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A CHARACTERISATION OF THE ABSOLUTE QUASI-RETRACTS IN A CLASS OF ACYCLIC CONTINUA

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The class of quasi-retracts of AR's (AQR's) was first considered by Stallings, who observed that every AQR has the fixed point property. More recently, it has been shown that the class of AQR's is closed with respect to the operations of taking cones, suspensions, or products with AR's, and that no AQR separates a Euclidean space. In this paper we show that every AQR is acyclic, and we obtain a simple, direct characterization of the AQR's in a certain class of acyclic continua.

1. Quasi-retracts of AR's

The class of almost continuous functions was introduced by Stallings [10] for the purpose of studying the fixed point property. A function $f: Y \to Y$ is said to be almost continuous if, for every neighbourhood U of the graph $\Gamma(f)$ in $Y \times Y$, there exists a map (continuous function) $g: Y \to Y$ such that $\Gamma(g) \subseteq U$. Stallings observed the following:

1) If Y is a Hausdorff space with the fixed point property, then in fact every almost continuous function $f: Y \to Y$ has a fixed point; and

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2) If $g: Y \to Y$ is continuous and $f: Y \to Y$ is almost continuous, then the composition gf is almost continuous.

It follows that, if Y is an AR (compact metric) and $q: Y \to Y$ is an almost continuous function for which $q(Y)$ is compact and $q^2 = q$, then $q(Y)$ has the fixed point property. We call such a function q a quasi-retraction, and say that the compactum $X = q(Y)$ is a quasi-retract of Y.

Stallings asked the following question, which he attributed to Borsuk: Is every acyclic planar continuum a quasi-retract of a disk? An affirmative answer would imply that every acyclic planar continuum has the fixed point property, thus solving a long-standing problem. This question went unanswered until 1981, when Akis [1] observed that, if X is a quasi-retract of Y, then the cone $C(X)$ is a quasi-retract of $C(Y)$. Hence, if X is a quasi-retract of a disk D (or any AR), then $C(X)$ is a quasi-retract of $C(D)$, and therefore both X and $C(X)$ have the fixed point property. Earlier, Knill [7] had shown that, if X is the disk with spiral (Figure 2a), then $C(X)$ does not have the fixed point property. Thus, X cannot be a quasi-retract of a disk.

Despite the negative answer to the Borsuk-Stallings question, quasi-retractions may yet be useful, in conjunction with other techniques, for attacking the problem concerning the fixed point property of acyclic planar continua. For example, it was shown independently by Bell [3] and by Sieklucki [9] thet, if there exists an acyclic planar continuum X without the fixed point property, then X must contain an indecomposable subcontinuum in its boundary. If it could be shown that such continua are quasi-retracts of disks, the problem would be solved.

Furthermore, the study of quasi-retracts of AR's can lead to other interesting results in fixed point theory. For instance, it was shown in [1] that, if X is a quasi-retract of Y, then the product $X \times E$ with any AR is a quasi-retract of $Y \times E$. Thus, for X a quasi-retract of an AR, not only do the spaces X and $C(X)$ have the fixed point property, but so also does the product of X with any AR.

The following proposition is a particularly useful tool for identifying quasi-retracts:

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PROPOSITION 1.1. [1]. *Let* $Y = Y_0 \supseteq Y_1 \supseteq \dots$ *be a nested sequence of compacta, with* $\cap Y_n = X$ *, such that for each* n *there exists a retraction* $r_n : Y_{n-1} \to Y_n$ *with* $r_n(Y_{n-1} \setminus Y_n) \subseteq X$ *. Then* X *is a quasi-retract of* Y *.*

Proof. Let $q: Y \to X$ be the function defined by $q = \lim_{n\to\infty} r_n r_{n-1} \dots r_1$. Then for each neighbourhood U of $\Gamma(q)$ in $Y \times Y$, $\Gamma(r_n r_{n-1} \dots r_1) \subseteq U$ for large n . Thus q is a quasi-retraction.

