A PAPPUS TYPE THEOREM IN THE AFFINE GROUP

R. Paré

(received February 12, 1968)

Introduction. In [3] H. Schwerdtfeger embedded the one-dimensional affine group over the real numbers in the projective plane. The relationship between group-theoretical properties and geometrical concepts was studied.

In this paper the methods of [3] are used to prove Pappus' theorem. In the final section we give a similar theorem for (4n+2)-gons.

This paper is a generalization of part of my master's thesis, written under the direction of Professor H. Schwerdtfeger.

1. Preliminaries. Let $\mathcal{G}$ be the affine group over the real numbers, i.e.

$$\{ (a, b) \mid a, b \in \mathbf{R}, a \neq 0 \}$$

with the operation

$$(a, b)(a', b') = (aa', ab' + b) .$$

Embed $\mathcal{G}$ in the cartesian model of the projective plane, π, by associating with the element $(a, b) \in \mathcal{G}$, the point with the cartesian coordinates (a, b).

There are two "exceptional" lines of π whose points are not elements of $\mathcal{G}$, the y-axis and the line at infinity, which we shall call $\mathcal{L}_0$ and $\mathcal{L}_\infty$ respectively. Two lines that intersect in a point of $\mathcal{L}_0$ are said to be 0-parallel and two lines that intersect in a point of $\mathcal{L}_\infty$ are said to be ∞-parallel. The point of intersection of $\mathcal{L}_0$ and $\mathcal{L}_\infty$ will be called $\mathfrak{U}$.

Given a line in the projective plane, we shall call the set of all elements of $\mathcal{G}$ which correspond to points on this line, a $\mathcal{G}$-line. If P and Q are two points of the plane, we shall represent the $\mathcal{G}$-line through P and Q by (P; Q). In what follows, we shall identify a $\mathcal{G}$-line with the line to which it corresponds.

Canad. Math. Bull. vol 11, no. 4, 1968.

https://doi.org/10.4153/CMB-1968-065-5 Published online by Cambridge University Press

In [3] it is shown that all $\mathcal{G}$-lines are the cosets of normalizers of single elements. It is easily seen that if $A, B \in \mathcal{G}$, then

$$(A; B) = \mathfrak{n}(AB^{-1})B = B\,\mathfrak{n}(B^{-1}A)$$

where by definition

$$\mathfrak{n}(C) = \{X \in \mathcal{G} \mid XC = CX\}.$$

Two ∞-parallel lines are shown to be left cosets of the same normalizer whereas 0-parallel lines are the right cosets. It is also shown that $\mathfrak{H} = \mathfrak{n}((1, 1))$ is the commutator subgroup of $\mathcal{G}$. It is easily seen that $\mathfrak{H}$ must pass through $\mathfrak{U}$.

The conjugate classes of $\mathcal{G}$ are $\{I\}$, $\mathfrak{H} \setminus \{I\}$, and the proper cosets of $\mathfrak{H}$, where I is the unit element of $\mathcal{G}$.

2. In this section we prove a proposition needed for our proof of Pappus' theorem. First we prove the following special case:

PROPOSITION 1. Let $\mathfrak{a}_1, \mathfrak{a}_3 \in \mathcal{L}_0$ and $\mathfrak{a}_2, \mathfrak{a}_4 \in \mathcal{L}_\infty$ with $\mathfrak{a}_i \neq \mathfrak{U}$, $i = 1, 2, 3, 4$. Let $A_i \in (\mathfrak{a}_i ; \mathfrak{a}_{i+1})$, $i = 1, 2, 3$, then $A_1A_2^{-1}A_3 \in (\mathfrak{a}_4 ; \mathfrak{a}_1)$.

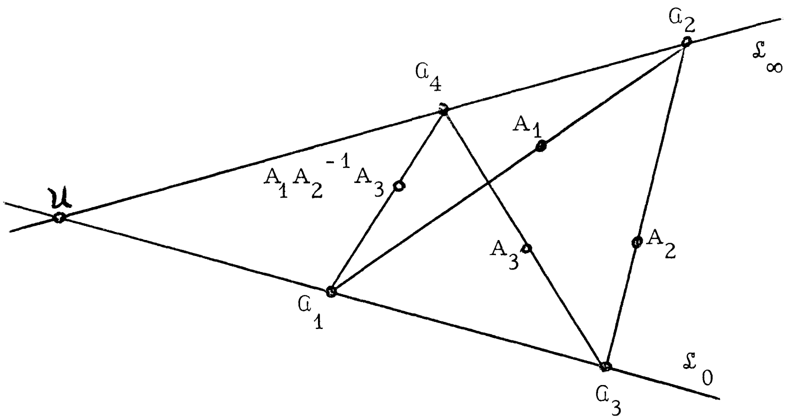

Figure 1

Proof. Let the line $(\mathfrak{a}_1 ; \mathfrak{a}_2) = A_1\mathfrak{n}(B)$ for some $B \in \mathcal{G}$. $A_2\mathfrak{n}(B)$ is ∞-parallel to $A_1\mathfrak{n}(B)$ and $A_2 \in A_2\mathfrak{n}(B)$, thus $A_2\mathfrak{n}(B) = (\mathfrak{a}_2 ; \mathfrak{a}_3)$. (See Figure 1.) It is easily shown that

