove that $y_n(i) \to 0$ as $n \to \infty$ for all $i \neq 1$. By (7), we get $y_n(1) \to 1/b_1 = y(1)$. From (6) and Lemma 1.1 it follows that $\|y_n - y\|_N^0 \to 0$ as $n \to \infty$.

IV. $x(1) = b_1$, $x(i) < b_i$ for $i \neq 1$ and $p_1^-(b_1) = \infty$.

In this case, $N \in \delta_2^0$ is not necessary. We get from the proof of Theorem 1.1 that the unique support functional at $x$ is given by $y = (1/b_1, 0, 0, \ldots)$.

If $x(i) < e_i$, by Lemma 1.3(i), we get $y_n(i) \to 0$ as $n \to \infty$. If $x(i) \geqslant e_i$, by Lemma 1.3(ii),

$$\lim_{n\to\infty} (y_n(1)) p_i(x(i)) - y_n(i) p_1^-(x(1)) \geqslant 0.$$

But $p_i(x(i)) < \infty$, $y_n(1) \leqslant N_1^{-1}(1)$, $y_n(1)p_i(x(i)) < \infty$. So, $y_n(i) \to 0$ as $n \to \infty$ for $i \neq 1$. By (7),

$$y_n(1) \to \frac{1}{b_1} = y(1), \quad \text{that is,} \quad y_n(i) \to y(i) \quad (i = 1, 2, \ldots).$$

Now, we shall consider two subcases to finish the proof in this case.

IV.1. For an infinite number of $n$ we have $1 = \|y\|_N^0 = \sum_{i=1}^{\infty} y_n(i) b_i$.

By (7),

$$1 = \lim_{n\to\infty} \sum_{i=1}^{\infty} x(i) y_n(i) = \lim_{n\to\infty} x(1) y_n(1) = x(1)/b_1.$$

Hence $\lim_{n\to\infty} \sum_{i=2}^{\infty} y_n(i) b_i = 0$. Consequently

$$\lim_{n\to\infty} \|y_n - y\|_N^0 = \lim_{n\to\infty} \left\| (0, y_n(2), y_n(3), \ldots) \right\|_N^0 \leqslant \lim_{n\to\infty} \sum_{i=2}^{\infty} y_n(i) b_i = 0.$$

IV.2. There is an infinite number of $n$ for which

$$1 = \|y_n\|_N^0 = \frac{1}{k_n} \big(1 + \rho_N(k_n y_n)\big).$$

First, we shall prove that $k_n \to \infty$. Otherwise, we can assume (passing to a subsequence if necessary) that $k_n \to k_0 < \infty$. Since $k_0/b_1 < \infty = p_1^-(b_1)$, there exists $c > 0$ such that $M_1(b_1) + N_1(k_0/b_1) \geqslant b_1(k_0/b_1) + c = k_0 + c$. Hence

$$\begin{aligned} 1 &= \lim_{n\to\infty} \frac{1}{k_n} \big(1 + \rho_N(k_n y_n)\big) \geqslant \lim_{n\to\infty} \frac{1}{k_n} \Big(1 + N_1\big(k_n y_n(1)\big)\Big) \\ &= \frac{1}{k_0} \big(1 + N_1(k_0/b_1)\big) \geqslant \frac{1}{k_0} \big(M_1(b_1) + N_1(k_0/b_1)\big) \geqslant \frac{k_0 + c}{k_0} = 1 + \frac{c}{k_0}. \end{aligned}$$

This is a contradiction. Notice that

$$\lim_{n\to\infty} \frac{N_1\big(k_n y_n(1)\big)}{k_n} \geqslant \lim_{n\to\infty} \frac{N_1(k_n/b_1)}{k_n} \geqslant \lim_{n\to\infty} \frac{(k_n/b_1)\cdot b_1 - M_1(b_1)}{k_n} = 1,$$

whence

$$1 = \lim_{n\to\infty} \frac{1}{k_n} \big(1 + \rho_N(k_n y_n)\big) \geqslant 1 + \lim_{n\to\infty} \frac{1}{k_n} \sum_{i=2}^{\infty} N_i\big(k_n y_n(i)\big).$$

