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ALGEBRAIC INNER DERIVATIONS ON OPERATOR ALGEBRAS

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Introduction. Let A be a C^*-algebra, let p be a polynomial over $\mathbf{C}$, and let a in $M(A)$ (the multiplier algebra of A) be such that $p(\text{ad } a) = 0$. In this paper we study the following problem: when does there exist λ in $Z(M(A))$ (the centre of $M(A)$) such that $p(a - \lambda) = 0$? The first result of this type known to us is due to I. N. Herstein [**7**], who showed that for a simple ring with identity, such a λ always exists when p is of the form $p(x) = x^k$ for some positive integer k. Later, in [**8**], C. R. Miers showed that the result is true for any primitive unital C^*-algebra and any polynomial whatever. It was also shown in [**8**] that if A is a unital C^*-algebra acting on H and p is any polynomial, then such a λ exists in the larger algebra $Z(A'')$. In particular, the strict result holds for any von Neumann algebra, A.

In this paper, we show that for any unital C^*-algebra, A, the result holds "locally" over $\hat{A}$. The problem of patching the local solutions together gives rise to an obstruction in $\check{H}^1(\hat{A}, G_p)$, the first Čech cohomology group of $\hat{A}$ with coefficients in the sheaf of continuous G_p-valued functions where G_p is the additive subgroup of $\mathbf{C}$ generated by the roots of p. We show that the obstruction can be non-trivial and so the general result fails. However, when $p(x) = x^k$ for some positive integer k, we have $G_p = \{0\}$ so that the obstruction vanishes, and in fact, the problem can always be solved in this case. Finally, we show that the general result holds for C^*-algebras which can be realized as the norm-continuous sections of certain bundles.

1. Preliminaries. All Hilbert spaces in this paper are complex and the word *operator* will always mean bounded linear operator on some Hilbert space. For an operator a on a Hilbert space H, we will let $\sigma(a)$ denote the spectrum of a, that is, the set $\{z \in \mathbf{C} | (z1 - a) \text{ is not invertible}\}$. By the

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point spectrum of a, $\sigma_p(a)$, we will mean the set of eigenvalues for a. By $\mathscr{B}(H)$ we will mean the algebra for all operators on H.

If E and F are finite subsets of $\mathbf{C}$ and $\delta > 0$, then we write $E \overset{\delta}{\subseteq} F$ if for each $z \in E$ there is a $w \in F$ with $|z - w| \leqq \delta$. We also define the Hausdorff distance between E and F to be:

$$\operatorname{dist}(E, F) = \inf\{\delta > 0 | E \overset{\delta}{\subseteq} F \text{ and } F \overset{\delta}{\subseteq} E\}.$$

We will use $\#(E)$ to denote the cardinality of E.

If X is a topological space and H is a Hilbert space, we will let $C^b_{*s}(X, \mathscr{B}(H))$ denote the C^*-algebra of all bounded *-strongly continuous functions from X into $\mathscr{B}(H)$. We will let $C^b(X)$ denote the commutative C^*-algebra of all continuous bounded functions from X into $\mathbf{C}$.

In general, our notations and definitions for C^*-algebras will coincide with [**3**].

We note that since $A \subseteq M(A) \subseteq A^{**}$, the second dual of A, a simple continuity argument shows that for $a \in M(A)$, $p(\text{ad } a_{|A}) = 0$ if and only if $p(\text{ad } a) = 0$ on $M(A)$. Therefore, replacing A with $M(A)$ allows us to assume that A has an identity.

We thank Iain Raeburn for the reference [**4**].

2. Semicontinuity of spectrum. In this section we show that the set-valued mapping $\pi \to \sigma(\pi(a))$ is semicontinuous on $\hat{A}$ in a very strong sense whenever ad a satisfies a polynomial identity.

2.1. LEMMA. *Let a be an operator on a Hilbert space H such that $p(\text{ad } a) = 0$ for some polynomial p. Then $\sigma(a) = \sigma_p(a)$ is contained in a translate of the roots of p.*

Proof. By [**8**, Theorem 3] there is an element λ_0 in $\sigma_p(a)$ and $p(a - \lambda_0) = 0$ for all such λ_0. Since

$$\sigma(a) = \sigma(a - \lambda_0) + \lambda_0 \quad \text{and} \quad \sigma_p(a) = \sigma_p(a - \lambda_0) + \lambda_0$$

we may well assume that $p(a) = 0$.

With this assumption, let $\lambda \in \sigma(a)$. Then, by the division algorithm

$$p(x) = (x - \lambda)q(x) + p(\lambda)$$

so that

$$0 = p(a) = (a - \lambda)q(a) + p(\lambda).$$

If $p(\lambda) \neq 0$ then we get

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$$(a - \lambda)^{-1} = \frac{-1}{p(\lambda)} q(a)$$

which contradicts $\lambda \in \sigma(a)$. Thus, $p(\lambda) = 0$, and $\sigma(a)$ is contained in the roots of p. Now, let p_0 be the minimal polynomial of a so that for $\lambda \in \sigma(a)$ we have $p_0(x) = (x - \lambda)q(x)$ where $q(a) \neq 0$. Thus, $0 = p_0(a) = (a - \lambda)q(a)$ and we can choose $\xi \in H$ with $q(a)\xi \neq 0$. Clearly, $q(a)\xi$ is an eigenvector for a with eigenvalue λ, so that $\lambda \in \sigma_p(a)$.

