6

Parabolic G-Bundles and Equivariant G-Bundles

Let G be a simple, simply-connected algebraic group with maximal torus H and let $(\Sigma, \vec{p} = (p_1, \ldots, p_s))$ be an s-pointed smooth projective irreducible curve (of any genus g). Fix a maximal compact subgroup K of G. Then the set of K-orbits $K/\operatorname{Ad} K$ in K under the adjoint action is parameterized by the fundamental alcove Φ_o (cf. Lemma 6.1.1). Recall that a parabolic G-bundle $(E, \vec{\tau}, \vec{\sigma})$ consists of a principal G-bundle $E \to \Sigma$ together with markings $\vec{\tau} = (\tau_1, \ldots, \tau_s)$, for $\tau_j \in \Phi_o$, and a section σ_j of E_{p_j}/P_j over p_j, for each $1 \leq j \leq s$, where $P_j := P(\tau_j)$ is the standard parabolic subgroup such that its Levi subgroup $L(\tau_j)$ containing H has for its simple roots $S_{\tau_j} := \{\alpha_i : \alpha_i(\tau_j) = 0\}$. We define the parabolic semistability (and parabolic stability) of $(E, \vec{\tau}, \vec{\sigma})$ in Definition 6.1.4(d). This definition generalizes the standard definition of parabolic semistability (and stability) for parabolic vector bundles (cf. Exercise 6.1.E.7). In particular, when $s = 0$, we recover the definition of semistability and stability of the G-bundle $E \to \Sigma$ (cf. Definition 6.1.4(b)) generalizing the corresponding notion for vector bundles (cf. Definition 6.1.4(a)). We show that a G-bundle $E \to \Sigma$ is semistable if and only if its adjoint bundle $\operatorname{ad} E$ is semistable (cf. Lemma 6.1.5).

For an algebra R over $\mathbb{C}$, let $\mathbb{D}_R = \operatorname{Spec} R[[t]]$ denote the formal disc. Let a finite group A act on $\mathbb{D} := \operatorname{Spec} \mathbb{C}[[t]]$ and let $\mathscr{E} \to \mathbb{D}_R$ be an A-equivariant principal G-bundle, which is trivial as a G-bundle, where A acts on $\mathbb{D}_R$ with the trivial action on R. Then, as proved in Theorem 6.1.9, there exists a G-bundle trivialization of $\mathscr{E}$ in which the A-action is the product action, i.e., there exists an A-equivariant G-bundle isomorphism inducing the identity on the base: $\mathscr{E} \underset{\sim}{\xrightarrow{\varphi}} \mathbb{D}_R \times G$ such that the action of A on $\mathbb{D}_R \times G$ is given by

$$\gamma \odot (x, g) = (\gamma x, \theta_\gamma(x(0))g),$$

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where $x(0)$ is the image of x in Spec R and θ_γ: Spec $R \to G$ is a morphism. Moreover, for any $x^o \in$ Spec R, the group homomorphism $\theta(x^o)\colon A \to G$, $\gamma \mapsto \theta_\gamma(x^o)$, is unique up to a conjugation, which is called the *type of $\mathscr{E}$ over x^o*. The proof of Theorem 6.1.9 uses non-abelian group cohomology. In fact, Theorem 6.1.9 is true for any connected affine algebraic group.

Let $\vec{\tau} = (\tau_1, \ldots, \tau_s)$ be a set of rational markings, i.e., $\tau_j = \bar{\tau}_j/d_j$, for some positive integers d_j and $\mathrm{Exp}(2\pi i \bar{\tau}_j) = 1$. As in Theorem 6.1.8, we fix a Galois cover $\pi\colon \hat{\Sigma} \to \Sigma$ with signature the pair $\vec{p}$ and the sequence $\vec{d} = (d_1, \ldots, d_s)$ with finite Galois group A. We also fix inverse images $\{\hat{p}_j \in \pi^{-1}(p_j)\}_{1\le j\le s}$ and generators $\vec{\gamma} = (\gamma_1, \ldots, \gamma_s)$ of the cyclic isotropy groups $(A_{\hat{p}_1}, \ldots, A_{\hat{p}_s})$. Thus, $A_{\hat{p}_j}$ is of order d_j. As earlier in Section 1.1, let **Alg** be the category of algebras over $\mathbb{C}$ and **Set** the category of sets. Define the functor $\mathscr{F}^{A,\vec{\tau}}_{G,\hat{\Sigma}^*}$: **Alg** $\to$ **Set** by

$$\mathscr{F}^{A,\vec{\tau}}_{G,\hat{\Sigma}^*}(R) = \{(\hat{E}_R, \hat{\sigma}_R) : \hat{E}_R \text{ is an } A\text{-equivariant } G\text{-bundle over } \hat{\Sigma}_R$$
$$\text{such that } \hat{E}_{R_{|\hat{\Sigma}\times x}} \text{ has local type } \vec{\tau} \text{ for any } x\in \operatorname{Spec} R \text{ and } \hat{\sigma}_R \text{ is an}$$
$$A\text{-equivariant section of } \hat{E}_R \text{ over } (\hat{\Sigma}^*)_R\}/ \text{ isomorphisms,}$$

where $\Sigma^* := \Sigma\backslash\vec{p}, \hat{\Sigma}^* := \pi^{-1}(\Sigma^*)$, A acts trivially on R and $\hat{\Sigma}_R := \hat{\Sigma} \times$ Spec R.

For any parabolic subgroup P of G, consider the parahoric subgroup scheme $\mathcal{P} \subset \bar{G}((t))$ defined by $\mathcal{P} := ev_0^{-1}(P)$, under the evaluation map $ev_0\colon \bar{G}[[t]] \to G$ at $t = 0$ (cf. Exercise 1.3.E.11). Let t_j be the formal parameter at $p_j \in \Sigma$ defined by identity (1) of Definition 6.1.11. Then, we prove (cf. Theorem 6.1.12) that, if $\theta(\tau_j) < 1$ for all j (for the highest root θ), the functor $\mathscr{F}^{A,\vec{\tau}}_{G,\hat{\Sigma}^*}$ is representable, represented by the ind-scheme $\bar{X}_{\vec{P}} = \Pi^s_{j=1}\bar{X}_G(P_j)$, where $P_j := P(\tau_j)$ is defined in the first paragraph.

Similar to the definition of the stack $\mathbf{Bun}_G(\Sigma)$ as in Chapter 5, define the groupoid fibration over $\mathfrak{S}$ of A-equivariant G-bundles $\mathbf{Bun}^{A,\vec{\tau}}_G(\hat{\Sigma})$ of local type $\vec{\tau}$, whose objects are A-equivariant G-bundles E_S over $\hat{\Sigma} \times S$ (with the trivial action of A on S) such that $E_{S|_{\hat{\Sigma}\times t}}$ (for any $t \in S$) is of local type $\vec{\tau}$ (cf. Definition 6.1.14). Let $\bar{X}_{\vec{P}} := \Pi^s_{j=1} \bar{X}_G(P_j)$ and let $\bar{\Gamma}$ be the ind-affine group variety with $\mathbb{C}$-points $\Gamma := \mathrm{Mor}(\Sigma^*, G)$, where Σ^* and P_j are as in the above paragraph. Then $\bar{\Gamma}$ acts on $\bar{X}_{\vec{P}}$ by the left multiplication on each factor via its Laurent series expansion in the formal coordinates t_j. With this notation, there exists an equivalence of categories between $\mathbf{Bun}^{A,\vec{\tau}}_G(\hat{\Sigma})$ and the quotient stack $\left[\bar{\Gamma}\backslash\bar{X}_{\vec{P}}\right]$ (cf. Theorem 6.1.15). In particular, $\mathbf{Bun}^{A,\vec{\tau}}_G(\hat{\Sigma})$ is isomorphic to the stack $\mathbf{Parbun}_G(\Sigma, \vec{P})$ of quasi-parabolic G-bundles over $(\Sigma, \vec{p})$ of type

$\vec{P} := (P_1, \ldots, P_s)$ (defined in Chapter 5) and hence it is a smooth (algebraic) stack. Specializing this result to the fiber over a point, we get (cf. Theorem 6.1.17) that there is a natural set-theoretic bijection between the set $\mathrm{Bun}_G^{A,\vec{\tau}}(\hat{\Sigma})$ of isomorphism classes of A-equivariant G-bundles over $\hat{\Sigma}$ of local type $\vec{\tau}$ and the set $\mathrm{Parbun}_G(\Sigma, \vec{P})$ of isomorphism classes of quasi-parabolic G-bundles of type $\vec{P}$ over $(\Sigma, \vec{p})$. Under this bijection, A-semistable (resp. A-stable) G-bundles over $\hat{\Sigma}$ correspond to the parabolic semistable (resp. parabolic stable) bundles over Σ with respect to the markings $\vec{\tau}$. This reduces the problem of studying the quasi-parabolic moduli stack (resp. parabolic semistable moduli *space*, resp. parabolic stable moduli *space*) of parabolic G-bundles over $(\Sigma, \vec{p})$ to that of the moduli stack (resp. semistable moduli *space*, resp. stable moduli *space*) of (non-parabolic) A-equivariant G-bundles over a cover $\hat{\Sigma}$ of Σ with Galois group A.

Let us assume now that G, more generally, is a connected reductive group and Σ continues to be a smooth irreducible projective curve. In Section 6.2, we prove the existence and uniqueness of Harder–Narasimhan (for short HN) reduction of a G-bundle over Σ. Let $\pi : E \to \Sigma$ be a G-bundle. Then, a P-subbundle $E_P \subset E$ for a standard parabolic subgroup P of G is called a *Harder–Narasimhan reduction* if the associated L-bundle $E_P(L)$, obtained from the P-bundle E_P via the extension of the structure group $P \to P/U \simeq L$, is semistable, where L is the Levi subgroup of P containing H and U is the unipotent radical of P. Moreover, we require that for any nontrivial character λ of P such that $\lambda \in \bigoplus_{i=1}^{\ell} \mathbb{Z}_+\alpha_i$ (in particular, λ is trivial restricted to the identity component of the center of G),

$$\deg\left(E_P \times^P \mathbb{C}_\lambda\right) > 0.$$

By virtue of Theorem 6.2.3, such a reduction exists and is unique. Moreover, for a G-bundle E over Σ, and an embedding of connected reductive groups $G \hookrightarrow G'$, the HN reduction of E coincides with the HN reduction of $E(G')$ intersected with E (cf. Theorem 6.2.6 for a more precise statement). As a consequence, it is shown (cf. Corollary 6.2.7) that if $E(G')$ is semistable, then so is E. Further, if E is semistable and G is not contained in any proper (not necessarily standard) parabolic subgroup of G', then $E(G')$ is semistable. As another consequence of HN reduction, an A-equivariant G-bundle over $\hat{\Sigma}$ is A-semistable if and only if it is semistable (cf. Exercise 6.2.E.4). By virtue of Exercise 6.1.E.15, a vector bundle over Σ is polystable (where polystability is defined in Definition 6.1.4(c)) if and only if it is a direct sum of stable vector bundles of the same slope. In Exercise 6.2.E.2 the HN reduction of vector bundles is discussed.