https://doi.org/10.4153/CMB-1968-065-5 Published online by Cambridge University Press

$$A_2 \mathfrak{h}(B) = \mathfrak{h}(A_2 B A_2^{-1}) A_2 .$$

Now $\mathfrak{h}(A_2BA_2^{-1})A_3$ is 0-parallel to $\mathfrak{h}(A_2BA_2^{-1})A_2$ and $A_3 \in \mathfrak{h}(A_2BA_2^{-1})A_3$, thus

$$\mathfrak{h}(A_2BA_2^{-1})A_3 = (\mathfrak{a}_3;\ \mathfrak{a}_4) .$$

Consider the line $\mathfrak{h}(A_1BA_1^{-1})A_1A_2^{-1}A_3$. This line is 0-parallel to

$$\mathfrak{h}(A_1BA_1^{-1})A_1 = A_1\mathfrak{h}(B) = (\mathfrak{a}_1; \mathfrak{a}_2).$$

Also

$$\mathfrak{h}(A_1BA_1^{-1})A_1A_2^{-1}A_3 = A_1A_2^{-1}A_3\,\mathfrak{h}(A_3^{-1}A_2BA_2^{-1}A_3)$$

is ∞-parallel to

$$A_3\mathfrak{h}(A_3^{-1}A_2BA_2^{-1}A_3) = \mathfrak{h}(A_2BA_2^{-1})A_3 = (\mathfrak{a}_3;\ \mathfrak{a}_4) .$$

Therefore

$$\mathfrak{h}(A_1BA_1^{-1})\ A_1A_2^{-1}A_3 = (\mathfrak{a}_4 ; \mathfrak{a}_1)$$

and consequently $A_1A_2^{-1}A_3 \in (\mathfrak{a}_4; \mathfrak{a}_1)$.

The following proposition is easily proved by induction using Proposition 1.

PROPOSITION 2. _Let_ $\mathfrak{a}_i \in \mathcal{L}_0$ _and_ $\mathfrak{a}_{i+1} \in \mathcal{L}_\infty$ _for_ $i = 1, 3, 5, \ldots, 2n-1$, _and suppose_ $\mathfrak{a}_i \neq \mathfrak{U}$ _for_ $i = 1, 2, \ldots, 2n$. _If_ $A_i \in (\mathfrak{a}_i; \mathfrak{a}_{i+1})$ _for_ $i = 1, 2, \ldots, 2n-1$, _then the product_ $A_1A_2^{-1}A_3 \ldots A_{2n-2}^{-1}A_{2n-1} \in (\mathfrak{a}_{2n};\mathfrak{a}_1)$.

3. _Pappus' theorem_.

THEOREM. _If the vertices of a simple hexagon lie alternately on two straight lines, then the three pairs of opposite sides intersect in collinear points_.

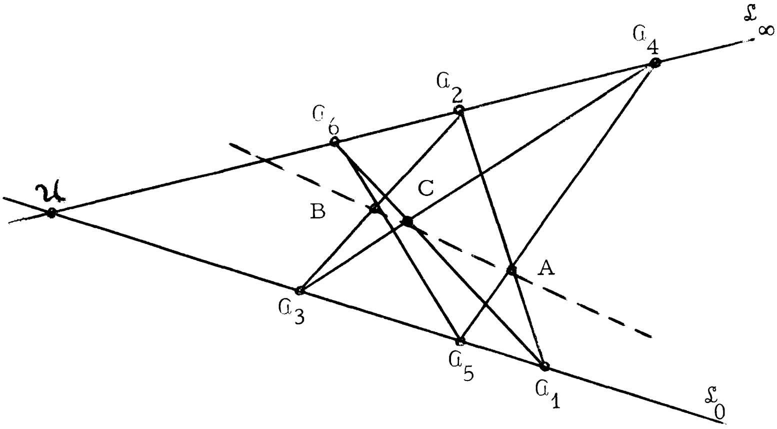

Figure 2

Proof. Since there exist collineations transforming any two distinct lines into any other two distinct lines, there will be no loss in generality if we suppose that the vertices of the hexagon lie on $\mathcal{L}_0$ and $\mathcal{L}_\infty$.