Before proceeding further, we should note the following topological invariance property for quasi-retracts of AR's :

PROPOSITION 1.2. [1]. *Let* $X \subseteq Y$ *and* $X' \subseteq Y'$ *, where* Y *and* Y' *are* AR*'s , and* X *is homeomorphic to* X' *. Then* X *is a quasi-retract of* Y *if and only if* X' *is a quasi-retract of* Y' *.*

DEFINITION. *A compactum* X *is an absolute quasi-retract (AQR) if it is homeomorphic to a quasi-retract of the Hilbert cube* Q *.*

By (1.2), any compactum which imbeds as a quasi-retract of <u>some</u> AR is an absolute quasi-retract, and is in fact a quasi-retract of <u>every</u> AR in which it imbeds.

In this paper we characterise the absolute quasi-retracts in a certain class of acyclic continua with the fixed point property, which includes many interesting planar examples. We need the following general result, which will also be used to show that every AQR is an acyclic continuum:

PROPOSITION 1.3. (see [2]). *Let* $q: Y \to Y$ *be a quasi-retraction of an* AR *, with* $X = q(Y)$ *. Then for each neighbourhood* U *of* $\Gamma(q)$ *in* $Y \times Y$ *, there exists a map* $g: Y \to Y$ *such that* $\Gamma(g) \subset U$ *and* $g|X = id$ *.*

Proof. Let $\lambda: Y \times Y \times I \to Y$ be an <u>equiconnecting map</u> for Y , that is, $\lambda(y_1, y_2, 0) = y_1$, $\lambda(y_1, y_2, 1) = y_2$, and $\lambda(y, y, t) = y$ for all $y_1, y_2, y \in Y$ and $t \in I$ (see [5]). Let Δ_X denote the diagonal in $X \times X$. Since $\Delta_X \subseteq \Gamma(q) \subseteq U$, there exists an open neighbourhood W of Δ_X in $Y \times Y$ such that for each $(w, w') \in W$ and $t \in I$, $(w, \lambda(w, w', t)) \in U$.

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Consider $V = W \cup (Y \setminus X) \times Y$; note that V is a neighbourhood of $\Gamma(q)$ in $Y \times Y$. By our hypothesis on q , there exists a map $f: Y \to Y$ with $\Gamma(f) \subseteq U \cap V$. Then $\{(x, f(x)) : x \in X\} \subseteq W$, and by continuity of f there exists a neighbourhood G of X in Y such that $\{(y, f(y)) : y \in G\} \subseteq W$. Let $\alpha: Y \to I$ be a Urysohn map with $\alpha(X) = 0$ and $\alpha(Y \setminus G) = 1$.

We now define the desired map $g: Y \to Y$ by the formula $g(y) = \lambda(y, f(y), \alpha(y))$. Clearly, $g|X = id$. For any $y \in Y \setminus G$, we have $(y, g(y)) = (y, f(y)) \in U$. And for $y \in G$, we have $(y, f(y)) \in W$, which implies that $(y, g(y)) = (y, \lambda(y, f(y), \alpha(y))) \in U$. Thus $\Gamma(g) \subseteq U$.

Rosen [8] has shown that if a compactum $X \subseteq \mathbb{R}^n$ is the quasi-retract of an n-cell, then X does not separate $\mathbb{R}^n$. A stronger result is readily obtainable from (1.3):

COLLARY 1.4. *Every AQR can be imbedded in* Q *as the intersection of a nested sequence of Hilbert cubes.*

Proof. Let $q: Q \to Q$ be a quasi-retraction. We show first that $X = q(Q)$ has trivial shape, that is, every map $f: X \to P$ into a polyhedron is null-homotopic. Let $F: N(X) \to P$ be an extension of f over some neighbourhood of X in Q . By (1.3), there exists a map $g: Q \to N(X)$ with $g|X = id$. Then the extension $Fg: Q \to P$ of f is null-homotopic, and so is f . Chapman's shape-complement theorem [4] shows that every trivial shape compactum can be imbedded in Q as the intersection of a nested sequence of Hilbert cubes.

2. Characterizing the *AQR*'s in a class of acyclic continua

Let S be the unit circle in the complex plane, and let K denote an AR . Let $e: [0, \infty) \to S$ be the covering map defined by $e(t) = e^{2\pi i t}$. For a map $\lambda: S \to K$, let $\hat{\lambda} = \lambda e: [0, \infty) \to K$, and define the continuum $X(\lambda) \subseteq [0, \infty] \times K$ by

$$X(\lambda) = \{(t, \hat{\lambda}(t)) : 0 \le t < \infty\} \cup \{\infty\} \times K .$$

Thus, $X(\lambda)$ may be considered as the disjoint union $[0, \infty) \cup K$, with the topology defined by the collection of basic open sets $\{U : U$ is open

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in $[0, \infty)\} \cup \{V \cup [\hat{\lambda}^{-1}(V) \cap (N, \infty)] : V$ is open in K and $N < \infty\}$.