Section 6.3 is devoted to the classical result of Narasimhan–Seshadri on topological construction of stable and polystable vector bundles over Σ and its generalization to any connected reductive G. For any homomorphism from the fundamental group $\rho\colon \pi_1(\Sigma) \to G$, we get a holomorphic G-bundle

$$E_\rho := \tilde{\Sigma} \times^{\pi_1(\Sigma)} G,$$

where $\tilde{\Sigma}$ is the simply-connected cover of Σ. By the Serre's GAGA principle, E_ρ is an algebraic G-bundle over Σ. If $\operatorname{Im}\rho$ lies in a compact subgroup of G, then ρ is called a *unitary* homomorphism and E_ρ is called a *unitary G-bundle*. The homomorphism ρ is called *irreducible* if $\operatorname{Im}\rho$ is not contained in any proper (not necessarily standard) parabolic subgroup of G. Then, by Proposition 6.3.4, E_ρ is a semistable G-bundle if ρ is unitary. Further, for a unitary ρ, E_ρ is a stable G-bundle if and only if ρ is irreducible. In fact, we prove a generalization of these results for equivariant bundles. It is shown that for a unitary representation V of $\pi_1(\Sigma)$, the subspace $V^{\pi_1(\Sigma)}$ of $\pi_1(\Sigma)$-invariants in V is canonically isomorphic with the space of global sections of the corresponding vector bundle over Σ (cf. Lemma 6.3.6 for its equivariant generalization). This leads to the result that for two unitary homomorphisms ρ, ρ', the corresponding bundles E_ρ and $E_{\rho'}$ are isomorphic if and only if ρ is conjugate to ρ' (cf. Corollary 6.3.7 for its equivariant generalization). A classification of topological G-bundles over Σ is obtained in Lemma 6.3.10. For a unitary representation ρ of $\pi_1(\Sigma)$, the dimension of the group cohomology $H^1(\pi_1(\Sigma), \operatorname{ad}\rho)$ is calculated in Corollary 6.3.14.

Let K be a compact connected Lie group (which we take to be a maximal compact subgroup of G). For any integer $g \geq 1$, let F_g be the free group on the symbols $\{a_1, b_1, a_2, b_2, \ldots, a_g, b_g\}$. Define the map

$$\beta : K^{2g} \to [K, K],\ \big((h_1,k_1),(h_2,k_2),\ldots,(h_g,k_g)\big) \mapsto \Pi_{i=1}^{g}[h_i,k_i].$$

Any $\bar{\rho} = \big((h_1,k_1),\ldots,(h_g,k_g)\big) \in K^{2g}$ determines a group homomorphism $\tilde{\rho}\colon F_g \to K$ taking $a_i \mapsto h_i$ and $b_i \mapsto k_i$. If $\bar{\rho} \in \beta^{-1}(e)$, then the homomorphism $\tilde{\rho}$ descends to a group homomorphism $\rho\colon \pi_1(\Sigma) \to K$, where g is the genus of Σ. For any $\bar{\rho} \in \beta^{-1}(e)$, $\operatorname{Ker}((d\beta)_{\bar{\rho}})$ is determined in Proposition 6.3.15 and identified with the space of 1-cocycles of $\pi_1(\Sigma)$ with coefficients in $\operatorname{ad}\rho$. As a corollary, we get that $M_g(K) := \{\bar{\rho} \in \beta^{-1}(e) : \rho \text{ is irreducible}\}$ is an $\mathbb{R}$-analytic (smooth) manifold of dimension $(2g-1)\dim K + \dim \mathfrak{z}$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$ (cf. Corollary 6.3.16). Moreover, $M_g(K)$ parameterizes an $\mathbb{R}$-analytic family of holomorphic G-bundles over Σ. It is shown in Proposition 6.3.18 that the infinitesimal deformation map for this family is surjective. In particular, this family is complete at each of its points (cf. Theorem 6.3.20).

As proved in Proposition 6.3.30, let $\mathscr{F} \to \Sigma \times T$ be a $\mathbb{C}$-analytic family of stable G-bundles over Σ parameterized by a $\mathbb{C}$-analytic space T. Then, the subset

$$T_u := \left\{t \in T : \mathscr{F}_t \simeq E_\rho \text{ for some unitary representation } \rho \text{ of } \pi_1(\Sigma) \text{ in } G\right\}$$

is a closed subset of T. Moreover, for any $\mathbb{C}$-analytic family $\mathscr{F}' \to \Sigma \times T$ of G-bundles, by Lemma 6.3.31 (resp. Exercise 6.3.E.9), the subset $T_s := \left\{t \in T : \mathscr{F}'_t \text{ is a stable } G\text{-bundle}\right\}$ (resp. T_{ss} defined as T_s by replacing 'stable' by 'semistable') is an open subset which is complement of a (closed) $\mathbb{C}$-analytic subset of T. Further, for any $\mathbb{R}$-analytic family $\mathscr{F}$ of G-bundles over Σ parameterized by an $\mathbb{R}$-analytic space T,

$$T_o := \left\{t \in T : \mathscr{F}_t \simeq E_\rho \text{ for some irreducible representation } \rho \text{ of } \pi_1(\Sigma) \text{ in } K\right\}$$

is an open subset of T (cf. Corollary 6.3.21). The above results lead finally to the following fundamental Theorem 6.3.35.

Theorem Let G be a connected reductive group and let E be a holomorphic G-bundle over a smooth irreducible projective curve Σ of genus $g \geq 2$. Then E is polystable of degree 0 (i.e., $E \times^G \mathbb{C}_\chi$ has degree 0 for any character χ of G) if and only if $E \simeq E_\rho$ (as holomorphic G-bundles) for a unitary representation $\rho\colon \pi_1(\Sigma) \to G$.

We further have the following equivariant generalization of the Narasimhan–Seshadri Theorem 6.3.35 (cf. Theorem 6.3.41).

Theorem Let $\hat{\Sigma}$ be an irreducible smooth projective curve with faithful action of a finite group A such that $\Sigma := \hat{\Sigma}/A$ has genus $g \geq 2$. Then an A-equivariant G-bundle $\hat{E}$ over $\hat{\Sigma}$ is A-unitary if and only it is A-polystable of degree 0.

In particular, an A-equivariant G-bundle over $\hat{\Sigma}$ is A-polystable if and only if it is polystable.

We also prove the following result (cf. Proposition 6.3.42).

Proposition Let $\hat{E}$ be an A-equivariant G-bundle over $\hat{\Sigma}$ such that $\hat{\Sigma}/A$ has genus ≥ 2 and let $\theta\colon G \to \mathrm{GL}_V$ be a representation with finite kernel, where G is a connected semisimple group. Then the vector bundle $\hat{E}(V)$ is A-unitary if and only if $\hat{E}$ is A-unitary.

6.1 Identification of Parabolic G-Bundles with Equivariant G-Bundles

Let G be a simple, connected, simply-connected algebraic group over $\mathbb{C}$ and let $(\Sigma, \vec{p})$ be an s-pointed (for any $s \geq 1$) smooth projective irreducible curve (of any genus g), where $\vec{p} = (p_1, \ldots, p_s)$. *Unless otherwise stated to the contrary, this will be our tacit assumption during this Section 6.1.* Fix a maximal compact subgroup K of G. Following the notation from Section 1.2, define the *fundamental alcove*:

$$\Phi_o = \{h \in \mathfrak{h} : \alpha_i(h) \geq 0 \quad \text{and} \quad \theta(h) \leq 1, \quad \text{for all the simple roots} \ \ \alpha_i\},$$

where θ is the highest root.

For any semisimple element $x \in \mathfrak{g}$, define the corresponding *Kempf's parabolic subalgebra*

$$\mathfrak{p}(x) := \{v \in \mathfrak{g} : \lim_{t \to -\infty} \operatorname{Ad}(\operatorname{Exp}(tx)) \cdot v \text{ exists in } \mathfrak{g}\},$$

and let $P(x)$ be the corresponding parabolic subgroup of G.

Then, for $h \in \Phi_o$, $P(h)$ is the standard parabolic subgroup such that its Levi subgroup $L(h)$ containing H has for its simple roots $S_h := \{\alpha_i : \alpha_i(h) = 0\}$.

We recall the following well-known result (cf. (Helgason, 1978, Chap. VII, Theorem 7.9)).

Lemma 6.1.1 *The map*

$$\Phi_o \to K/\operatorname{Ad}K, \quad h \mapsto [\operatorname{Exp}(2\pi i h)],$$

is a bijection, where $K/\operatorname{Ad}K$ *denotes the set of* K*-orbits in* K *under the adjoint action and* $[\operatorname{Exp}(2\pi i h)]$ *denotes the* K*-orbit of* $\operatorname{Exp}(2\pi i h)$.

Definition 6.1.2 Let $E \to \Sigma$ be a principal G-bundle (cf. Example C.4(d)). A *parabolic structure* on E (with respect to the pointed curve $(\Sigma, \vec{p})$) consists of:

(a) Markings (called *parabolic weights*) $\vec{\tau} = (\tau_1, \ldots, \tau_s)$, for $\tau_j \in \Phi_o$, where τ_j is 'attached' to the point p_j, and
(b) A section σ_j of E_{p_j}/P_j over p_j, for each $1 \leq j \leq s$, where $P_j := P(\tau_j)$ and E_{p_j} is the fiber of E over p_j.
Denote $\vec{\sigma} := (\sigma_1, \ldots, \sigma_s)$.

A G-bundle $E \to \Sigma$ with the above additional structures (a) and (b) is called a *parabolic* G*-bundle over* $(\Sigma, \vec{p})$ *with markings* $\vec{\tau}$ and denoted by $(E,\vec{\tau},\vec{\sigma})$. Thus, a parabolic G-bundle over $(\Sigma,\vec{p})$ is nothing but a quasi-parabolic G-bundle over $(\Sigma,\vec{p})$ of type $\vec{P} := (P_1,\dots,P_s)$ (cf. Definition 5.1.4) together with the markings $\vec{\tau}$.

Similarly, a family of parabolic G-bundles parameterized by a scheme S is a G-bundle $\mathscr{E}$ over $\Sigma \times S$ consisting of:

(a′) markings $\vec{\tau} = (\tau_1,\dots,\tau_s)$ as in (a), and
(b′) a section σ_j^S of $(\mathscr{E}|_{p_j\times S})/P_j$, for each $1 \le j \le s$.

Let $\mathscr{E}_1$ and $\mathscr{E}_2$ be two families of parabolic G-bundles with the same markings $\vec{\tau}$ (parameterized by schemes S_1 and S_2, respectively). By a morphism $\varphi\colon \mathscr{E}_1 \to \mathscr{E}_2$ of families of parabolic G-bundles, we simply mean a morphism of the underlying quasi-parabolic G-bundles (cf. Definition 5.1.4).

Definition 6.1.3 (a) Let P be a standard parabolic subgroup with the Levi subgroup $L = L_P$ containing the maximal torus H (with Lie algebra $\mathfrak{h}$). Let $S_P \subset \{\alpha_1,\dots,\alpha_\ell\}$ be the set of simple roots for L. Then the set $X(P)$ of characters of P (i.e., algebraic group homomorphisms $P \to \mathbb{G}_m$) can be identified with

$$\mathfrak{h}^*_{\mathbb{Z},P} := \{\lambda \in \mathfrak{h}^* : \lambda(\alpha_i^\vee) \in \mathbb{Z}\ \forall \text{ simple roots } \alpha_i \text{ and } \lambda(\alpha_i^\vee) = 0\ \forall \alpha_i \in S_P\} \tag{1}$$

under $\chi \mapsto \dot{\chi}(1)|_{\mathfrak{h}}$. We often identify χ with $\dot{\chi}(1)|_{\mathfrak{h}}$ and write it additively.