2.2. LEMMA. *Let $\{a_n\}$ be a bounded net of operators on a Hilbert space H and let p be a polynomial. Suppose that there is a net $\{\lambda_n\}$ in $\mathbf{C}$ and elements a in $\mathcal{B}(H)$, λ in $\mathcal{C}$ with*

*(1) $a_n \to a$ *-strongly*

(2) $\lambda_n \in \sigma(a_n)$ for all n, and $\lambda \in \sigma(a)$

(3) $p(a - \lambda) = 0 = p(a_n - \lambda_n)$ for all n.

Then, each point in $\sigma(a)$ is the limit of points in $\cup \sigma(a_n)$.

Proof. If not, then there is a γ in $\sigma(a)$, a subnet $\{a_k\}$ of $\{a_n\}$, and an $r > 0$ such that

$$N_r(\gamma) \cap \sigma(a_k) = \emptyset \quad \text{for all } k \quad \text{where}$$

$$N_r(\gamma) = \{z \in \mathbf{C} | \quad |\lambda - z| < r\}.$$

Now, since $\{\lambda_n\}$ is bounded by 2) we can choose a subnet and by relabelling assume that $\lambda_k \to \lambda_\infty \in \mathbf{C}$ so that $(a_k - \lambda_k) \to (a - \lambda_\infty)$ *-strongly. Since

$$N_r(\gamma - \lambda_\infty) \cap \sigma(a_k - \lambda_\infty) = \emptyset \quad \text{for all } k,$$

we eventually have

$$N_{r/2}(\gamma - \lambda_\infty) \cap \sigma(a_k - \lambda_k) = \emptyset.$$

We let $b_k = a_k - \lambda_k$ and $b = a - \lambda_\infty$. We note that $p(b_k) = p(a_k - \lambda_k) = 0$ and $p(b_k) \to p(b)$ *-strongly so that $p(b) = 0$. Moreover,

$$\lambda_0 = (\gamma - \lambda_\infty) \in \sigma(a - \lambda_\infty) = \sigma(b).$$

In summary, we have a bounded net of operators $\{b_k\}$ such that

(1) $b_k \to b$ *-strongly,

(2) $p(b) = 0 = p(b_k)$ for all k,

(3) there is a $\lambda_0 \in \sigma(b)$ and an $s > 0$ such that

$$N_s(\lambda_0) \subset \sigma(b_k) = \emptyset \quad \text{for all } k.$$

Now, by (2), $\sigma(b) \subseteq$ (roots of p) so that we can write:

(i) $p(x) = (x - \lambda_0)^m q(x)$ where $q(\lambda_0) \neq 0$ and hence
(ii) $q(x) = (x - \lambda_0)d(x) + q(\lambda_0)$.

From (i), we observe that

$$0 = p(b_k) = (b_k - \lambda_0)^m q(b_k)$$

and since $(b_k - \lambda_0)$ is invertible we see that $q(b_k) = 0$ for all k. Now, from (ii) we see that

$$0 = q(b_k) = (b_k - \lambda_0)d(b_k) + q(\lambda_0)$$

so that

$$(b_k - \lambda_0)^{-1} = \frac{-1}{q(\lambda_0)} d(b_k) \quad \text{for all } k.$$

Finally, we observe

$$\begin{aligned} 1 &= (b_k - \lambda_0)(b_k - \lambda_0)^{-1} \\ &= (b_k - \lambda_0) \frac{-1}{q(\lambda_0)} d(b_k) \\ &\rightarrow (b - \lambda_0) \frac{-1}{q(\lambda_0)} d(b) \text{ *-strongly,} \end{aligned}$$

which implies that $(b - \lambda_0)$ is invertible, contradicting $\lambda_0 \in \sigma(b)$.

2.3. LEMMA. *Let* $\{a_n\}$, a, $\{\lambda_n\}$, λ, *and* p *satisfy the hypotheses of Lemma* 2.2. *Then there is a* $\delta > 0$, *depending only on* p, *and an* n_0 *such that*:

(1) $\sigma(a) \overset{\delta}{\subseteq} \sigma(a_n)$ *for all* $n \geqq n_0$,

(2) *for each* λ *in* $\sigma(a)$ *and each* $n \geqq n_0$ *there is a unique* λ_n *in* $\sigma(a_n)$ *with* $|\lambda - \lambda_n| < \delta$.

Proof. Let

$$\delta = \frac{1}{2} \min\{|\gamma_i - \gamma_j| \mid \gamma_i \neq \gamma_j \text{ are roots of } p\}.$$

That (1) holds is immediate from Lemma 2.2. To see (2), it follows from (1) that such λ_n's exist. If two such existed, say λ_n, λ'_n, then we would have $|\lambda_n - \lambda'_n| < 2\delta$. But, $\sigma(a_n)$ is contained in a translate of the roots of p, so that $\lambda_n \neq \lambda'_n$ would imply $|\lambda_n - \lambda'_n| \geqq 2\delta$. Hence, $\lambda_n = \lambda'_n$.