Each $X(\lambda)$ is acyclic, since it is the intersection of a nested sequence of AR's in $[0, \infty] \times K$. Also, each $X(\lambda)$ has the fixed point property (in fact, the product of $X(\lambda)$ with any AR has the fixed point property). Note that for nonconstant λ , $X(\lambda)$ is not an AR , or even locally connected. In what follows, we obtain a partial answer to the general question:

When is $X(\lambda)$ an AQR?

PROPOSITION 2.1. *If* $\lambda: S \to K$ *factors through an arc, then* $X(\lambda)$ *is an* AQR .

Proof. Let $\alpha: S \to I$ and $\sigma: I \to K$ be maps such that $\lambda = \sigma\alpha$, and assume that $\alpha(S) = I$. Let M be the mapping cylinder of σ . We consider M as the disjoint union $M = ([0, \infty) \times I) \cup K$ (where $\{\infty\} \times I$ is identified with $\sigma(I)$ in K) . Since K is an AR , so is M . Define $\hat{\alpha} = \alpha e: [0, \infty) \to I$, and let G be the graph of $\hat{\alpha}$. Then $G \cup K \subset M$ is homeomorphic to $X(\lambda)$. Choose an increasing sequence $\{t_n\}$ in $[0, \infty)$ such that $t_0 = 0$, $\hat{\alpha}(t_1) = \hat{\alpha}(t_3) = \ldots = 1$, $\hat{\alpha}(t_2) = \hat{\alpha}(t_4) = \ldots = 0$, and $t_n \to \infty$. Set $M_0 = M$ and $M_1 = M_0 \setminus \{(t, y) : 0 \leqq t \leqq t_1 , \hat{\alpha}(t) < y \leqq 1\}$. For each $n \geqq 1$, set $M_{2n} = M_{2n-1} \setminus \{(t, y) : t_{2n-2} \leqq t \leqq t_{2n}, 0 \leqq y < \hat{\alpha}(t)$ and $M_{2n+1} = M_{2n} \setminus \{(t, y) : t_{2n-1} \leqq t \leqq t_{2n+1}, \hat{\alpha}(t) < y \leqq 1\}$. Then $G \cup K = \cap M_i$ is an AQR by (1.1) .

DEFINITION. *A map* $\omega: S \to K$ *is free if there exists a map* $\theta: S \to S$ *such that* ω *and* $\omega\theta$ *have disjoint graphs.*

THEOREM 2.2. *Let* $\omega: S \to K$ *be a free map. Then for any map* $\psi: S \to S$ *, the continuum* $X(\omega\psi)$ *is an* AQR *if and only if* ψ *is inessential.*

Proof. If ψ is inessential, it factors through an arc, and therefore $X(\omega\psi)$ is an AQR by (2.1).

Now suppose that ψ is essential. To show that $X(\omega\psi)$ cannot be an AQR , we first construct an AR space M which serves as a compact

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mapping cylinder for $\hat{\omega} = \omega e: [0, \infty) \to K$, and which contains a copy of $X(\omega\psi)$. We may assume that $\deg \psi = m > 0$. Let $H = \{(t, y) : 0 \leqq t < \infty , 0 \leqq y \leqq mt\}$. Take $M = H \cup K$, with the topology defined by basic open sets of the following types:

1) open sets U in H ; and
2) sets $V \cup \{(t, y) : t > N$ and $\hat{\omega}(y) \in V\}$, for V open in K and $N < \infty$.

Since M is a compact Hausdorff space with a countable base, it is metrisable. For each $y < \infty$, let $M_y = (H \cap [0, \infty) \times [0, y]) \cup K$. Note that M_y is topologically the mapping cylinder of $\hat{\omega}|[0, y]$, hence an AR . Since M admits arbitrarily small deformation retractions onto the subspaces M_y , M is itself an AR (see [6], Theorem 5.3).