Let $\{\omega_1,\dots,\omega_\ell\}$ denote the set of fundamental weights, i.e.,

$$\omega_i(\alpha_j^\vee) = \delta_{i,j}, \quad 1 \le i,j \le \ell. \tag{2}$$

Then

$$\mathfrak{h}^*_{\mathbb{Z},P} = \bigoplus_{\alpha_i \notin S_P} \mathbb{Z}\,\omega_i.$$

Recall that the standard maximal parabolic subgroups Q_k are parameterized by $1 \le k \le \ell$, where Q_k is the unique standard parabolic subgroup with

$$S_{Q_k} := \{\alpha_1,\dots,\hat{\alpha}_k,\dots,\alpha_\ell\}.$$

For a standard parabolic subgroup P, let $W_P \subset W$ be the Weyl group of its Levi subgroup L_P.

(b) Let $E \to \Sigma$ be a principal G-bundle and let $f\colon G \to G'$ be a homomorphism of algebraic groups. Then, by $E(G')$ we mean the principal G'-bundle $E \times^G G' \to \Sigma$, where G acts on G' via the left multiplication through the morphism f and G' acts on $E \times^G G'$ via the right multiplication on the G'-factor.

(c) Let $E \to \Sigma$ be a principal G-bundle. For any parabolic subgroup P of G and $\chi \in X(P)$, define the line bundle over E/P:

$$\mathscr{L}_P(\chi) = \mathscr{L}_P(\chi, E) = E \times^P \mathbb{C}_{\chi^{-1}} \to E/P,$$

where $\mathbb{C}_{\chi-1}$ is the 1-dimensional representation of P associated to the character χ^{-1}.

Let $T_v(E/P)$ be the relative tangent bundle of E/P over Σ (consisting of tangent vectors of E/P along the fibers of the bundle $E/P \to \Sigma$). Then, $T_v(E/P)$ can canonically be identified with the vector bundle $E \times^P (\mathfrak{g}/\mathfrak{p})$ over E/P, where $\mathfrak{p} := \text{Lie}\, P$ and P acts on $\mathfrak{g}/\mathfrak{p}$ via the adjoint action.

Definition 6.1.4 (Semistable bundles) (a) A vector bundle $\mathscr{V}$ over Σ is defined to be *semistable* (resp. *stable*) if for any subbundle $(0) \subsetneq \mathscr{W} \subsetneq \mathscr{V}$,

$$\mu(\mathscr{W}) \leq \mu(\mathscr{V}) \ \ (\text{resp. } \mu(\mathscr{W}) < \mu(\mathscr{V})), \tag{1}$$

where the slope $\mu(\mathscr{V}) := \deg(\mathscr{V})/\text{rank}(\mathscr{V})$ and deg denotes the first Chern class.

Thus, a vector bundle $\mathscr{V}$ is semistable (resp. stable) if and only if $\mathscr{V} \otimes \mathscr{L}$ is semistable (resp. stable) for any line bundle $\mathscr{L}$ over Σ.

(b) A G-bundle $E \to \Sigma$ is called *semistable* (resp. *stable*) if for any standard maximal parabolic subgroup Q_k of G $(1 \leq k \leq \ell)$ and any section μ of $E/Q_k \to \Sigma$,

$$\deg \mu^* \left(\mathscr{L}_{Q_k}(-\omega_k)\right) \leq 0 \left(\text{resp. } \deg \mu^* \left(\mathscr{L}_{Q_k}(-\omega_k)\right) < 0\right). \tag{2}$$

Observe that the trivial bundle $\Sigma \times G \to \Sigma$ is semistable.

Alternatively, a G-bundle $E \to \Sigma$ is called *semistable* (resp. *stable*) if for any standard proper parabolic subgroup P of G and any section μ of $E/P \to \Sigma$,

$$\deg \mu^* (T_v(E/P)) \geq 0 \ \ (\text{resp. } > 0).$$

By Exercise 6.1.E.4, these two definitions are equivalent.

These alternative definitions remain valid for any connected reductive group G provided we take the fundamental weights ω_k to vanish on the center $Z(\mathfrak{g})$ $(\subset \mathfrak{h})$ of $\mathfrak{g}$ and we replace ω_k by some positive multiple $d\omega_k$ so that $d\omega_k$ is a character of T.

By Exercise 6.1.E.5, a vector bundle $\mathscr{V}$ over Σ is semistable (resp. stable) if and only if the associated frame bundle $F(\mathscr{V})$ (which is a principal GL_n-bundle for $n = \text{rank}\, \mathscr{V}$) is semistable (resp. stable).

(c) As in (b), let G be a connected reductive group. Then, a G-bundle E over Σ is called *polystable* if it has a reduction E_L to a Levi subgroup L

(of a parabolic subgroup P of G) such that the L-bundle E_L is stable and for any character χ of L which is trivial restricted to the center of G, we have

$$\deg\left(E_L \times^L \mathbb{C}_\chi\right) = 0,$$

where $\mathbb{C}_\chi$ is the 1-dimensional representation of L given by the character χ.

A vector bundle $\mathscr{V}$ over Σ of rank r is called *polystable* if the associated frame bundle $F(\mathscr{V})$ is polystable as a GL_r-bundle. By Exercise 6.1.E.15, $\mathscr{V}$ is polystable if and only if it is a direct sum of stable vector bundles all of which have the same slope.

By Theorem 6.1.7, $\operatorname{ad} E$ is polystable if E is so, where $\operatorname{ad} E := E \times^G \mathfrak{g}$. Thus, by Exercise 6.1.E.15, $\operatorname{ad} E$ is semistable and hence E is semistable by Lemma 6.1.5.

(d) Let $(E, \vec{\tau}, \vec{\sigma})$ be a parabolic G-bundle over $(\Sigma, \vec{p})$. Then, it is called *parabolic semistable* (resp. *parabolic stable*) if for any standard maximal parabolic subgroup Q_k $(1 \leq k \leq \ell)$ and any section μ of $E/Q_k \to \Sigma$, we have

$$\deg \mu^*\left(\mathscr{L}_{Q_k}(-\omega_k)\right) + \sum_{j=1}^{s} \omega_k(w_j^{-1}\tau_j) \leq 0 \quad (\text{resp. } < 0), \tag{3}$$

where $\bar{w}_j := W_{P_j} w_j W_{Q_k} \in W_{P_j}\backslash W/W_{Q_k}$ is the unique element such that taking any $e_j \in E_{p_j}$ and writing $\sigma_j = e_j g_j P_j$ and $\mu(p_j) = e_j h_j Q_k$, for some $g_j, h_j \in G$, we have

$$h_j \in g_j P_j w_j Q_k. \tag{4}$$

(It is easy to see that $\bar{w}_j$ does not depend upon the choices of e_j, g_j and h_j. This $\bar{w}_j$ is called the *relative position* of μ with respect to the quasi-parabolic structure at p_j.)

The number on the left side of (3) is called the *parabolic degree* (denoted $\operatorname{pardeg} \mu^*\mathscr{L}_{Q_k}(-\omega_k)$) of the parabolic bundle E with respect to the section μ and the line bundle $\mathscr{L}_{Q_k}(-\omega_k)$ for the parabolic markings $\vec{\tau} = (\tau_1, \ldots, \tau_s)$.

An equivalent characterization of parabolic semistability (resp. parabolic stability) for vector bundles is given in Exercise 6.1.E.7.

Lemma 6.1.5 *Let G be a connected reductive group and let $E \to \Sigma$ be a G-bundle. If the adjoint vector bundle*

$$\operatorname{ad} E := E \times^G \mathfrak{g}$$

is semistable (resp. stable), then so is E.

In fact, by Theorem 6.1.7, we see that if E is semistable, then so is $\operatorname{ad} E$. *Thus, semistability of E is equivalent to that of* $\operatorname{ad} E$.

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In general E being stable does not necessarily imply that ad E *is stable even when G is a simple group (cf. Exercise 6.3.E.10).*

Proof Let P be a standard proper parabolic subgroup of G and let $E_P \subset E$ be a P-subbundle obtained from a section μ of $E/P \to \Sigma$ (cf. Lemma 5.1.2). Now, by definition, $\mu^*(T_v(E/P))$ is a quotient of the adjoint bundle ad E. But, $\deg(\operatorname{ad} E) = 0$ (since G acts trivially on $\wedge^{\text{top}}(\mathfrak{g})$ under the adjoint action) and since ad E is semistable (resp. stable), by assumption, we get $\deg \mu^*(T_v(E/P)) \geq 0$ (resp. > 0). This proves the lemma. □

Remark 6.1.6 A G-bundle can be thought of as a parabolic G-bundle for $s = 0$. Further, in this case, parabolic semistable (resp. stable) bundle is nothing but a semistable (resp. stable) bundle.

We recall the following result without proof from Ramanan and Ramanathan (1984, Theorem 3.18). The proof in the same has a gap, but a modified proof is given in Balaji and Parameswaran (2003, Proposition 6 and Remarks 17, 18).

Theorem 6.1.7 *Let $f: G \to G'$ be a homomorphism between connected reductive groups such that $f(Z^o(G)) \subset Z^o(G')$, where $Z^o(G)$ denotes the identity component of the center of G. Then, if $E \to \Sigma$ is a semistable (resp. polystable) G-bundle, then so is $E(G')$ obtained from E by extension of the structure group to G' (cf. Definition 6.1.3(b)).*

In particular, for any semistable (resp. polystable) G-bundle E, ad E *is a semistable (resp. polystable) vector bundle (cf. Exercise 6.1.E.5).*

We recall the following result. To prove the result, by Selberg (1960, Lemma 8), any finitely generated linear group Γ has a normal torsion-free subgroup Γ_o of finite index in Γ. Moreover, observe that if Γ acts faithfully on the upper half plane $\mathbb{H}$ (resp. $\mathbb{A}^1(\mathbb{C})$) with all its Γ-orbits closed and the action of Γ is properly discontinuous on a nonempty Γ-stable open subset, then Γ_o acts fixed point freely on $\mathbb{H}$ (resp. $\mathbb{A}^1(\mathbb{C})$). To prove this, realize $\mathbb{H} = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2$ and thus $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ in this case. In the case of $\mathbb{A}^1(\mathbb{C})$, observe that the group of variety automorphisms of the affine line:

$$\operatorname{Aut}(\mathbb{A}^1(\mathbb{C})) = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : a \in \mathbb{C}^*, b \in \mathbb{C} \right\}$$

acting on $\mathbb{A}^1(\mathbb{C}) = \{[z : 1] : z \in \mathbb{C}\}$ as a subset of $\mathbb{P}^1(\mathbb{C})$. An element $\gamma = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \in \operatorname{Aut}(\mathbb{A}^1(\mathbb{C}))$ is of infinite order if and only if either $a = 1$ and $b \neq 0$ or $a \in \mathbb{C}^*$ is of infinite order (in the multiplicative group). Now, using the results from Serre (1992, §6.4) the following result is obtained.

Theorem 6.1.8 *Let $(\Sigma, \vec{p})$ be a smooth irreducible projective s-pointed curve for $s \geq 1$ and let $\vec{d} = (d_1, \ldots, d_s)$ be a set of integers $d_i \geq 2$ attached to $\vec{p}$.*

$$\text{We assume that if } \Sigma = \mathbb{P}^1, \text{ then } s \geq 3. \tag{1}$$

Then there exists a smooth irreducible projective curve $\hat{\Sigma}$ and a Galois cover $\pi : \hat{\Sigma} \to \Sigma$ with finite Galois group A such that A acts freely on $\hat{\Sigma}\backslash\pi^{-1}(\bar{p})$ and the isotropy subgroup $A_{\hat{p}_i}$ (for any $\hat{p}_i \in \pi^{-1}(p_i)$) is cyclic of order d_i, where $\bar{p} \subset \Sigma$ denotes the subset $\{p_1, \ldots, p_s\}$. The set $\vec{p}$ together with $\vec{d}$ is called signature on Σ.