2.4. COROLLARY. *Let* $\{a_n\}$, a, $\{\lambda_n\}$, λ, *and* p *satisfy the hypotheses of Lemma* 2.2. *Then there is a* $\delta > 0$ (*depending only on* p), *and an* n_0 *such that*

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for each $n \geqq n_0$ *there is a unique subset* F_n *of* $\sigma(a_n)$ *such that*

$$\operatorname{dist}(F_n, \sigma(a)) < \delta.$$

Moreover, $\operatorname{dist}(F_n, \sigma(a)) \to 0$.

Remark. Let $\{e_n\}$ be a sequence of nonzero projections such that $e_n \to 0$ *-strongly. Then, everything in sight satisfies $p(x) = x^2 - x$ while $\sigma(e_n) = \{0, 1\}$ for all n, but $\sigma(0) = \{0\}$. Hence, we cannot expect that the subsets $F_n \subseteq \sigma(a_n)$ defined above will be equal to $\sigma(a_n)$, in general.

2.5. LEMMA. *Let* $\{a_n\}$, a $\{\lambda_n\}$, λ, *and* p *satisfy the hypotheses of Lemma* 2.2. *Let* δ, n_0 *and* $\{F_n\}$ *be as in Corollary* 2.4. *Then there is an* $n_1 \geqq n_0$ *so that for all* $n \geqq n_1$, F_n *is a translate of* $\sigma(a)$.

Proof. Let

$$\mathcal{F} = \{S|S \text{ is a subset of the roots of } p \text{ but is not a translate of } \sigma(a)\}.$$

Clearly, $\mathcal{F}$ is a finite set. Now, we let

$$\eta = \inf\{\operatorname{dist}(S + \lambda, \sigma(a))|S \in \mathcal{F}, \lambda \in \mathbf{C}\}.$$

To see that $\eta > 0$, suppose $\eta = 0$ then we can choose sequences $\{S_k\}$ from $\mathcal{F}$ and $\{\lambda_k\}$ from $\mathbf{C}$ such that

$$\operatorname{dist}(S_k + \lambda_k, \sigma(a)) \to 0.$$

Since $\mathcal{F}$ is finite, we can assume by choosing a subsequence that $\{S_k\}$ is constant and equal to S and since $\{\lambda_k\}$ is bounded we can assume that it converges to some $\lambda \in \mathbf{C}$. Then,

$$\operatorname{dist}(S + \lambda, \sigma(a)) = \lim_k \operatorname{dist}(S + \lambda_k, \sigma(a)) = 0$$

so that $S \notin \mathcal{F}$, a contradiction. So, we have $\eta > 0$.

Now, for each $n \geqq n_0$, either F_n is a translate of $\sigma(a)$, or it is not. If it is not, then it is a translate of some set in $\mathcal{F}$. If the $\{F_n\}$ were not eventually translates of $\sigma(a)$, then we could choose a subnet $\{F_m = S_m + \lambda_m\}$ where $S_m \in \mathcal{F}$ and $\lambda_m \in \mathbf{C}$. But, then by Corollary 2.4

$$0 = \lim_m \operatorname{dist}(F_m, \sigma(a)) = \lim_m \operatorname{dist}(S_m + \lambda_m, \sigma(a)) \geqq \eta$$

a contradiction. Therefore, there is an $n_1 \geqq n_0$ so that for $n \geqq n_1$, F_n is a translate of $\sigma(a)$.

2.6. LEMMA. *Let* X *be a topological space and let* $a \in C^b_{*s}(X, \mathcal{B}(H))$

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where H is a Hilbert space. Let p be a polynomial and suppose that $p(\text{ad } a) = 0$. *Then, for each* x_0 *in X there is a neighbourhood* $N(x_0)$ *of* x_0 *so that for all x in* $N(x_0)$,

(1) $\sigma(a(x_0)) \overset{\delta/2}{\subseteq} \sigma(a(x))$, *where* δ *is defined in Lemma* 1.3,

(2) *there is a unique set* $F_x \subseteq \sigma(a(x))$ *which is a translate of* $\sigma(a(x_0))$ *and*

$$\operatorname{dist}(F_x, \sigma(a(x_0))) < \frac{\delta}{2},$$

(3) *and if* $\sigma(a(x_0)) = F_x - \xi_x$ *as described in* (2) *the map* $x \to \xi_x$ *is continuous.*

Proof. First we observe that

$$\mathcal{O} = \{x \in X | \sigma(a(x_0)) \overset{\delta/2}{\subseteq} \sigma(a(x))\}$$

contains a neighbourhood of x_0. For, otherwise we can choose a filter of neighbourhoods of x_0 shrinking to $\{x_0\}$ and points in these neighbourhoods which miss $\mathcal{O}$. This is impossible by Lemma 2.3. Call this first neighbourhood $N_1(x_0)$. So, by Lemma 2.4 for each x in $N_1(x_0)$ there is a unique set $F_x \subseteq \sigma(a(x))$ with

$$\operatorname{dist}(F_x, \sigma(a(x_0))) < \frac{\delta}{2}.$$

By a similar application of Lemma 2.5 we can find another neighbourhood $N(x_0) \subseteq N_1(x_0)$ so that, in addition, F_x is a unique translate of $\sigma(a(x_0))$ for all x in $N(x_0)$.