There exists a map $\tilde{\psi}: [0, \infty) \to [0, \infty)$ such that $e\tilde{\psi} = \psi e$; we may assume without loss of generality that $\tilde{\psi}(0) = 0$. Since $\deg \psi = m$, $\tilde{\psi}(t + 1) = \tilde{\psi}(t) + m$ for all t . Choose $a \geqq 1$ such that $ma \geqq \tilde{\psi}(t)$ for all $0 \leqq t \leqq 1$. Let $G = \{(a + t, \tilde{\psi}(t)) : t \geqq 0\} \subseteq H$. Then $G \cup K \subseteq M$ is homeomorphic to $X(\omega\psi)$, so if $X(\omega\psi)$ is an AQR , there exists a quasi-retraction of M onto $G \cup K$.

Let $L = \{(t, y) : a + 1 \leqq t < \infty , y = m(t - a - 1) \subset H\}$. Note that $L \cap G = \emptyset$.

Since ω is free, there exists a map $\theta: S \to S$, necessarily of degree 1 , such that ω and $\omega\theta$ have disjoint graphs. Let $\tilde{\theta}: [0, \infty) \to [0, \infty)$ be a map such that $e\tilde{\theta} = \theta e$.

Let $p: H \to [0, \infty)$ denote the projection onto the second coordinate. Define $F \subset M \times M$ to be the closure of the subset

$\{(w, z) \in H \times L : \tilde{\theta}(p(w)) = p(z)$ or, $p(w) = 0$ and $0 \leqq p(z) \leqq \tilde{\theta}(0)\}$.

Then $F \subset M \times L \cup (K \times K \setminus \Delta)$. Thus, the graph of any quasi-retraction of M onto $G \cup K$ is disjoint from F .

Assuming the existence of such a quasi-retraction we then obtain by (1.3) a map $f: M \to M$ with $\Gamma(f) \cap F = \emptyset$ and $f|G \cup K = id$. Note that the quotient space $D = M / K = H \cup \{\infty\}$ is topologically a disk, with K

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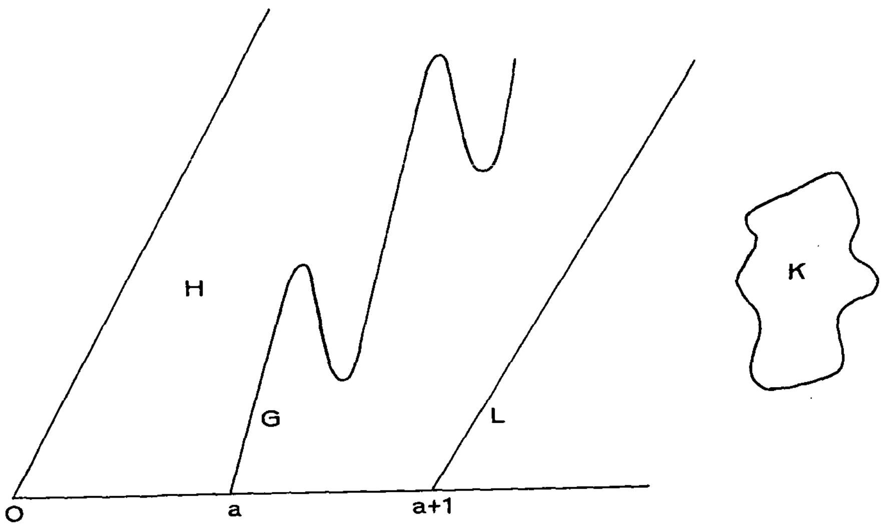

FIGURE 1

corresponding to a boundary point ∞ . Since $f|K = id$, f induces a map $\phi: D \to D$ with $\phi(\infty) = \infty$. For each $t \geq 0$, consider the arc $A_t = ([a + t, \infty) \times \tilde{\psi}(t)) \cup \{\infty\} \subset D$. Note that ϕ fixes the endpoints of A_t , and for the point z_t in L such that $p(z_t) = \widetilde{\theta\psi}(t)$, z_t is not in $\phi(A_t) \cup A_t$. Let γ_t denote the element of $\pi_1(D \setminus \{z_t\})$ determined by combining the maps $id|A_t$ and $\phi|A_t$.