Conversely, any smooth irreducible projective curve $\hat{\Sigma}$ with faithful action of a finite group A gives rise to such an example by taking $\Sigma = \hat{\Sigma}/A$ and $\vec{p} = (p_1, \ldots, p_s)$ in Σ consists of ramification points. Here, $\vec{d} = (d_1, \ldots, d_s)$ is the set of integers ≥ 2 such that d_i is the order of the isotropy group for any point $\hat{p}_i \in \hat{\Sigma}$ over p_i.

Even though, given $(\Sigma, \vec{p})$ and $\vec{d}$, $\hat{\Sigma}$ is *not* unique, we will fix one such $\hat{\Sigma}$ in the sequel.

Let A be a finite group acting on the formal disc $\mathbb{D} := \text{Spec}(\mathbb{C}[[t]])$ and let $R \in$ **Alg** (cf. Section 1.1). Then, A acts on $\mathbb{D}_R := \text{Spec}(R[[t]])$ with the trivial action of A on R, by observing that $R[[t]] = \varprojlim_n \big(R \otimes_{\mathbb{C}} (\mathbb{C}[[t]]/\langle t^n \rangle)\big)$.

Theorem 6.1.9 *Let G be any connected affine algebraic group (not necessarily semisimple) and let $\mathscr{E} \to \mathbb{D}_R$ be an A-equivariant principal G-bundle, which is trivial as a G-bundle. Then, there exists a G-bundle trivialization of $\mathscr{E}$ in which the A-action is the product action, in the sense that there exists an A-equivariant G-bundle isomorphism inducing the identity on the base:*

$$\mathscr{E} \underset{\sim}{\overset{\varphi}{\to}} \mathbb{D}_R \times G$$

such that the action of A on $\mathbb{D}_R \times G$ is given by

$$\gamma \odot (x, g) = (\gamma x, \theta_\gamma(x(0))g), \text{ for } \gamma \in A, x \in \mathbb{D}_R \text{ and } g \in G, \tag{1}$$

where $x(0)$ is the image of x in Spec*R induced from the embedding $R \to R[[t]]$ and $\theta_\gamma :$* Spec*$R \to G$ is a morphism.*

Moreover, for any $x^o \in$ Spec*R, the group homomorphism $\theta(x^o): A \to G$, $\gamma \mapsto \theta_\gamma(x^o)$, is unique up to a conjugation, which is called the type of $\mathscr{E}$ over x^o.*

If $R = \mathbb{C}$, so that Spec*R is a point, we simply call θ the* type of $\mathscr{E}$.

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Proof Pick any G-bundle trivialization $\mathscr{E} \xrightarrow[\sim]{\beta} \mathbb{D}_R \times G$. Then, the action of A transports via β to an action given by

$$\gamma \cdot (x,g) = (\gamma x, \alpha_\gamma(\gamma x)g), \quad \text{for } \gamma \in A, x \in \mathbb{D}_R, g \in G, \tag{2}$$

where $\alpha_\gamma : \mathbb{D}_R \to G$ is a morphism. Since $\mathscr{E}$ is an A-equivariant G-bundle, we get

$$\alpha_{\gamma_1\gamma_2}(x) = \alpha_{\gamma_1}(x)\alpha_{\gamma_2}(\gamma_1^{-1}x), \text{ for } \gamma_1,\gamma_2 \in A, x \in \mathbb{D}_R. \tag{3}$$

Thus, thinking of α_γ as an element of $G(R[[t]])$, we get a 1-cochain $\alpha: A \to G(R[[t]])$, $\gamma \mapsto \alpha_\gamma$, for the group A with coefficients in $G(R[[t]])$, with the trivial action of A on G. Moreover, α is a 1-cocycle by (3) (cf. (Serre, 1997, Chap. I, §5.1)).

Evaluation at $t = 0$ gives rise to an A-equivariant algebra homomorphism $R[[t]] \to R$ and hence an A-equivariant group homomorphism

$$e^o : G(R[[t]]) \to G(R).$$

Composing $e^o \circ \alpha$, we get a 1-cocycle

$$\alpha^o : A \to G(R) \hookrightarrow G(R[[t]]).$$

Let $G(R[[t]])^+$ be the kernel of e^o. Clearly, e^o is surjective (due to the inclusion $G(R) \hookrightarrow G(R[[t]])$). The exact sequence

$$1 \to G(R[[t]])^+ \to G(R[[t]]) \xrightarrow{e^o} G(R) \to 1$$

gives rise to an exact sequence of pointed sets in non-abelian group cohomology (cf. (Serre, 1997, Proposition 38, §5.5)):

$$H^1(A, G(R[[t]])^+) \to H^1(A, G(R[[t]])) \xrightarrow{\hat{e^o}} H^1(A, G(R)). \tag{4}$$

We next show that $\hat{e}^o$ is a one-to-one map. To prove this, by Serre (1997, Chap. I, Corollary 2, §5.5), it suffices to show that for any 1-cocycle $\beta: A \to G(R[[t]])$,

$$H^1(A, G(R[[t]])^+_\beta) \text{ is trivial,} \tag{5}$$

where $G(R[[t]])^+_\beta$ denotes the same group $G(R[[t]])^+$ but with a twisted action of A via β:

$$\gamma \odot_\beta f = \beta(\gamma)(\gamma \cdot f)\beta(\gamma)^{-1}, \text{ for } \gamma \in A \text{ and } f \in G(R[[t]])^+.$$

We first prove by induction on $n \geq 1$ that

$$H^1\left(A, G\left(R[[t]]/\langle t^n\rangle\right)^+_\beta\right) \quad \text{is trivial,} \tag{6}$$

where $G\left(R[[t]]/\langle t^n\rangle\right)^+$ is the kernel of the surjective homomorphism $G\left(R[[t]]/\langle t^n\rangle\right) \to G(R)$ and $G\left(R[[t]]/\langle t^n\rangle\right)^+_\beta$ denotes the same group with twisted action of A via the image of β in $G\left(R[[t]]/\langle t^n\rangle\right)$. Clearly, (6) for $n = 1$ is trivial. Now, consider the exact sequence of A-groups:

$$1 \to (K_n(R))_\beta \to G\left(R[[t]]/\langle t^{n+1}\rangle\right)^+_\beta \xrightarrow{\pi_n^R} G\left(R[[t]]/\langle t^n\rangle\right)^+_\beta \to 1, \tag{7}$$

where $K_n(R)$ is the kernel of π_n^R. By Exercise 6.1.E.1, π_n^R is surjective with kernel isomorphic (as a group) to the $\mathbb{C}$-vector space $R \otimes_{\mathbb{C}} \left(\mathfrak{g} \otimes_{\mathbb{C}} \frac{t^n\mathbb{C}[[t]]}{t^{n+1}\mathbb{C}[[t]]}\right)$. Next, observe that any element $\gamma \in A$ acts on $(K_n(R))_\beta$ via a $\mathbb{C}$-linear isomorphism. Thus, by Hochschild and Serre (1953, Proposition 6),

$$H^1(A, (K_n(R))_\beta) = 0. \tag{8}$$

From the cohomology sequence (analogue of (4)) associated to the coefficient sequence (7) of A-groups, and using (6) (valid by the induction hypothesis) and (8), we get that

$$H^1\left(A, G\left(R[[t]]/\langle t^{n+1}\rangle\right)^+_\beta\right) = 0,$$

completing the induction and hence (6) is proved for all $n \geq 1$ and any 1-cocycle $\beta\colon A \to G(R[[t]])$.

Since $M_N(R[[t]]) \simeq \varprojlim_n M_N\left(R[[t]]/\langle t^n\rangle\right)$, by considering an embedding $G \hookrightarrow M_N$ and the equations defining G, it is easy to see that

$$G(R[[t]])^+ \simeq \varprojlim G\left(R[[t]]/\langle t^n\rangle\right)^+. \tag{9}$$

Consider the isomorphism of varieties induced from the exponential map (cf. Exercise 6.1.E.1):

$$\mathrm{Exp}\colon \mathfrak{g} \otimes \left(t\mathbb{C}[[t]]/\langle t^n\rangle\right) \to G\left(\mathbb{C}[[t]]/\langle t^n\rangle\right)^+.$$

It induces a bijection

$$\begin{aligned}\mathfrak{g} \otimes \left(tR[[t]]/\langle t^n\rangle\right) &\simeq \mathrm{Mor}\left(\mathrm{Spec}\, R, \mathfrak{g} \otimes \left(t\mathbb{C}[[t]]/\langle t^n\rangle\right)\right)\\ &\simeq \mathrm{Mor}\left(\mathrm{Spec}\, R, G\left(\mathbb{C}[[t]]/\langle t^n\rangle\right)^+\right)_\beta, \quad f \mapsto \mathrm{Exp}\circ f\\ &\xrightarrow[\sim]{\theta} G\left(R[[t]]/\langle t^n\rangle\right)^+_\beta,\end{aligned}$$

where the bijection θ is obtained by using Exercises 1.3.E.10 and 1.3.E.6.

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The bijection θ allows us to transport the action of A on $G\,(R[[t]]/\langle t^n\rangle)^+_\beta$ to that on $\mathfrak{g}\otimes(tR[[t]]/\langle t^n\rangle)$. Moreover, it is easy to see that any $\gamma\in A$ acts on $\mathfrak{g}\otimes(tR[[t]]/\langle t^n\rangle)$ via a $\mathbb{C}$-linear isomorphism. Thus, $\left[\mathfrak{g}\otimes(tR[[t]]/\langle t^n\rangle)\right]^A$ is a linear subspace. From this we immediately see that the canonical map

$$\left[G\left(R[[t]]/\langle t^{n+1}\rangle\right)^+_\beta\right]^A \to \left[G\left(R[[t]]/\langle t^n\rangle\right)^+_\beta\right]^A \quad \text{is surjective.}$$

Thus, by Exercise 6.1.E.2, (6) and (9), we get

$$H^1\left(A, G(R[[t]])^+_\beta\right) = 0,$$

for any 1-cocycle $\beta\colon A\to G(R[[t]])$. This proves (5) and hence the map (cf. (4))

$$\hat{e}^o\colon H^1(A, G(R[[t]])) \to H^1(A, G(R)) \quad \text{is one-to-one.}$$

We return to the 1-cocycle α as at the beginning of the proof. Clearly, $\hat{e}^o([\alpha]) = \hat{e}^o([\alpha^o])$, where $[\alpha]$, $[\alpha^o]\in H^1(A, G(R[[t]])$ denote the cohomology classes of α and α^o, respectively. Since $\hat{e}^o$ is one-to-one, we get

$$[\alpha] = [\alpha^o], \tag{10}$$

i.e., there exists a $\tau\in G(R[[t]]) = \mathrm{Mor}(\mathbb{D}_R, G)$ such that

$$\begin{aligned} &\tau(\gamma x)^{-1}\alpha_\gamma(\gamma x)\tau(x) = \alpha_\gamma(x(0)), \text{ for all } x\in\mathbb{D}_R, \gamma\in A,\\ &\qquad\text{since } (\gamma x)(0) = x(0) \end{aligned} \tag{11}$$

(cf. (Serre, 1997, Chap. I, §5.1)).