Now, we let $\sigma(a(x_0)) = F_x - \xi_x$ be as described above: it remains to see that $x \to \xi_x$ is continuous on $N(x_0)$. To this end, let $x' \in N(x_0)$ and let $\{x_n\}$ be a net in $N(x_0)$ converging to x'. Now, eventually there exist sets $E_{x_n} \subseteq \sigma(a(x_n))$ such that

$$E_{x_n} \to \sigma(a(x')) \quad \text{and} \quad \operatorname{dist}(E_{x_n}, \sigma(a(x'))) < \frac{\delta}{2}.$$

To see that $F_{x_n} \subseteq E_{x_n}$, let $\beta \in F_{x_n}$ so there is a unique $\beta_0 \in \sigma(a(x_0))$ with $|\beta - \beta_0| < \delta/2$. Also, there is a unique β' in $F_{x'}$ so that $|\beta' - \beta_0| < \delta/2$. Thus, $|\beta - \beta'| < \delta$ and there is a $\widetilde{\beta}$ in E_{x_n} with $|\beta' - \widetilde{\beta}| < \delta/2$ so that

$$|\beta - \widetilde{\beta}| < 3\delta/2 < 2\delta.$$

But, this implies $\beta = \widetilde{\beta}$ as $E_{x_n} \subseteq \sigma(a(x_n))$ is contained in a translate of the roots of p. Thus, $F_{x_n} \subseteq E_{x_n}$ eventually. Since $E_{x_n} \to \sigma(a(x'))$ and each

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point ξ in $\sigma(a(x_0))$ has associated with it a unique net of points $\{\lambda_n\}$ in the $\delta/2$-ball about ξ such that $\lambda_n \in F_{x_n}$ for each n, we see that F_{x_n} converges to some subset $F \subseteq \sigma(a(x'))$. But, then

$$\text{dist}(F, \sigma(a(x_0))) < \delta/2 \quad \text{and} \quad \text{dist}(F_{x'}, \sigma(a(x_0))) < \delta/2$$

implies that $F = F_{x'}$, i.e., $F_{x_n} \to F_{x'}$. It easily follows that $\xi_{x_n} \to \xi_{x'}'$.

The following theorem is the best that one can do for a general C^*-algebra A and a general polynomial: it shows that the problem can always be solved "locally" over $\hat{A}$. As we shall see, these local solutions cannot always be patched together to form a global solution.

2.7. THEOREM. *Let A be a unital C^*-algebra, p a polynomial, and $a \in A$ such that $p(\text{ad } a) = 0$. Then for each $\pi_0 \in \hat{A}$, there is a neighbourhood N of π_0 such that if I is the ideal in A vanishing on $(\hat{A}\backslash N)$ there exists a $\lambda \in Z(M(I))$ with $p(\bar{a} - \lambda) = 0$ where $\bar{a}$ is the multiplier of I determined by a.*

Proof. Following [**4**], let X be the space of "railway representations" of A on some sufficiently large Hilbert space H. Then, the evaluation map represents A isomorphically as a C^*-subalgebra of $C^b_{*s}(X, \mathcal{B}(H))$. Now, by construction, $(A(x))''$ is a type I factor for each $x \in X$, and the obvious map $X \to \hat{A}$ is an open, continuous surjection. To see that $p(\text{ad } a) = 0$ when a is considered as an element $C^b_{*s}(X, \mathcal{B}(H))$, it suffices to see that $p(\text{ad } a(x)) = 0$ on $\mathcal{B}(H)$ for each $x \in X$. However, in this case,

$$H = H_x^1 \otimes H_x^2 \quad \text{and} \quad (A(x))'' = \mathcal{B}(H_x^1) \otimes 1$$

and so $a(x) = a_x' \otimes 1$. As $p(\text{ad } a(x)) = 0$ on $(A(x))''$, we have

$$p(\text{ad } a_x') = 0 \quad \text{on } \mathcal{B}(H_x^1).$$

Since $\mathcal{B}(H) = \mathcal{B}(H_x^1) \mathbin{\bar{\otimes}} \mathcal{B}(H_x^2)$ we see that on $\mathcal{B}(H)$,

$$p(\text{ad } a(x)) = p(\text{ad}(a_x' \otimes 1)) = p(\text{ad } a_x') \otimes \text{id} = 0.$$

Now, we apply Lemma 2.6 at a point x_0 in X whose image in $\hat{A}$ is π_0 and obtain a neighbourhood $N(x_0)$ of x_0 as described. Now, fix $\gamma \in \sigma(a(x_0))$, and as we have $\sigma(a(x_0)) = F_x - \xi_x$ on $N(x_0)$ we see that $\gamma = \lambda_x - \xi_x$ on $N(x_0)$ and so $\lambda_x = \gamma + \xi_x$ is continuous on $N(x_0)$. Since $\lambda_x \in F_x \subseteq \sigma(a(x))$ we also see that

$$p(a(x) - \lambda_x) = 0 \quad \text{on } N(x_0)$$

by Lemma 2.1 and [**8**, Theorem 3]. Now, let N be the image of $N(x_0)$ in $\hat{A}$ so that N is open about π_0. Then, the spectrum of the ideal