Let $\mathcal{G}_1, \mathcal{G}_3, \mathcal{G}_5 \in \mathcal{L}_0$ and $\mathcal{G}_2, \mathcal{G}_4, \mathcal{G}_6 \in \mathcal{L}_\infty$ be the vertices of the hexagon. We may assume that $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4, \mathcal{G}_5, \mathcal{G}_6, \mathfrak{U}$ are all different, for otherwise the hexagon has coinciding sides and the theore then is trivial.

Let

$$\{A\} = (\mathcal{G}_1; \mathcal{G}_2) \cap (\mathcal{G}_4; \mathcal{G}_5)$$

$$\{B\} = (\mathcal{G}_2; \mathcal{G}_3) \cap (\mathcal{G}_5; \mathcal{G}_6)$$

$$\{C\} = (\mathcal{G}_3; \mathcal{G}_4) \cap (\mathcal{G}_6; \mathcal{G}_1)$$

(see Figure 2).

By Proposition 2, $AB^{-1}CA^{-1}B \in (\mathcal{G}_1; \mathcal{G}_6)$. Also $AB^{-1}CA^{-1}B\mathfrak{H} = C\mathfrak{H}$, since $\mathfrak{G}/\mathfrak{H}$ is abelian, thus $AB^{-1}CA^{-1}B \in C\mathfrak{H}$. But $C\mathfrak{H}$ passes through $\mathfrak{U}$ and $(\mathcal{G}_1; \mathcal{G}_6)$ does not, therefore they cut in a point of $\mathfrak{G}$.

Consequently $\{AB^{-1}CA^{-1}B\} = C\mathcal{H} \cap (\mathcal{G}_1; \mathcal{G}_6)$ but $\{C\} = C\mathcal{H} \cap (\mathcal{G}_1; \mathcal{G}_6)$,

therefore $C = AB^{-1}CA^{-1}B$

$$\Longrightarrow AB^{-1}C = CB^{-1}A \quad \Longrightarrow \quad AB^{-1}CB^{-1} = CB^{-1}AB^{-1}$$

$$\Longrightarrow \quad CB^{-1} \in h(AB^{-1}) \Longrightarrow C \in h(AB^{-1})B;$$

therefore A, B, C are collinear.

4. Generalization.

Definition. In what follows a non-degenerate simple m-gon will mean a polygon consisting of m distinct points and m distinct lines joining these points in their given order.

THEOREM. Given two lines $\mathcal{L}_1$ and $\mathcal{L}_2$ and 2n points A_i not on $\mathcal{L}_1$ or $\mathcal{L}_2$, then there exists a line $\mathcal{L}$ such that if a non-degenerate simple (4n+2)-gon with vertices, $\mathcal{G}_i$, lying alternately on $\mathcal{L}_1$ and $\mathcal{L}_2$, is such that

$$\{A_i\} = (\mathcal{G}_i; \mathcal{G}_{i+1}) \cap (\mathcal{G}_{2n+1+i}; \mathcal{G}_{2n+2+i})$$

for $i \leqslant 2n$, then $(\mathcal{G}_{2n+1}; \mathcal{G}_{2n+2})$ and $(\mathcal{G}_{4n+2}; \mathcal{G}_1)$ cut on $\mathcal{L}$.

(Figure 3 illustrates the case where n = 2.)

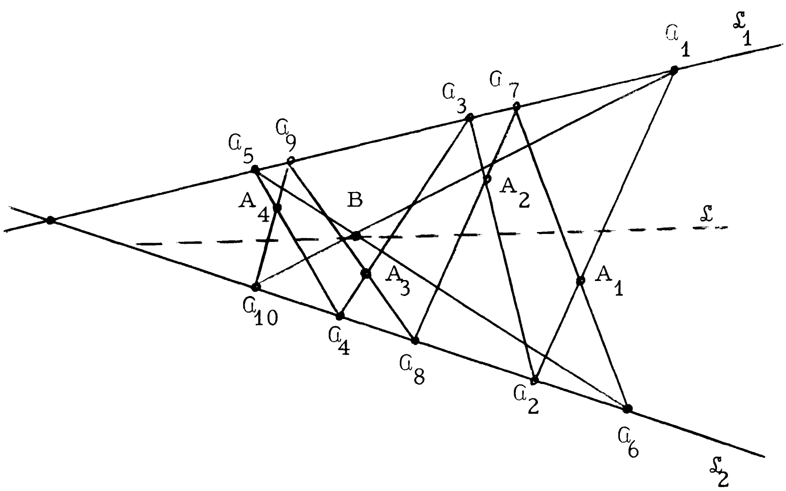

Figure 3

https://doi.org/10.4153/CMB-1968-065-5 Published online by Cambridge University Press

Proof. As in § 3, we shall assume that $\mathcal{L}_1 = \mathcal{L}_0$ and $\mathcal{L}_2 = \mathcal{L}_\infty$.