Define a G-bundle isomorphism

$$\mathbb{D}_R\times G \xrightarrow{\hat{\tau}} \mathbb{D}_R\times G, \;\; (x,g)\mapsto (x, \tau(x)^{-1}g).$$

Then, the action of A on the range transported via $\hat{\tau}$ (to be denoted $\odot$) becomes (cf. (2))

$$\begin{aligned} \gamma\odot(x,g) &= \hat{\tau}(\gamma\cdot\hat{\tau}^{-1}(x,g))\\ &= \hat{\tau}(\gamma\cdot(x,\tau(x)g))\\ &= \hat{\tau}(\gamma x, \alpha_\gamma(\gamma x)\tau(x)g)\\ &= (\gamma x, \tau(\gamma x)^{-1}\alpha_\gamma(\gamma x)\tau(x)g)\\ &= (\gamma x, \alpha_\gamma(x(0))g), \quad \text{by (11).} \end{aligned}$$

Taking $\theta_\gamma = e^o(\alpha_\gamma)$, we get the first part of the theorem.

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To prove the uniqueness of $\theta(x^o)$ up to a conjugation for any $x^o \in \operatorname{Spec} R$, let

$$\mathbb{D}_R \times G \xrightarrow{\delta} \mathbb{D}_R \times G, \ \ \delta(x,g) = (x, \bar{\delta}(x)g), \quad \text{for} \quad x \in \mathbb{D}_R, g \in G,$$

be an A-equivariant G-bundle isomorphism such that A acts on the domain by (1) and on the range by

$$\gamma \odot' (x,g) = (\gamma x, \theta'_\gamma(x(0))g), \quad \text{for} \quad \gamma \in A, x \in \mathbb{D}_R \quad \text{and} \quad g \in G,$$

where $\bar{\delta} \colon \mathbb{D}_R \to G$ is a morphism. In particular, for $x^o \in \operatorname{Spec} R \subset \operatorname{Spec} R[[t]]$,

$$\begin{aligned} \delta(\gamma \odot (x^o,g)) &= \delta(\gamma x^o, \theta_\gamma(x^o)g) \\ &= (\gamma x^o, \bar{\delta}(\gamma x^o)\theta_\gamma(x^o)g) \\ &= (x^o, \bar{\delta}(x^o)\theta_\gamma(x^o)g), \end{aligned} \tag{12}$$

since A acts trivially on $\operatorname{Spec} R$. On the other hand, from the A-equivariance of δ, we get

$$\begin{aligned} \delta(\gamma \odot (x^o,g)) &= \gamma \odot' \delta((x^o,g)) \\ &= \gamma \odot' (x^o, \bar{\delta}(x^o)g) \\ &= (x^o, \theta'_\gamma(x^o)\bar{\delta}(x^o)g). \end{aligned} \tag{13}$$

Comparing (12) and (13), we get

$$\bar{\delta}(x^o)\theta_\gamma(x^o)\bar{\delta}(x^o)^{-1} = \theta'_\gamma(x^o).$$

Thus, $\theta'(x^o) \colon A \to G$ is a conjugate of $\theta(x^o)$, proving the theorem. □

The above theorem justifies the following.

Definition 6.1.10 Let G be as at the beginning of this section.

(a) Let $(\Sigma, \vec{p})$ be an s-pointed curve as in Theorem 6.1.8 (in particular, it satisfies (1) of Theorem 6.1.8) and let $\vec{d} = (d_1, \dots, d_s)$ be a set of positive integers attached to $\vec{p}$. Fix a Galois cover $\pi : \hat{\Sigma} \to \Sigma$ with Galois group A as guaranteed by Theorem 6.1.8. We also fix preimages $\vec{\hat{p}} = (\hat{p}_1, \dots, \hat{p}_s)$ in $\hat{\Sigma}$ of $\vec{p}$ and generators $\vec{\gamma} = (\gamma_1, \dots, \gamma_s)$ of the isotropy groups $(A_{\hat{p}_1}, \dots, A_{\hat{p}_s})$. Observe that $A_{\hat{p}_i}$ are cyclic groups, being subgroups of $\operatorname{Aut}(T_{\hat{p}_i}(\hat{\Sigma}))$.

For any A-equivariant principal G-bundle $\hat{E}$ over $\hat{\Sigma}$, $\hat{E}_{|\mathbb{D}_{\hat{p}_j}}$ is trivial as a G-bundle (e.g., by Theorem 5.2.5), where $\mathbb{D}_{\hat{p}_j} \subset \hat{\Sigma}$ is the formal disc around $\hat{p}_j$. Since $\hat{p}_j$ is fixed by $A_{\hat{p}_j}$ (in particular, it acts on $\mathbb{D}_{\hat{p}_j}$), $\hat{E}_{|\mathbb{D}_{\hat{p}_j}}$ is an $A_{\hat{p}_j}$-equivariant trivial G-bundle. Thus, by Theorem 6.1.9, we get a homomorphism (the type of $\hat{E}_{|\mathbb{D}_{\hat{p}_j}}$) $\theta_j : A_{\hat{p}_j} \to G$ (unique up to a conjugation). Moreover, any conjugate of θ_j can be realized as θ_j with respect to some G-bundle

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trivialization of $\hat{E}_{|\mathbb{D}_{\hat{p}_j}}$. Let $\tau_j \in \Phi_o$ be the unique element such that $\operatorname{Exp}(2\pi i \tau_j)$ is conjugate to $\theta_j(\gamma_j)$ (cf. Lemma 6.1.1). Define the *local type* of $\hat{E}$ to be the sequence

$$\vec{\tau} = (\tau_1, \dots, \tau_s).$$

Observe that τ_j *does* depend upon the choice of the generator γ_j of $A_{\hat{p}_j}$.

(b) Let $(\Sigma, \vec{p})$ be an s-pointed curve as in Theorem 6.1.8 (in particular, it satisfies (1) of Theorem 6.1.8). Let $\vec{\tau} = (\tau_1, \dots, \tau_s)$ be a set of markings (cf. Definition 6.1.2) with τ_j rational points of Φ_o, i.e., we can write $\tau_j = \bar{\tau}_j/d_j$, for some positive integers d_j and $\operatorname{Exp}(2\pi i \bar{\tau}_j) = 1$.

As in Theorem 6.1.8, we fix a Galois cover $\pi : \hat{\Sigma} \to \Sigma$ with finite Galois group A associated to $(\Sigma, \vec{p})$ and the sequence $\vec{d} = (d_1, \dots, d_s)$, ignoring those p_i with $d_i = 1$. We also fix inverse images $\{\hat{p}_j \in \pi^{-1}(p_j)\}_{1 \le j \le s}$ and generators $\vec{\gamma} = (\gamma_1, \dots, \gamma_s)$ of the cyclic isotropy groups $(A_{\hat{p}_1}, \dots, A_{\hat{p}_s})$.

We make the following definition similar to Definition 5.2.6.

Definition 6.1.11 With the above notation; in particular, $s \ge 1$ and $s \ge 3$ if $\Sigma = \mathbb{P}^1$, define the functor $\mathscr{F}^{A,\vec{\tau}}_{G,\hat{\Sigma}^*} : \textbf{Alg} \to \textbf{Set}$ by

$$\begin{aligned}\mathscr{F}^{A,\vec{\tau}}_{G,\hat{\Sigma}^*}(R) = \{(\hat{E}_R, \hat{\sigma}_R) : \hat{E}_R &\text{ is an } A\text{-equivariant principal } G\text{-bundle over } \hat{\Sigma}_R \\ &\text{such that } \hat{E}_{R_{|\hat{\Sigma}\times x}} \text{ has local type } \vec{\tau} \text{ for any } x \in \operatorname{Spec} R \text{ and } \hat{\sigma}_R \text{ is an} \\ &A\text{-equivariant section of } \hat{E}_R \text{ over } (\hat{\Sigma}^*)_R\}/\sim,\end{aligned}$$

where $\Sigma^* := \Sigma \backslash \{p_1, \dots, p_s\}$, $\hat{\Sigma}^* := \pi^{-1}(\Sigma^*)$, A acts trivially on R and $(\hat{E}_R, \hat{\sigma}_R) \sim (\hat{E}'_R, \hat{\sigma}'_R)$ if there exists an isomorphism $\hat{\theta}_R$ of A-equivariant G-bundles:

$$\begin{array}{ccc} \hat{E}_R & \xrightarrow{\hat{\theta}_R} & \hat{E}'_R \\ & \searrow \quad \swarrow & \\ & \hat{\Sigma}_R & \end{array}$$

such that $\hat{\theta}_R \circ \hat{\sigma}_R = \hat{\sigma}'_R$. We denote the equivalence class of $(\hat{E}_R, \hat{\sigma}_R)$ by $[\hat{E}_R, \hat{\sigma}_R]$.

Choose a local parameter $\hat{t}_j$ of $\hat{\Sigma}$ around $\hat{p}_j$ such that the generator γ_j of the isotropy group $A_{\hat{p}_j}$ acts on the function $\hat{t}_j$ via

$$\gamma_j \cdot \hat{t}_j = e^{-\frac{2\pi i}{d_j}} \hat{t}_j \quad \text{and} \quad t_j := (\hat{t}_j)^{d_j} \tag{1}$$

is a local parameter for Σ at p_j. Such a local parameter $\hat{t}_j$ exists.

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For any one-parameter subgroup $\tau\colon \mathbb{G}_m \to G$, recall the definition of *Kempf's parabolic subgroup*

$$Q(\tau) := \left\{ g \in G : \lim_{z\to 0} \tau(z) g \tau(z)^{-1} \text{ exists in } G \right\}. \tag{2}$$

If semisimple $x \in \mathfrak{g}$ is such that $\mathrm{Exp}(2\pi i d x) = 1$ for some positive integer d, then

$$\bar{\sigma}_x\colon \mathbb{C} \to G,\ \ t \mapsto \mathrm{Exp}(dtx)$$

descends to a one-parameter subgroup $\sigma_x\colon \mathbb{C}^* \to G$, where $\mathbb{C}^* := \mathbb{C}/2\pi i\mathbb{Z}$. In this case, it is easy to see that $\mathrm{Lie}(Q(\sigma_x)) = \mathfrak{p}(x)$, where $\mathfrak{p}(x)$ is defined at the beginning of this section.

For any parabolic subgroup P of G, define the parahoric subgroup scheme $\mathcal{P} \subset \bar{G}((t))$ as in Exercise 1.3.E.11 by

$$\mathcal{P} := ev_0^{-1}(P), \ \text{ under the evaluation map } \ ev_0\colon \bar{G}[[t]] \to G \ \text{ at } \ t = 0. \tag{3}$$

Analogous to Proposition 5.2.7, we have the following.

Theorem 6.1.12 *Let the notation and assumptions be as in the above definition. Assume further that $\theta(\tau_j) < 1$ for each $1 \le j \le s$, where θ is the highest root of G. Then, the functor $\mathscr{F}^{A,\vec{\tau}}_{G,\hat{\Sigma}^*}$ is representable, represented by the ind-scheme (cf. Exercise 1.3.E.11)*

$$\bar{X}_{\vec{P}} = \Pi_{j=1}^{s}\, \bar{X}_G(P_j),$$

where P_j is the standard parabolic subgroup $P(\tau_j)$ of G as at the beginning of this section and $\bar{X}_G(P_j)$ is the partial infinite flag variety, which is an ind-projective variety as in Exercise 1.3.E.11.

Proof We need to prove that for any $R \in \mathbf{Alg}$, $\mathscr{F}^{A,\vec{\tau}}_{G,\hat{\Sigma}^*}(R)$ is canonically isomorphic with $\bar{X}_{\vec{P}}(R) = \mathrm{Mor}(\mathrm{Spec}\, R, \bar{X}_{\vec{P}})$.