$$I = \{b \in A | \pi(b) = 0 \text{ if } \pi \notin N\}$$

is canonically identified with N. To see that λ defines an element of $Z(M(I))$ it suffices to see that λ is constant on equivalence classes in $N(x_0)$ for then we would have λ defining an element of $C^b(N)$ which equals $Z(M(I))$ by the Dauns-Hoffman Theorem [**2**]. So, we suppose that x_1, x_2 are in $N(x_0)$ and the type I representations $b \to b(x_1)$ and $b \to b(x_2)$ are quasi-equivalent. Then there is an isomorphism Φ: $(A(x_1))'' \to (A(x_2))''$ so that $\Phi(b(x_1)) = b(x_2)$ for all $b \in A$. In particular, $\Phi(a(x_1)) = a(x_2)$ and so $a(x_1)$ has the same spectrum as $a(x_2)$. By Lemma 2.6 this implies that $F_{x_1} = F_{x_2}$ and so $\xi_{x_1} = \xi_{x_2}$. That is,

$$\lambda_{x_1} = \gamma + \xi_{x_1} = \gamma + \xi_{x_2} = \lambda_{x_2}$$

and we are done.

2.8. *Čech cohomology*. Let G be a (discrete) abelian group and X a topological space. Let $q \geqq 0$ be an integer. We denote by $\check{H}^q(X, G)$ the qth Čech cohomology group of X with coefficients in the sheaf of continuous G-valued functions on X (the so-called constant sheaf). We use [**11**, Chapter 2] as a reference for sheaf cohomology.

2.9. THEOREM. *Let A be a unital C^*-algebra, p a polynomial, d an inner derivation of A such that $p(d) = 0$. Let G_p be the additive subgroup of* **C** *generated by the roots of p. There is an element $\eta(d)$ in $\check{H}^1(\hat{A}, G_p)$ such that $\eta(d) = 0$ if and only if there is a polynomial q whose roots lie in G_p and an $a \in A$ with $d = \text{ad } a$ so that $q(a) = 0$.*

Proof. Let $d = \text{ad } a$, $a \in A$. First we show the existence of the element $\eta(d) \in \check{H}^1(\hat{A}, G_p)$. By the previous theorem we can find an open cover $\{N_i\}$ of $\hat{A}$ and continuous maps $\lambda_i: N_i \to \mathbf{C}$ such that

$$\lambda_i(\pi) \in \sigma(\pi(a)) \quad \text{for } \pi \in N_i.$$

Now, for each pair (i, j) with $N_i \cap N_j \neq \emptyset$, define

$$\gamma_{ij}(\pi) = \lambda_i(\pi) - \lambda_j(\pi)$$

which is in the spectrum of $\text{ad}(\pi(a))$ and so must be a root of p. We observe that $\{\gamma_{ij}\}$ trivially satisfies the 1-cocycle equation

$$\gamma_{jk} - \gamma_{ik} + \gamma_{ij} = 0 \quad \text{on } N_i \cap N_j \cap N_k.$$

Let $\eta(\text{ad } a)$ be the class of $\{\gamma_{ij}\}$ in $\check{H}^1(\hat{A}, G_p)$. To see that $\eta(\text{ad } a)$ depends only on a and not on the various choices we have made, let $\{N'_\alpha\}$ be some other open cover of $\hat{A}$ and suppose we have corresponding $\{\lambda'_\alpha\}$ and $\{\gamma'_{\alpha\beta}\}$ chosen as above. In order to compare $[\{\gamma_{ij}\}]$ and $[\{\gamma'_{\alpha\beta}\}]$ in $\check{H}^1(\hat{A}, G_p)$

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we must take a common refinement of the covers $\{N_i\}$ and $\{N'_\alpha\}$ and then restrict the $\{\gamma_{ij}\}$ and the $\{\gamma'_{\alpha\beta}\}$ to the corresponding intersections of pairs. However, this is clearly the same as first restricting the $\{\lambda_i\}$ and the $\{\lambda'_\alpha\}$ and then taking appropriate differences. Thus, we may assume that our covers are the same, say $\{N_i\}$. Then, for each i we have a map

$$t_i = (\lambda_i - \lambda'_i):N_i \rightarrow (\text{roots of } p) \subseteq G_p.$$

We easily compute that $\gamma_{ij} = t_i + \gamma'_{ij} - t_j$ on $N_i \cap N_j$. That is, $\{\gamma_{ij}\}$ differs from $\{\gamma'_{ij}\}$ by the trivial 1-cocycle $(t_i - t_j)$ and so $\eta(\text{ad } a) = [\{\gamma_{ij}\}] = [\{\gamma'_{ij}\}]$ is well-defined. To see that $\eta(d)$ is independent of the representation of d, let $\lambda \in Z(A) = C^b(\hat{A})$ and choose $\{N_i\}$, $\{\lambda_i\}$ as above to define $\eta(d)$. Then $\{N_i\}$, $\{\lambda_i + \lambda|_{N_i}\}$ constitute an appropriate choice to define $\eta(\text{ad}(a + \lambda))$. Thus,

$$\eta(\text{ad}(a + \lambda)) = [\{(\lambda_i + \lambda) - (\lambda_j + \lambda)\}] = [\{\lambda_i - \lambda_j\}]$$
$$= \eta(\text{ad } a).$$

Suppose there exists a polynomial q whose roots lie in G_p, $d = \text{ad } a$, with $q(a) = 0$. If we choose $\{N_i\}$, $\{\lambda_i\}$ as above then $\lambda_i(\pi) \in \sigma(\pi(a)) \subseteq$ roots of $q \subseteq G_p$ for all π in N_i. Consequently, $\{\gamma_{ij}\} = \{\lambda_i - \lambda_j\}$ is by definition a trivial 1-cocycle so that $\eta(d) = 0$.