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So, $\lim_{n\to\infty}(1/k_n)\sum_{i=2}^{\infty}N_i\big(k_ny_n(i)\big)=0$. Consequently

$$\begin{aligned}\lim_{n\to\infty}\|y_n-y\|_N^0 &= \lim_{n\to\infty}\Big\|\big(0,y_n(2),y_n(3),\dots\big)\Big\|_N^0\\ &\leqslant \lim_{n\to\infty}\frac{1}{k_n}\Big(1+\sum_{i=2}^{\infty}N_i\big(k_ny_n(i)\big)\Big)=0.\end{aligned}$$

V. $x(1)=b_1$, $x(i)<b_i$ for $i\neq 1$ and $\rho_M(x)<1$.

By the proof of Theorem 1.1, the unique support functional at $x$ is given by $y=(1/b_1,0,0,\dots)$. Assume to the contrary that $(y_n)$ does not converge coordinatewise for $i\neq 1$. Then we can assume that there is $i\neq 1$ such that $y_n(i)\geqslant c>0$ for all $n\in\mathbb{N}$. Since $x(i)<b_i$, one can find $u>0$ with $x(i)<u<b_i$ satisfying $\sum_{j\neq i}M_j\big(x(j)\big)+M_i(u)\leqslant 1$. Define $\overline{x}$ with $\overline{x}(j)=x(j)$ for $j\neq i$ and $\overline{x}(i)=u$. Then $\rho_M(\overline{x})\leqslant 1$. But

$$\begin{aligned}\langle\overline{x},y_n\rangle &= \langle x,y_n\rangle+y_n(i)\big(u-x(i)\big)\geqslant\langle x,y_n\rangle+c\big(u-x(i)\big)\\ &\to 1+c\big(u-x(i)\big).\end{aligned}$$

This contradicts the inequality $\langle\overline{x},y_n\rangle\leqslant\|\overline{x}\|_M\|y_n\|_N^0=1$. So, $y_n(i)\to 0$ as $n\to\infty$ for $i\neq 1$. Using (7), we get $y_n(1)\to y(1)=1/b_1$. We divide the remaining part of the proof into two subcases.

V-1. $\|y_n\|_N^0=\sum_{i=1}^{\infty}b_iy_n(i)$ for an infinite number of $n$.

Similarly to case IV-1 we can prove that $\|y_n-y\|_N^0\to 0$ as $n\to\infty$.

V-2. $\|y_n\|_N^0=(1/k_n)\big(1+\rho_N(k_ny_n)\big)$ for an infinite number of $n$.

We shall show that in this case $\limsup_{n\to\infty}k_n=\infty$. Otherwise, we can assume that $k_n\to k_0<\infty$, whence

$$\begin{aligned}1 &= \lim_{n\to\infty}\frac{1}{k_n}\big(1+\rho_N(k_ny_n)\big)\geqslant\lim_{n\to\infty}\frac{1}{k_n}\Big(1+N_1\big(k_ny_n(1)\big)\Big)\\ &=\frac{1}{k_0}\Big(1+N_1\Big(\frac{k_0}{b_1}\Big)\Big)=\frac{1}{k_0}\Big(M_1(b_1)+N_1\Big(\frac{k_0}{b_1}\Big)+1-M_1(b_1)\Big)\\ &\geqslant\frac{1}{k_0}\Big(b_1\cdot\frac{k_0}{b_1}+1-\rho_M(x)\Big)=1+\frac{1-\rho_M(x)}{k_0},\end{aligned}$$

a contradiction, which shows that $\limsup_{n\to\infty}k_n=\infty$. Now, we can prove that $\|y_n-y\|_N^0\to 0$ similarly to case IV-2. ∎

As an immediate consequence of Theorem 1.2 we get the following:

**Corollary 1.1.** *The following assertions are equivalent:*

(i) $l_M$ *is strongly smooth,*

(ii) $l_M$ *is very smooth,*

(iii) $l_M$ *is smooth and* $N\in\delta_2^0$.

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