Define the map $\mathfrak{H}\colon \mathscr{F}^{A,\vec{\tau}}_{G,\hat{\Sigma}^*}(R) \to \bar{X}_{\vec{P}}(R)$ as follows. Let $\hat{\mathbb{D}}_j := \mathrm{Spec}\, \mathbb{C}[[\hat{t}_j]]$ be the formal disc around $\hat{p}_j$. Let $[\hat{E}_R, \hat{\sigma}_R] \in \mathscr{F}^{A,\vec{\tau}}_{G,\hat{\Sigma}^*}(R)$. Recall that there exists an algebra $R' \in \mathbf{Alg}$ and a surjective étale morphism $\varphi\colon \mathrm{Spec}\, R' \to \mathrm{Spec}\, R$ such that the G-bundle $\hat{E}_{R'_{|(\hat{\mathbb{D}}_j)_{R'}}}$ is trivial for each $1 \le j \le s$, where $\bar{\varphi} := \mathrm{Id}_{\hat{\Sigma}} \times \varphi\colon \hat{\Sigma}_{R'} \to \hat{\Sigma}_R$ and $\hat{E}_{R'} := \bar{\varphi}^*(\hat{E}_R)$ (cf. Theorem 5.2.5). Moreover, by Theorem 6.1.9, we can assume that the action of $A_{\hat{p}_j}$ on $\hat{E}_{R'_{|(\hat{\mathbb{D}}_j)_{R'}}}$ is the 'product action' in the sense that there exists a section $\hat{\mu}_j = \hat{\mu}_{j,R'}$ of $\hat{E}_{R'_{|(\hat{\mathbb{D}}_j)_{R'}}}$ such that the generator γ_j of the stabilizer $A_{\hat{p}_j} \subset A$ acts on $\hat{\mu}_j$ via

$$\gamma_j \cdot \hat{\mu}_j = \hat{\mu}_j \cdot \mathrm{Exp}(2\pi i \tau_j). \tag{1}$$

Write as sections

$$\hat{\mu}_j^* = \hat{\sigma}_{R'}^* \cdot \hat{\beta}_j, \quad \text{for} \quad \hat{\beta}_j \in G(R'((\hat{t}_j))), \tag{2}$$

where $\hat{\sigma}_{R'}$ is the section of $\hat{E}_{R'_{|(\hat{\Sigma}^*)_{R'}}}$ obtained from the pull-back of $\hat{\sigma}_R$, and

$$\hat{\sigma}_{R'}^* := \hat{\sigma}_{R'_{|\,\mathrm{Spec}\, R'((\hat{t}_j))}}, \quad \hat{\mu}_j^* := \hat{\mu}_{j\,|\,\mathrm{Spec}\, R'((\hat{t}_j))}.$$

By (1), it is easy to see that

$$\gamma_j \cdot \hat{\beta}_j = \hat{\beta}_j \cdot \mathrm{Exp}(2\pi i \tau_j). \tag{3}$$

Define the transition function

$$\beta_j := \hat{\beta}_j \cdot (\hat{t}_j)^{\bar{\tau}_j} \in G(R'((\hat{t}_j))). \tag{4}$$

From identity (1) of Definition 6.1.11 and identity (3), it is easy to see that

$$\gamma_j \cdot \beta_j = \beta_j. \tag{5}$$

Thus, β_j descends to an element of $G(R'((t_j)))$. If we take a different section $\hat{\mu}'_j$ of $\hat{E}_{R'_{|(\hat{\mathbb{D}}_j)_{R'}}}$, then we can write

$$\hat{\mu}'_j = \hat{\mu}_j \cdot \hat{f}_j, \quad \text{for some} \quad \hat{f}_j \in G(R'[[\hat{t}_j]]).$$

Hence,

$$\hat{\beta}_j \hat{f}_j = \hat{\beta}'_J.$$

Moreover, if $\hat{\mu}'_j$ also satisfies (1), then we see that

$$\gamma_j \cdot \hat{f}_j = \mathrm{Exp}(2\pi i \tau_j)^{-1} \cdot \hat{f}_j \cdot \mathrm{Exp}(2\pi i \tau_j). \tag{6}$$

Conversely, for any $\hat{f}_j \in G(R'[[\hat{t}_j]])$ satisfying (6), the section $\hat{\mu}_j \cdot \hat{f}_j$ of $\hat{E}_{R'_{|(\hat{\mathbb{D}}_j)_{R'}}}$ satisfies condition (1). Let

$$f_j := (\hat{t}_j)^{-\bar{\tau}_j} \cdot \hat{f}_j \cdot (\hat{t}_j)^{\bar{\tau}_j}.$$

Then, by (6) and identity (1) of Definition 6.1.11,

$$\gamma_j \cdot f_j = f_j. \tag{7}$$

Thus, $f_j \in G(R'((t_j)))$. We next claim that

$$f_j \in \mathcal{P}_j(R') := ev_0^{-1}(P_j(R')), \tag{8}$$

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where $ev_0 \colon G(R'[[t_j]]) \to G(R')$ is the map induced from the evaluation at $t_j = 0$ (cf. Exercise 1.3.E.11). Let

$$\hat{f}_j^o := \widehat{ev}_0(\hat{f}_j), \quad \text{where} \quad \widehat{ev}_0 : G(R'[[\hat{t}_j]]) \to G(R').$$

Then, by (6), $\hat{f}_j^o \in Z_{\mathrm{Exp}(2\pi i \tau_j)}(R')$, where $Z_{\mathrm{Exp}(2\pi i \tau_j)}$ is the centralizer scheme of $\mathrm{Exp}(2\pi i \tau_j)$ in G and $Z_{\mathrm{Exp}(2\pi i \tau_j)}(R')$ is its R'-rational points. Now, $Z_{\mathrm{Exp}(2\pi i \tau_j)}$ is the Levi subgroup L_j of G containing H with roots $\beta \in \Delta$ such that $\beta(\tau_j) = 0$, where $\Delta \subset \mathfrak{h}^*$ is the set of roots of G. (We have used the assumption here that $|\beta(\tau_j)| < 1$ for all $\beta \in \Delta$) Thus, we get

$$(\hat{t}_j)^{-\bar{\tau}_j} \cdot \hat{f}_j^o \cdot (\hat{t}_j)^{\bar{\tau}_j} = \hat{f}_j^o \quad \text{and} \quad \hat{f}_j^o \in L_j(R') \subset P_j(R'). \tag{9}$$

Think of $\hat{f}_j^o \in G(R') \subset G(R'[[\hat{t}_j]])$. Then $\hat{\zeta}_j := (\hat{f}_j^o)^{-1} \cdot \hat{f}_j \colon \mathrm{Spec}(R'[[\hat{t}_j]]) \to G$ has image inside the big cell $H \times \Pi_{\alpha\in\Delta} U_\alpha$ (fixing an ordering of Δ so that all the positive roots appear first and then all the negative roots or vice versa), where U_α is the one-parameter unipotent subgroup corresponding to the root α. (To prove this observe that $((\hat{f}_j^o)^{-1} \cdot \hat{f}_j)_{|\,\mathrm{Spec}\, R'}$ is the constant map going to $e \in G$.) Decompose the morphism $\hat{\zeta}_j = (\hat{\zeta}_j(0), \hat{\zeta}_j(\alpha))_{\alpha\in\Delta}$, where $\hat{\zeta}_j(0)$ (resp. $\hat{\zeta}_j(\alpha)$) is the component of $\hat{\zeta}_j$ in H (resp. U_α). Then, for any $\alpha \in \Delta$,

$$\zeta_j(\alpha) := (\hat{t}_j)^{-\bar{\tau}_j} \cdot \hat{\zeta}_j(\alpha) \cdot (\hat{t}_j)^{\bar{\tau}_j} \in (\hat{t}_j)^{-\alpha(\bar{\tau}_j)+1} R'[[\hat{t}_j]], \tag{10}$$

where we have identified $\epsilon_\alpha \colon \mathbb{G}_a \xrightarrow{\sim} U_\alpha$ satisfying $h\epsilon_\alpha(z)h^{-1} = \epsilon_\alpha(\alpha(h)z)$, for any $z \in \mathbb{G}_a$ and $h \in H$ (cf. (Jantzen, 2003, Part II, §1.2)). (Observe that the '+1' in the exponent of $\hat{t}_j$ in (10) appears due to the fact that $\hat{\zeta}_{j|\,\mathrm{Spec}\, R'}$ is the constant map with image e and hence $\hat{\zeta}_j(\alpha) \in \hat{t}_j R'[[\hat{t}_j]]$.)

By (7) and (9) (since $\gamma_j \cdot \hat{f}_j^o = \hat{f}_j^o$) we get

$$\gamma_j \cdot \zeta_j = \zeta_j, \quad \text{where} \quad \zeta_j := (\hat{t}_j)^{-\bar{\tau}_j} \cdot \hat{\zeta}_j \cdot (\hat{t}_j)^{\bar{\tau}_j}.$$

In particular,

$$\gamma_j \cdot \zeta_j(\alpha) = \zeta_j(\alpha) \ \ (\text{i.e., } \zeta_j(\alpha) \in R'((t_j))) \ \text{and} \quad \gamma_j \cdot \zeta_j(0) = \zeta_j(0) = \hat{\zeta}_j(0). \tag{11}$$

By the assumption $\theta(\tau_j) < 1$, we get (since $\alpha(\bar{\tau}_j) \in \mathbb{Z}$)

$$-(d_j - 2) \le -\alpha(\bar{\tau}_j) + 1 \le d_j, \quad \text{for any root} \quad \alpha \in \Delta. \tag{12}$$

Moreover, for any (negative) root α which is not a root of P_j,

$$2 \le -\alpha(\bar{\tau}_j) + 1. \tag{13}$$

By (1) of Definition 6.1.11, since $t_j = (\hat{t}_j)^{d_j}$, the exponents of $\hat{t}_j$ in $\zeta_j(\alpha)$ (for any $\alpha \in \Delta$) are multiples of d_j by (11). Hence, by (10)–(13), $\zeta_j \in \mathcal{P}_j(R')$ and hence so is $f_j \in \mathcal{P}_j(R')$ by (9). This proves (8). Thus, by (8), associated to $[\hat{E}_{R'}, \hat{\sigma}_{R'}]$, we get a well-defined element

$$\bar{\beta} = (\bar{\beta}_1, \dots, \bar{\beta}_s) \in \Pi_{j=1}^s \left(G(R'((t_j)))/\mathcal{P}_j(R')\right), \tag{14}$$

i.e., it does not depend upon the choice of the trivializations $(\hat{\mu}_j)_{1 \le j \le s}$ satisfying (1), where $\bar{\beta}_j := \beta_j \cdot \mathcal{P}_j(R')$ and β_j is defined by (4).

Consider the canonical injective map (cf. Exercise 1.3.E.11):

$$i_j(R') \colon G(R'((t_j)))/\mathcal{P}_j(R') \to \operatorname{Mor}\left(\operatorname{Spec} R', \bar{X}_G(P_j)\right).$$

Let $\bar{\beta}'$ be the image of $\bar{\beta}$ in $\operatorname{Mor}(\operatorname{Spec} R', \bar{X}_{\vec{P}})$.

Considering $\operatorname{Spec} R' \underset{\operatorname{Spec} R}{\times} \operatorname{Spec} R'$ as in the proof of Proposition 5.2.7, from the uniqueness of $\bar{\beta}'$ we get a well-defined element in $\bar{X}_{\vec{P}}(R) := \operatorname{Mor}(\operatorname{Spec} R, \bar{X}_{\vec{P}})$. This gives our sought-after map $\mathfrak{H} \colon \mathscr{F}^{A,\vec{\tau}}_{G,\hat{\Sigma}^*}(R) \to \bar{X}_{\vec{P}}(R)$.