Conversely, suppose $\eta(d) = 0$ with $d = \text{ad } a'$. As $\hat{A}$ is compact we can find a finite open cover $\{N_i\}_{i=1}^n$ and continuous functions $\lambda_i:N_i \rightarrow \mathbf{C}$ so that

$$\lambda_i(\pi) \in \sigma(\pi(a')) \quad \text{for } \pi \in N_i.$$

Since the cocycle $\{\gamma_{ij}\} = \{\gamma_i - \gamma_j\}$ is trivial we can assume that there are continuous maps $t_i:N_i \rightarrow G_p$ so that $\{\gamma_{ij}\} = \{t_i - t_j\}$. Because G_p is a discrete subset of $\mathbf{C}$, the set $t_i^{-1}(Z_0)$ is open for each $Z_0 \in G_p$ and so

$$\{t_i^{-1}(Z_0)|Z_0 \in G_p, i = 1, \cdots, n\}$$

is an open cover of $\hat{A}$ from which we can extract a finite subcover. That is, by refining our original cover we can assume that each t_i is constant on N_i.

Let

$$q(x) = p(x - t_1)p(x - t_2) \cdots p(x - t_m)$$

so that the roots of $q \subseteq G_p$. Let $\zeta_i = \lambda_i - t_i$ on N_i. Then, on $N_i \cap N_j$ we have

$$\zeta_i - \zeta_j = (\lambda_i - t_i) - (\lambda_j - t_j) = \gamma_{ij} - \gamma_{ij} = 0$$

and so there is a $\zeta \in C^b(\hat{A}) = Z(A)$ with $\zeta|_{N_i} = \zeta_i$. To see that $q(a' - \zeta) = 0$, let $\pi \in \hat{A}$ be arbitrary. Choose i so that $\pi \in N_i$, then

$$\begin{aligned}\pi(q(a' - \zeta)) &= q(\pi(a') - \zeta(\pi)) \\ &= q(\pi(a') - \zeta_i(\pi)) = q(\pi(a') - \lambda_i(\pi) + t_i) \\ &= p(\pi(a') - \lambda_i(\pi) + t_i - t_i) \prod_{i \neq j} p(\pi(a') \\ &\qquad - \lambda_i(\pi) + \gamma_{ij}) \\ &= p(\pi(a') - \lambda_i(\pi)) \cdot f(\pi(a')) = 0\end{aligned}$$

by Lemma 2.1 and [**8**, Theorem 3]. Hence, $q(a' - \zeta) = 0$. Let $a = a' - \zeta$.

2.10. *Example*. We now produce an example where $\eta(\text{ad } a) \neq 0$. First we observe that if $Y \subseteq \hat{A}$, then the injection $i:Y \to \hat{A}$ produces a homorphism

$$i^*:\check{H}^1(\hat{A}, G_p) \to H^1(Y, G_p)$$

obtained by restricting cocycles, etc. to Y intersected with their domains. Thus, to see that $\eta(\text{ad } a) \neq 0$, it is sufficient to work on some appropriately chosen subset $Y \subseteq \hat{A}$ and show directly that

$$i^*(\eta(\text{ad } a)) \neq 0.$$

Let S^1 be the unit circle and M_2 the complex two-by-two matrices. Let S^+, S^- denote the closed upper and lower semicircles in S^1, respectively, and let $e:S^- \to M_2$ be a fixed continuous function such that

(1) $e(z)$ is a rank one projection for all z in S^-,

(2) $e(1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $e(-1) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Let

$$B = \{f:S^1 \to M_2 | f \text{ is continuous and}$$
$$f(z) = f^-(z)e(z) \text{ for } z \text{ in } S \text{ where } f^-:S^- \to \mathbf{C}\}.$$

Then, B, with pointwise operations is clearly a (non-unital) C^*-algebra with continuous trace and $\hat{B} = S^1$. Let $A = M(B)$ so that $i:S_1 \hookrightarrow \hat{A}$ in a natural way. Let $a \in B \subseteq A$ be the following element:

$$a(z) = \begin{cases} \begin{bmatrix} \tfrac{1}{2}\text{Re}(z) + \tfrac{1}{2} & 0 \\ 0 & \tfrac{1}{2}\text{Re}(z) - \tfrac{1}{2} \end{bmatrix} & , z \in S^+ \\ z \cdot e(z) & , z \in S^-. \end{cases}$$

For each $z \in S^1$ let π_z denote the corresponding irreducible representation of B (and hence A) and note that for z in S^-, π_z is on a one-dimensional space. Then, clearly we have:

$$\sigma(\pi_z(a)) = \begin{cases} \{(\tfrac{1}{2}\mathrm{Re}(z) + \tfrac{1}{2}), (\tfrac{1}{2}\mathrm{Re}(z) - \tfrac{1}{2})\} & , z \in S^+ \setminus \{1, -1\} \\ \{z\} & , z \in S^-. \end{cases}$$

Let $p(x) = x^3 - x$. Then for z in $S^+ \setminus \{1, -1\}$, $\pi_z(a)$ is the following translate of a projection:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (\tfrac{1}{2}\mathrm{Re}(z) - \tfrac{1}{2})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so that $p(\mathrm{ad}\ \pi_z(a)) = 0$. For $z \in S^-$, $\mathrm{ad}\ \pi_z(a) = 0$ already, so that $p(\mathrm{ad}\ a) = 0$.