Let

$$A_{2n}A_{2n-1}^{-1} \dots A_2A_1^{-1} = C$$

and

$$A_1^{-1}A_2 \dots A_{2n-1}^{-1}A_{2n} = D.$$

Since $\mathcal{G}/\mathcal{H}$ is commutative $C\mathcal{H} = D\mathcal{H}$ and thus C is a conjugate of D, i.e. there exists $E \in \mathcal{G}$ such that $C = EDE^{-1}$.

Let $\mathcal{L} = E\mathfrak{h}(D)$ and

$$(\mathfrak{a}_{2n+1}; \mathfrak{a}_{2n+2}) \cap (\mathfrak{a}_{4n+2}; \mathfrak{a}_1) = \{B\}.$$

We shall show that $B \in E\mathfrak{h}(D)$.

Note that $\mathfrak{a}_i \neq \mathfrak{U}$ for otherwise $\mathfrak{a}_j = \mathfrak{U}$ for all j, and our polygon would be degenerate. By Proposition 2, the product

$$A_1A_2^{-1}A_3 \dots A_{2n}^{-1}BA_1^{-1}A_2A_3^{-1} \dots A_{2n} = C^{-1}BD \in (\mathfrak{a}_{4n+2}; \mathfrak{a}_1).$$

But $C^{-1}BD \in B\mathcal{H}$ and $B\mathcal{H} \cap (\mathfrak{a}_{4n+2}; \mathfrak{a}_1) = \{B\}$ thus

$$C^{-1}BD = B$$

$$\Longrightarrow BDB^{-1} = C = EDE^{-1}$$

$$\Longrightarrow (E^{-1}B)D(E^{-1}B)^{-1} = D$$

$$\Longrightarrow E^{-1}B \in \mathfrak{h}(D)$$

$$\Longrightarrow B \in E\mathfrak{h}(D)$$

It would be interesting to find a geometrical description of the line $\mathcal{L}$ defined above. The following partial results are noteworthy.

(1) If $A_1, A_2, A_3, \ldots, A_{2n-1}$ are collinear, then $\mathcal{L}$ passes through A_{2n}.

Assume that $A_1, A_2, A_3, \ldots, A_{2n-1} \in L\eta(K)$, then $A_i = LK_i$ where $K_i \in \eta(K)$. We see that

$$C = A_{2n}K_{2n-1}^{-1}K_{2n-2}K_{2n-3}^{-1} \ldots K_2K_1^{-1}L^{-1}$$

and

$$D = K_1^{-1}K_2K_3^{-1} \ldots K_{2n-3}^{-1}K_{2n-2}K_{2n-1}^{-1}L^{-1}A_{2n}.$$

Now $\eta(K_i)$ is the line passing through K_i and I, and so is $\eta(K)$, thus $\eta(K_i) = \eta(K) = \eta(K_j)$, therefore K_i commutes with K_j and K_j^{-1}. We conclude that

$$C = A_{2n}DA_{2n}^{-1}$$

and thus $\mathcal{L} = A_{2n}\eta(D)$. Therefore $A_{2n} \in \mathcal{L}$.

(2) By relabeling the A_i, we show that if $A_2, A_3, \ldots, A_{2n}$ are collinear, then $\mathcal{L}$ passes through A_1.

(3) If all the A_i are collinear, then by combining (1) and (2) we see that $\mathcal{L}$ must pass through A_1 and A_{2n}. If $A_1 \neq A_{2n}$ then $\mathcal{L}$ must pass through all the A_i.

Pappus' theorem is a special case of this last result.

https://doi.org/10.4153/CMB-1968-065-5 Published online by Cambridge University Press

REFERENCES

1. A. Heyting, Axiomatic projective geometry (Amsterdam).

2. A.G. Kurosh, The theory of groups (New York).

3. H. Schwerdtfeger, Projective geometry in the one-dimensional affine group. Canad. J. Math. 16 (1964) 683-700.

4. O. Veblen and J.W. Young, Projective geometry, Vol. 1 (Boston 1910).

McGill University

https://doi.org/10.4153/CMB-1968-065-5 Published online by Cambridge University Press