We now prove that $\mathfrak{H}$ is a bijection. We first prove that $\mathfrak{H}$ is injective. Take $(\hat{E}_R, \hat{\sigma}_R)$, $(\hat{E}'_R, \hat{\sigma}'_R) \in \mathscr{F}^{A,\vec{\tau}}_{G,\hat{\Sigma}^*}(R)$ such that their images under $\mathfrak{H}$ coincide. Choose a surjective étale morphism $\varphi \colon \operatorname{Spec} R' \to \operatorname{Spec} R$ such that both the G-bundles $\hat{E}_{R'_{|(\hat{\mathbb{D}}_j)_{R'}}}$ and $\hat{E}'_{R'|(\hat{\mathbb{D}}_j)_{R'}}$ are trivial for each $1 \le j \le s$, where $\hat{E}_{R'}$, $(\hat{\mathbb{D}}_j)_{R'}$ are as at the beginning of this proof. Taking a section $\hat{\mu}_j$ (resp. $\hat{\mu}'_j$) of $\hat{E}_{R'_{|(\hat{\mathbb{D}}_j)_{R'}}}$ (resp. $\hat{E}'_{R'_{|(\hat{\mathbb{D}}_j)_{R'}}}$) satisfying (1), we get $\hat{\beta}_j$ (resp. $\hat{\beta}'_j$) defined by (2). From the injectivity of $i_j(R')$, we get that

$$\hat{\beta}'_j \cdot (\hat{t}_j)^{\bar{\tau}_j} \in \hat{\beta}_j \cdot (\hat{t}_j)^{\bar{\tau}_j} \mathcal{P}_j(R'), \quad \text{for all} \quad 1 \le j \le s,$$

i.e., there exists $f_j \in \mathcal{P}_j(R')$ such that

$$\hat{\beta}'_j = \hat{\beta}_j \cdot \hat{f}_j, \quad \text{where} \quad \hat{f}_j := (\hat{t}_j)^{\bar{\tau}_j} \cdot f_j \cdot (\hat{t}_j)^{-\bar{\tau}_j}. \tag{15}$$

It is easy to see from (3) that

$$\gamma_j \cdot \hat{f}_j = \operatorname{Exp}(2\pi i \tau_j)^{-1} \cdot \hat{f}_j \cdot \operatorname{Exp}(2\pi i \tau_j). \tag{16}$$

We next claim that $\hat{f}_j \in G(R'[[\hat{t}_j]])$. Similar to $\hat{f}^o_j$, consider $f^o_j := ev_0(f_j) \in P_j(R')$ under the evaluation map $G(R'[[t_j]]) \to G(R')$. Considering $\zeta_j := (f^o_j)^{-1} \cdot f_j$, it is easy to see (similar to the case of $\hat{\zeta}_j$ considered earlier) that

$$\hat{\zeta}_j := (\hat{t}_j)^{\bar{\tau}_j} \cdot \zeta_j \cdot (\hat{t}_j)^{-\bar{\tau}_j} \in G(R'[[\hat{t}_j]]). \tag{17}$$

Further,

$$\hat{f}_j = (\hat{t}_j)^{\bar{\tau}_j} \cdot f^o_j \cdot (\hat{t}_j)^{-\bar{\tau}_j} \cdot \hat{\zeta}_j. \tag{18}$$

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Since $(\hat{t}_j)^{\bar{\tau}_j}$ commutes with $L_j(R')$ (where L_j is the Levi subgroup of P_j containing H), it is easy to see that

$$(\hat{t}_j)^{\bar{\tau}_j} \cdot f_j^o \cdot (\hat{t}_j)^{-\bar{\tau}_j} \in G(R'[[\hat{t}_j]]). \tag{19}$$

Combining (17)–(19), we get that

$$\hat{f}_j \in G(R'[[\hat{t}_j]]), \quad \text{for all} \quad 1 \le j \le s. \tag{20}$$

For any $1 \le j \le s$, choose a set of coset representatives:

$$\left\{a_j^1 A_{\hat{p}_j}, \dots, a_j^{q_j} A_{\hat{p}_j}\right\} \quad \text{of} \quad A/A_{\hat{p}_j}.$$

For any $1 \le k \le q_j$, consider the formal disc $a_j^k \cdot \hat{\mathbb{D}}_j$ in $\hat{\Sigma}$ centered at $a_j^k \cdot \hat{p}_j$. Identify the disc $a_j^k \cdot \hat{\mathbb{D}}_j$ with $\hat{\mathbb{D}}_j$ under the action of a_j^k and transport the local parameter $\hat{t}_j$ of $\hat{\mathbb{D}}_j$ to $a_j^k \cdot \hat{\mathbb{D}}_j$ (still denoted by $\hat{t}_j$) under this identification.

Take the section $\hat{\mu}_j(k)$ (resp. $\hat{\mu}'_j(k)$) of $\hat{E}_{R'_{|(a_j^k \cdot \hat{\mathbb{D}}_j)_{R'}}}$ $\left(\text{resp. } \hat{E}'_{R'_{|(a_j^k \cdot \hat{\mathbb{D}}_j)_{R'}}}\right)$ defined by

$$\hat{\mu}_j(k)(a_j^k \cdot x) := a_j^k \cdot (\hat{\mu}_j(x)), \quad \text{for any} \quad x \in (\hat{\mathbb{D}}_j)_{R'},$$

and similarly for $\hat{\mu}'_j(k)$, where $\hat{\mu}_j$ and $\hat{\mu}'_j$ are any sections of $\hat{E}_{R'_{|(\hat{\mathbb{D}}_j)_{R'}}}$ and $\hat{E}'_{R'_{|(\hat{\mathbb{D}}_j)_{R'}}}$ respectively satisfying (1). Then, it is easy to see (since $\hat{\sigma}_{R'}$ is A-equivariant) that $\hat{\mu}_j(k)^* = \hat{\sigma}^*_{R'} \cdot \hat{\beta}_j$ as sections over $(a_j^k \hat{\mathbb{D}}_j^*)_{R'}$ for any $1 \le k \le q_j$ and similarly for $\hat{\mu}'_j(k)^*$. Thus, by the analogue of Proposition 5.2.7 with several punctures (for Σ^* replaced by $\hat{\Sigma}^*$) and using (15) and (20), we get that there exists a G-bundle isomorphism

$$\begin{array}{ccc} \hat{E}_{R'} & \xrightarrow[\sim]{\theta_{R'}} & \hat{E}'_{R'} \\ & {\scriptstyle \pi_{R'}} \searrow \quad \swarrow {\scriptstyle \pi'_{R'}} & \\ & \hat{\Sigma}_{R'} & \end{array}$$

taking $\hat{\sigma}_{R'}$ to $\hat{\sigma}'_{R'}$. Since $\hat{\sigma}_{R'}$ and $\hat{\sigma}'_{R'}$ are A-equivariant over $(\hat{\Sigma}^*)_{R'}$ (by assumption) and $\pi_{R'}^{-1}((\hat{\Sigma}^*)_{R'})$ is dense in $\hat{E}_{R'}$, we conclude that $\theta_{R'}$ is A-equivariant. From the uniqueness of $\theta_{R'}$ (since it is uniquely determined on $\pi_{R'}^{-1}((\hat{\Sigma}^*)_{R'})$), following the same argument as in the last part of the proof of Proposition 5.2.7, by considering the fiber product

$$\operatorname{Spec} R' \underset{\operatorname{Spec} R}{\times} \operatorname{Spec} R',$$

we conclude that $(\hat{E}_R, \hat{\sigma}_R)$ is isomorphic with $(\hat{E}'_R, \hat{\sigma}'_R)$ as A-equivariant G-bundles. This proves that $\mathfrak{H}$ is one-to-one.

We next prove that $\mathfrak{H}$ is surjective. Take a morphism $\delta\colon \operatorname{Spec} R \to \bar{X}_{\vec{P}}$. Then by Exercise 1.3.E.11 and the proof of Lemma B.2, there exists an fppf cover $\varphi\colon \operatorname{Spec} R' \to \operatorname{Spec} R$ such that the morphism $\delta_{R'} := \delta \circ \varphi\colon \operatorname{Spec} R' \to \bar{X}_{\vec{P}}$ lifts to a morphism $\hat{\delta}_{R'}\colon \operatorname{Spec} R' \to \Pi_{j=1}^{s} \bar{G}((t_j))$ giving rise to the elements $\beta_j \in G(R'((t_j)))$ by taking the projection of $\hat{\delta}_{R'}$ to the jth factor and using Lemma 1.3.2. Define

$$\hat{\beta}_j := \beta_j \cdot (\hat{t}_j)^{-\bar{\tau}_j} \in G(R'((\hat{t}_j))). \tag{21}$$

Consider the trivial G-bundle $E'_{R'}$ over $(\hat{\Sigma}^*)_{R'}$ with the trivial A-action, i.e.,

$$E'_{R'} = (\hat{\Sigma}^*)_{R'} \times G \to (\hat{\Sigma}^*)_{R'}$$

with

$$a \cdot (x,g) = (a \cdot x, g), \quad \text{for} \quad a \in A, x \in (\hat{\Sigma}^*)_{R'},\ g \in G.$$

Further, consider the $A_{\hat{p}_j}$-equivariant trivial G-bundle $E^j_{R'} = (\hat{\mathbb{D}}_j)_{R'} \times G \to (\hat{\mathbb{D}}_j)_{R'}$ with the action of the generator γ_j of $A_{\hat{p}_j}$ given by

$$\gamma_j \cdot (x,g) = (\gamma_j \cdot x, \operatorname{Exp}(2\pi i \tau_j) g), \quad \text{for} \quad x \in (\hat{\mathbb{D}}_j)_{R'} \quad \text{and} \quad g \in G.$$

There is an A-equivariant isomorphism of schemes

$$A \times^{A_{\hat{p}_j}} (\hat{\mathbb{D}}_j)_{R'} \to (\hat{F}_j)_{R'}, \quad [a,x] \mapsto a \cdot x,$$

where $\hat{F}_j := A \cdot \hat{\mathbb{D}}_j = \coprod_{k=1}^{q_j} (a_j^k \cdot \hat{\mathbb{D}}_j)$, $\{a_j^1, \dots, a_j^{q_j}\}$ is a set of coset representatives of $A/A_{\hat{p}_j}$ (as earlier) and $A_{\hat{p}_j}$ acts on $A \times (\hat{\mathbb{D}}_j)_{R'}$ diagonally:

$$\gamma \cdot (a,x) = (a \cdot \gamma^{-1}, \gamma \cdot x), \quad \text{for} \quad a \in A, \gamma \in A_{\hat{p}_j} \quad \text{and} \quad x \in (\hat{\mathbb{D}}_j)_{R'}.$$

Hence, an $A_{\hat{p}_j}$-equivariant G-bundle on $(\hat{\mathbb{D}}_j)_{R'}$ extends uniquely (unique up to a unique isomorphism) to an A-equivariant G-bundle on $(\hat{F}_j)_{R'}$ (cf. (Chriss and Ginzburg, 1997, §5.2.16)). In particular, the $A_{\hat{p}_j}$-equivariant G-bundle $E^j_{R'}$ extends uniquely to an A-equivariant G-bundle $\hat{E}^j_{R'}$ over $(\hat{F}_j)_{R'}$.