We now compute $i^*(\eta(\mathrm{ad}\ a))$, first noting that $G_p = \mathbf{Z}$. Fix $\epsilon_0 > 0$ and let

$$S_l = \{z \in S^1 \mid Re(z) < \epsilon_0\}$$

$$S_r = \{s \in S^1 \mid \mathrm{Re}(z) > -\epsilon_0\},$$

so that $\{S_l, S_r\}$ is an open cover of S^1. Define $\lambda_l : S_l \to \mathbf{C}$ and $\lambda_r : S_r \to \mathbf{C}$ via

$$\lambda_l = \begin{cases} (\tfrac{1}{2}\mathrm{Re}(z) - \tfrac{1}{2}) & , z \in S_l \cap S^+ \\ z & , z \in S_l \cap S^- \end{cases}$$

$$\lambda_r = \begin{cases} (\tfrac{1}{2}\mathrm{Re}(z) + \tfrac{1}{2}) & , z \in S_r \cap S^+ \\ z & , z \in S_r \cap S^-. \end{cases}$$

Clearly, λ_l and λ_r are continuous and $\lambda_l(z) \in \sigma(\pi_z(a))$ for all $z \in S_l$, and $\lambda_r(z) \in \sigma(\pi_z(a))$ for all $z \in S_r$. Now, $S_l \cap S_r$ consists of two components, one in S^+ and one in S^-. Clearly $\gamma_{rl} = \lambda_r - \lambda_l$ is 1 on the upper component and 0 on the lower component. It is easy to see that

$$i^*(\lambda(\mathrm{ad}\ a)) = [\{\gamma_{rl}, \gamma_{lr}, 0_{ll}, 0_{rr}\}] \neq 0 \quad \text{in } \check{H}^1(S^1, \mathbf{Z}):$$

since S_l and S_r are contractible and $S_l \cap S_r$ is homotopic to $\{i, -i\}$, we have that

$$\check{H}^1(S^1, \mathbf{Z}) = \check{H}^1(\{S_r, S_l\}, \mathbf{Z})$$

[**11**, Leray's Theorem], and so we need only show that

$$i^*(\eta(a)) \neq 0 \quad \text{in } \check{H}^1(\{S_r, S_l\}, \mathbf{Z}).$$

This is trivial, since otherwise we could choose continuous (and hence constant) integer-valued functions λ_l', λ_r' on S_l, S_r respectively, such that γ_{rl}

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$= \lambda_r - \lambda_i$ so that γ_{rl} would have to be constant, which it is not.

By Theorem 2.9, there cannot exist a $\lambda \in Z(A)(= C(S^1))$ so that $p(a - \lambda) = 0$.

2.11. COROLLARY. *Let A be a C^*-algebra, k be a positive integer and $a \in M(A)$ be such that* $(\text{ad } a)^k = 0$. *Then there is a $\lambda \in Z(M(A))$ so that* $(a - \lambda)^k = 0$.

Proof. We may assume that A is unital and $a \in A$. Let $p(x) = x^k$ so that $G_p = \{0\}$. This forces $\eta(\text{ad } a) = 0$ and so by Theorem 2.9 there is a q with $G_q = \{0\}$ and a λ with $q(a - \lambda) = 0$. But, then q is of the form $q(x) = x^m$ for some positive integer m. But,

$$(\pi(a) - \lambda(\pi))^m = 0 \quad \text{for a given } \pi \in \hat{A}$$

implies that $(\pi(a) - \lambda(\pi))$ is not invertible and so $\lambda(\pi) \in \sigma(\pi(a))$. Now, by Lemma 2.1 and [**8**, Theorem 3] we have

$$(\pi(a) - \lambda(\pi))^k = 0.$$

Since π was arbitrary we have $(a - \lambda)^k = 0$.

3. Continuity of spectrum.

3.1. *Definition*. Let A be a unital C^*-algebra, p a polynomial and $a \in A$ satisfy $p(\text{ad } a) = 0$. We say that the spectrum of a is *continuous over* $\hat{A}$ if for each $\pi_0 \in \hat{A}$ there is a neighbourhood N of π_0 so that for all $\pi \in N$, $\sigma(\pi(a))$ is a translate of $\sigma(\pi_0(a))$.

Note that by the proof of Theorem 2.7 we can always find a neighbourhood N of π_0 so that for all $\pi \in N$, $\sigma(\pi(a))$ contains a unique translate F_π of $\sigma(\pi_0(a))$ with F_π close to $\sigma(\pi_0(a))$ and $\sigma(\pi_0(a)) = F_\pi - \xi_\pi$ where $\xi:N \to \mathbf{C}$ is continuous. The hypothesis of continuity defined above implies that we can choose N so that

$$\sigma(\pi_0(a)) = \sigma(\pi(a)) - \xi_\pi.$$

That this is a crucial difference can be seen by the next theorem.