For $t = 0$, $p(\phi(A_0) \cap L) \subset (\tilde{\theta}(0), \infty)$, and it follows that γ_0 is nontrivial. On the other hand, for all sufficiently large t, γ_t is trivial. To see this, consider the arc $A_t^* = (A_t \cap H) \cup \widehat{\omega}\widetilde{\psi}(t) \subset M$. We have diam $(A_t^* \cup f(A_t^*)) \to 0$ as $t \to \infty$, while the distance between z_t and $\widehat{\omega}\widetilde{\psi}(t)$ remains bounded away from 0 . Thus for large t , the subset $A_t^* \cup f(A_t^*)$ contracts to a point in $M \setminus \{z_t\}$, which implies that γ_t is trivial. But since each of the subsets $\{t : \gamma_t$ is trivial$\}$ and

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$\{t : \lambda_t$ is nontrivial$\}$ is open, this contradicts the connectedness of $[0, \infty)$. Thus $X(\omega\psi)$ cannot be an AQR if ω is free and ψ is essential.

COROLLARY 2.3. *For every free map* $\omega: S \to K$, $X(\omega)$ *is not an* AQR.

In particular, neither the disc with a spiral (Figure 2a), nor the triod with a spiral (Figure 2b), is an AQR.

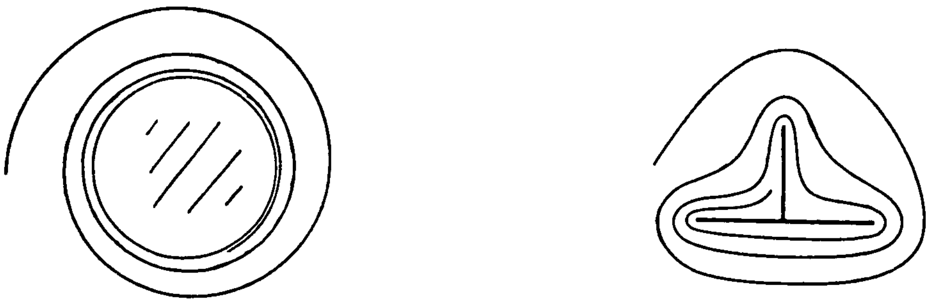

FIGURE 2

COROLLARY 2.4. *For any map* $\lambda: S \to K$ *such that* $\lambda(S)$ *is a simple closed curve,* $X(\lambda)$ *is an* AQR *if and only if* $\lambda: S \to \lambda(S)$ *is inessential.*

Proof. The inclusion map $\lambda(S) \hookrightarrow K$ is free, hence the result follows immediately from (2.2).

It follows from (2.4) that the spaces shown in Figures 3a and 3b are AQR's while the space shown in Figure 3c is not.

Let $B = \{(r, \theta) : r \leq 1\}$ be the unit disc in the plane, with $S = \partial B$, and consider the triod

$$T = \{(r, \theta) : r \leq 1 , \theta = 2n\pi/3 \text{ for some integer } n\} .$$

To each map $\lambda: S \to T$ we may associate a map $\bar{\lambda}: S \to S$ defined as follows:

1) $\bar{\lambda}$ agrees with λ over $\lambda^{-1}(\partial T)$;
2) for each subarc $J \subset S$ such that $\lambda(J) \cap \partial T = \lambda(\partial J) = \{p\}$, $\bar{\lambda}(J) = \{p\}$; and