3.2. THEOREM. *Let A be a unital C^*-algebra, p a polynomial and $a \in A$ satisfy* $p(\text{ad } a) = 0$. *If the spectrum of a is continuous over $\hat{A}$, then there is a $\lambda \in Z(A)$ with $p(a - \lambda) = 0$.*

Proof. Fix $\pi_0 \in \hat{A}$ and let

$$\mathcal{O}_{\pi_0} = \{\pi \in \hat{A} \mid \ \sigma(\pi(a)) \text{ is a translate of } \sigma(\pi_0(a))\}.$$

Clearly, $\mathcal{O}_{\pi_0}$ is open. To see that $\mathcal{O}_{\pi_0}$ is closed, let π be a limit point of $\mathcal{O}_{\pi_0}$.

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Since the spectrum is continuous at π there is a neighbourhood $N(\pi)$ such that for all $\rho \in N(\pi)$, $\sigma(\rho(a))$ is a translate of $\sigma(\pi(a))$. Since there is a ρ in $N(\pi) \cap \mathcal{O}_{\pi_0}$ we see that $\sigma(\pi(a))$ is a translate of $\sigma(\pi_0(a))$ so that $\pi \in \mathcal{O}_{\pi_0}$.

Thus, $\hat{A}$ is the disjoint union of closed and open set $\{\mathcal{O}_{\pi_i}\}$ of the above form. Now, for each $\pi \in \mathcal{O}_{\pi_0}$, there is a (necessarily unique) $\xi_\pi \in \mathbf{C}$ such that

$$\sigma(\pi_0(a)) = \sigma(\pi(a)) - \xi_\pi$$

and ξ is continuous in a neighbourhood of π_0. A simple argument using the uniqueness of ξ shows that, in fact, ξ is continuous on $\mathcal{O}_{\pi_0}$. Now, fix $\gamma \in \sigma(\pi_0(a))$ and define $\lambda_0(\pi) = \gamma + \xi_\pi$ so that $\lambda_0:\mathcal{O}_{\pi_0} \to \mathbf{C}$ is continuous and $\lambda_0(\pi) \in \sigma(\pi(a))$ for all $\pi \in \mathcal{O}_{\pi_0}$. Thus,

$$\|\lambda_0\| \leqq \|a\| \quad \text{and} \quad p(\pi(a) - \lambda_0(\pi)) = 0 \quad \text{for } \pi \in \mathcal{O}_{\pi_0}.$$

Since $\hat{A}$ is the disjoint union of such clopen sets $\mathcal{O}_{\pi_i}$, there is a continuous function $\lambda:\hat{A} \to C$ with

$$\|\lambda\| \leqq \|a\| \quad \text{and} \quad p(\pi(a) - \lambda(\pi)) = 0 \quad \text{for all } \pi \text{ in } \hat{A}.$$

Since $Z(A) = C^b(\hat{A})$ we are done.

We could now re-prove Corollary 2.11 by observing that $(\text{ad } a)^k = 0$ implies the spectrum of a is continuous.

3.3. COROLLARY. *Let A be a unital continuous trace C^*-algebra, p a polynomial and d a derivation of A satisfying $p(d) = 0$. Then there is an $a \in A$ with $p(a) = 0$ and $d = \text{ad } a$.*

Proof. By [**1**], there is an $a \in A$ with $d = \text{ad } a$. We shall show that the spectrum of a is continuous over $\hat{A}$. To do this, we need only work locally and so by the work of Fell [**5**], see also [**9**, Lemma 4.8], we can assume that $A = C(X, M_n)$ where $X = \hat{A}$ is compact. But, now $a \in A$ is a norm-continuous function on X and by [**6**, problem 86] we can easily deduce that

$$\#(\sigma(a(x))) = \#(\sigma(a(x_0)))$$

for all x in a neighbourhood of x_0 and so by Lemma 2.6, part (2), $\sigma(a(x))$ is a translate of $\sigma(a(x_0))$ in a neighbourhood of x_0.

3.4. *Remark.* Example 2.10 shows that we cannot generalize this result to arbitrary continuous trace C^*-algebras, even if we assume the irreducible representations are of bounded dimension. The result also fails for stable continuous trace C^*-algebras; we sketch an example.

Let $A = C^b_{*s}(S^1, \mathscr{B}(H))$ and let $a \in A$ be infinite projection-valued on $S^+ \backslash \{-1\}$, $\mathbf{C}1_H$-valued on S^- with $a(-1) = 0$ and $a(1) = 1_H$. Let $p(x) = x^3 - x$ and proceed as in Example 2.10.

In a fashion similar to 3.3 we can also prove the following theorem.

3.5. THEOREM. *Let X be a topological space, B a primitive unital C^*-algebra and E a locally trivial bundle over X with fibre B and structure group* Inn$B$ *in the norm topology. Let $A = \Gamma_b(E)$ be the C^*-algebra of bounded norm-continuous sections of E. Let p be a polynomial and let $a \in A$ be such that $p(\mathrm{ad}\ a) = 0$. Then there is a λ in $C^b(X) = Z(A)$ such that $p(a - \lambda) = 0$.*

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