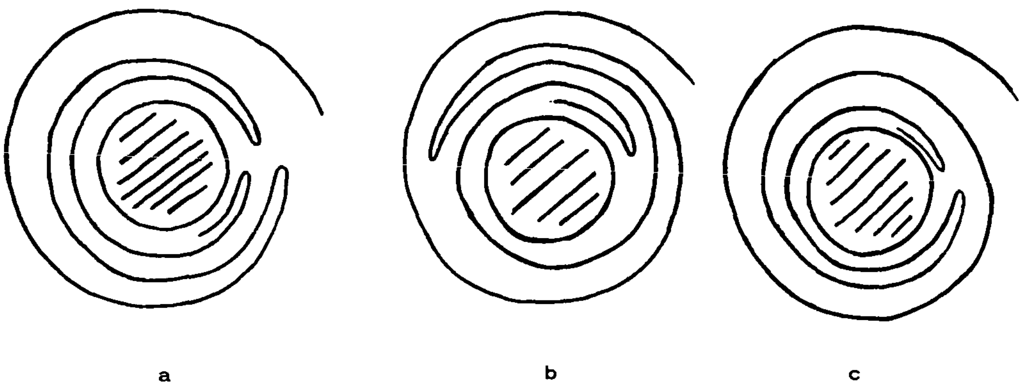

FIGURE 3

3) for each subarc $J \subset S$ such that $J \cap \lambda^{-1}(\partial T) = \partial J$ and $\lambda(\partial J) = \{p, q\}$, $\bar{\lambda}|J$ is a homeomorphism onto the subarc of S corresponding to $\{p, q\}$.

We say that $\lambda: S \to T$ is boundary inessential if the associated map $\bar{\lambda}$ is inessential. Note that it is quite easy to ascertain whether λ is boundary inessential.

DEFINITION. *The map* $\lambda: S \to T$ *is simple if, for every arc* $J \subset S$ *such that* $\lambda(J) \cap \partial T = \emptyset$, $\lambda(J)$ *is an arc.*

It is easily seen that the continuum $X(\lambda)$ is planar if and only if λ is simple.

COROLLARY.2.5. *For any map* $\lambda: S \to K$ *such that* $\lambda(S)$ *is a triod and* $\lambda: S \to \lambda(S)$ *is simple,* $X(\lambda)$ *is an AQR if and only if* λ *is boundary inessential.*

Proof. We may consider that $\lambda(S) = T$ as described above. Define a map $\omega: S \to T$ as follows:

1) $\omega(1, n\pi/3) = (0, 0)$ if n is odd ;

2) $\omega(1, n\pi/3) = (1, n\pi/3)$ if n is even; and

3) ω is one-to-one over each arc $\{(1, \theta) : n\pi/3 \leq \theta \leq (n + 1)\pi/3\}$.

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Note that ω is a free map. Clearly, every simple map $\lambda: S \to T$ factors through ω, that is, there exists a map $\psi: S \to S$ such that $\lambda = \omega\psi$. Moreover, the map ψ is homotopic to the associated map $\bar{\lambda}: S \to S$. Thus, $X(\lambda)$ is an AQR if and only if λ is boundary inessential.

It follows from (2.5) that the space shown in Figure 2b is not an AQR, while the spaces shown in Figure 4 are AQR's.

CONJECTURE. The corollary holds without the requirement that λ be simple.

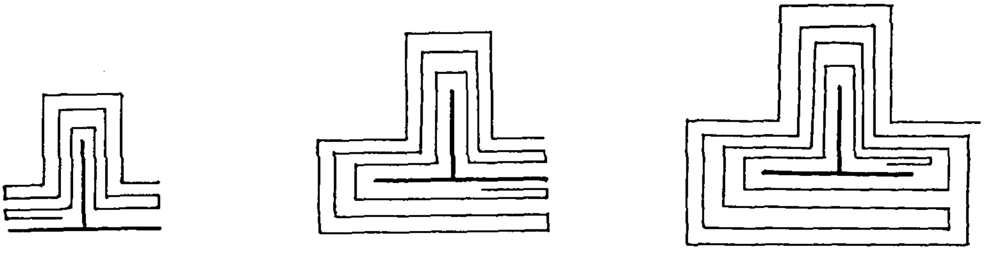

FIGURE 4

3. A question

The disc with a spiral (Figure 2a) is essentially the only known example of a planar continuum with the fixed point property, whose cone fails to have the property. (All other known examples are modifications of this one). It was this observation that prompted our consideration of the continua $X(\lambda)$ of Section 2. The result in (2.4) implies that if a continuum X is the disjoint union of a disc with a ray which approaches its boundary "inessentially", then the cone $C(X)$ <u>has</u> the fixed point property; a similar statement follows from (2.5), with the disc replaced by a triod. The following question naturally arises: if X is the disjoint union of a disc (or a triod) with a ray which approaches its boundary "essentially", can $C(X)$ have the fixed point property? In particular, if X is the continuum in Figure 2b, or the continuum in Figure 3c, does $C(X)$ have the fixed point property?

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