Identify the $A_{\hat{p}_j}$-equivariant bundles $E'_{R'}$ and $E^j_{R'}$ over the intersection $(\hat{\mathbb{D}}^*_j)_{R'} = (\hat{\mathbb{D}}_j)_{R'} \cap (\hat{\Sigma}^*)_{R'}$ via

$$\begin{aligned}\theta_j\colon E^j_{R'_{|(\hat{\mathbb{D}}^*_j)_{R'}}} &= (\hat{\mathbb{D}}^*_j)_{R'} \times G \to E'_{R'_{|(\hat{\mathbb{D}}^*_j)_{R'}}} \\ &= (\hat{\mathbb{D}}^*_j)_{R'} \times G,\ (x,g) \mapsto (x, \hat{\beta}_j(x) g), \text{for } x \in (\hat{\mathbb{D}}^*_j)_{R'} \text{ and } g \in G,\end{aligned}$$

where $\hat{\beta}_j \in G(R'((\hat{t}_j)))$ is defined by (21).

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Clearly, θ_j is an $A_{\hat{p}_j}$-equivariant isomorphism of G-bundles and hence gives rise to a unique A-equivariant isomorphism of G-bundles

$$\hat{\theta}_j : \hat{E}^j_{R'_{|(\hat{F}^*_j)_{R'}}} \to E'_{R'_{|(\hat{F}^*_j)_{R'}}}, \quad \text{where } (\hat{F}^*_j)_{R'} := (A \cdot \hat{\mathbb{D}}^*_j)_{R'}.$$

The A-equivariant G-bundles $E'_{R'}$ and $\{\hat{E}^j_{R'}\}_{1 \le j \le s}$ and the above isomorphisms allow us to get an A-equivariant G-bundle $\hat{E}_{R'}$ over $\hat{\Sigma}_{R'}$ via the 'descent' lemma (cf. the analogue of Lemma 5.2.3 for several punctures in $\hat{\Sigma}$). By the definition, $\hat{E}_{R'}$ is of local type $\{\tau_j\}_{1 \le j \le s}$ which comes equipped with an A-equivariant section $\hat{\sigma}_{R'}$ over $(\hat{\Sigma}^*)_{R'}$ given by

$$\hat{\sigma}_{R'}(x) = (x, 1) \quad \text{in} \quad E'_{R'}, \quad \text{for } x \in (\hat{\Sigma}^*)_{R'}.$$

Further, from the definition of $\mathfrak{H}$,

$$\mathfrak{H}\left(\left[\hat{E}_{R'}, \hat{\sigma}_{R'}\right]\right) = \delta_{R'}. \tag{22}$$

From the injectivity of $\mathfrak{H}$, $(\hat{E}_{R'}, \hat{\sigma}_{R'})$ satisfying (22) is unique (up to a unique isomorphism) and hence considering (as earlier) the fiber product

$$\text{Spec}\, R'' := \text{Spec}\, R' \underset{\text{Spec}\, R}{\times} \text{Spec}\, R'$$

with the two projection to $\text{Spec}\, R'$, we get (e.g., applying the analogue of Proposition 5.2.7 for $\hat{\Sigma}$ with several punctures) that $(\hat{E}_{R'}, \hat{\sigma}_{R'})$ descends to a G-bundle $(\hat{E}_R, \hat{\sigma}_R)$ over $\hat{\Sigma}_R$ with section over $(\hat{\Sigma}^*)_R$. Moreover, it is easy to see by considering $(\hat{E}_{R''}, \hat{\sigma}_{R''})$ that the A-equivariant structure on $\hat{E}_{R'}$ also descends to give an A-equivariant structure on $\hat{E}_R$ such that $\hat{\sigma}_R$ is A-equivariant . Thus, $(\hat{E}_R, \hat{\sigma}_R) \in \mathscr{F}^{A, \hat{\tau}}_{G, \hat{\Sigma}^*}(R)$, which maps to δ under $\mathfrak{H}$. This proves the surjectivity of $\mathfrak{H}$ and hence the theorem is fully established. □

Remark 6.1.13 In Balaji and Seshadri (2015), the restriction $\theta(\tau_j) < 1$ in Theorem 6.1.12 plays no role since by considering general parahoric subgroups of $G((t_j))$, their work is independent of the location of the weights τ_j in the fundamental alcove. However, the proof, in the case when $\theta(\tau_j)$ is allowed to be 1, is very similar to the proof given above.

It might be remarked that for any semisimple group G, the 'parahoric viewpoint' is a natural one since the 'unit group' of A-invariant local sections is a parahoric subgroup of a general kind.

Definition 6.1.14 Similar to the definition of the stack $\mathbf{Bun}_G(\Sigma)$ as in Definition 5.1.1, define the groupoid fibration of A-equivariant G-bundles $\mathbf{Bun}^{A, \vec{\tau}}_G(\hat{\Sigma})$ of local type $\vec{\tau}$ over the category $\mathfrak{S}$, whose objects are A-equivariant G-bundles E_S over $\hat{\Sigma} \times S$ (with the trivial action of A on S) such that $E_{S|_{\hat{\Sigma} \times t}}$ (for any $t \in S$) is of local type $\vec{\tau}$. By a morphism between

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two such bundles E_S (over $\hat{\Sigma} \times S$) and $E'_{S'}$ (over $\hat{\Sigma} \times S'$), we mean an $A \times G$-equivariant morphism $f : E_S \to E'_{S'}$ and a morphism $\bar{f} : S \to S'$ making the following diagram commutative:

$$\begin{array}{ccc} E_S & \xrightarrow{\ f\ } & E'_{S'} \\ \downarrow & & \downarrow \\ \hat{\Sigma} \times S & \xrightarrow[\mathrm{Id}_{\hat{\Sigma}} \times \bar{f}]{} & \hat{\Sigma} \times S' . \end{array}$$

The functor $\mathbf{Bun}_G^{A,\vec{\tau}}(\hat{\Sigma}) \to \mathfrak{S}$ takes $E_S \rightsquigarrow S$ and $f \rightsquigarrow \bar{f}$.

Let $\bar{X}_{\vec{P}} := \Pi_{j=1}^{s} \bar{X}_G(P_j)$ be as in Theorem 6.1.12 and let $\bar{\Gamma}$ be the ind-affine group variety as in Definition 5.2.9 and Lemma 5.2.10 with $\mathbb{C}$-points $\Gamma := \mathrm{Mor}(\Sigma^*, G)$. Then $\bar{\Gamma}$ acts on $\bar{X}_{\vec{P}}$ by the left multiplication on each factor via its Laurent series expansion in the coordinates t_j (cf. Corollary 5.2.11).

With the notation as above, we have the following.

Theorem 6.1.15 *Let $s \geq 1$ and $\vec{\tau}$ be as in Theorem 6.1.12. Then there exists an equivalence of categories over $\mathfrak{S}$ between the groupoid fibration $\mathbf{Bun}_G^{A,\vec{\tau}}(\hat{\Sigma})$ over $\mathfrak{S}$ of A-equivariant G-bundles of local type $\vec{\tau}$ and the quotient stack $\left[\bar{\Gamma}\backslash\bar{X}_{\vec{P}}\right]$ (cf. Example C.18(b)).*

In particular, $\mathbf{Bun}_G^{A,\vec{\tau}}(\hat{\Sigma})$ is isomorphic to the stack $\mathbf{Parbun}_G(\Sigma, \vec{P})$ of quasi-parabolic G-bundles over $(\Sigma, \vec{p})$ of type $\vec{P} := (P_1, \ldots, P_s)$ (cf. Definition 5.1.4) and hence it is a smooth (algebraic) stack.

(Even though $\mathbf{Parbun}_G(\Sigma, \vec{P})$ only depends upon $\vec{P}$, its isomorphism with $\mathbf{Bun}_G^{A,\vec{\tau}}(\hat{\Sigma})$ does depend upon the choice of $\vec{\tau}$.)

Proof The proof is parallel to the proof of Theorem 5.2.14. We first define a functor $\zeta : \mathbf{Bun}_G^{A,\vec{\tau}}(\hat{\Sigma}) \to [\bar{\Gamma}\backslash\bar{X}_{\vec{P}}]$. Let $\hat{E} = \hat{E}_S \to \hat{\Sigma} \times S \in \mathbf{Bun}_G^{A,\vec{\tau}}(\hat{\Sigma})$ (for any scheme $S \in \mathfrak{S}$). Define a $\mathbb{C}$-space functor $\hat{\mathscr{E}}_S$ as follows. For any $\mathbb{C}$-algebra R and an element in $S(R)$, i.e., a morphism $\varphi\colon \operatorname{Spec} R \to S$, define

$$\hat{\mathscr{E}}^o_S(\varphi) := \text{ set of } A\text{-equivariant sections of } \hat{E}_{\varphi_{|\hat{\Sigma}^*_R}},$$

where $\hat{E}_\varphi$ denotes the pull-back bundle $(\mathrm{Id}_{\hat{\Sigma}} \times \varphi)^*(\hat{E})$. Now, for any $\mathbb{C}$-algebra R, set

$$\hat{\mathscr{E}}^o_S(R) := \sqcup_{\varphi \in S(R)} \hat{\mathscr{E}}^o_S(\varphi).$$

Then, for an fppf R-algebra R', $\hat{\mathscr{E}}^o_S(R) \to \hat{\mathscr{E}}^o_S(R')$ is injective. Let $\hat{\mathscr{E}}_S$ be the sheafification of $\hat{\mathscr{E}}^o_S$ (cf. Lemma B.2). Then, we get a morphism

$$\hat{p} : \hat{\mathscr{E}}_S \to S.$$

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Further, for any $\varphi \in S(R)$, using Lemma 5.2.10, there is a canonical action $\hat{\mathscr{E}}^o_S(\varphi) \times \bar{\Gamma}(R) \to \hat{\mathscr{E}}^o_S(\varphi)$ giving rise to an action

$$\hat{\mathscr{E}}_S \times \bar{\Gamma} \to \hat{\mathscr{E}}_S. \tag{1}$$

(Here, we have used the identification $\mathrm{Mor}(\Sigma^*_R, G) \simeq \mathrm{Mor}_A(\hat{\Sigma}^*_R, G)$, where $\mathrm{Mor}_A(\hat{\Sigma}^*_R, G)$ denotes the set of A-invariant morphisms $\hat{\Sigma}^*_R \to G$.) Since A acts freely on $\hat{\Sigma}^*_S$ with quotient Σ^*_S, the A-equivariant G-bundle $\hat{E}_{S_{|\hat{\Sigma}^*_S}}$ is the pull-back of a unique G-bundle E_S over Σ^*_S with trivial A-action. Take an affine étale cover $S' \to S$ such that $E_{S'}$ over $\Sigma^*_{S'}$ is trivial (cf. Theorem 5.2.5), where $E_{S'}$ is the pull-back of E_S to $\Sigma^*_{S'}$. Then it is easy to see that $\hat{E}_{S'}$ (which is the pull-back of $\hat{E}_S$ to $\hat{\Sigma} \times S'$) satisfies $\hat{E}_{S'_{|\hat{\Sigma}^*_{S'}}} \simeq q^*(E_{S'})$ as A-equivariant G-bundles, where $q\colon \hat{\Sigma}^*_{S'} \to \Sigma^*_{S'}$ is the standard quotient map induced by $\pi\colon \hat{\Sigma} \to \Sigma$. In particular, $\hat{E}_{S'_{|\hat{\Sigma}^*_{S'}}}$ admits an A-equivariant section. Then $\hat{\mathscr{E}}_{S'} = S' \times_S \hat{\mathscr{E}}_S$ is isomorphic with $S' \times \bar{\Gamma}$ such that the induced action of $\bar{\Gamma}$ on $\hat{\mathscr{E}}_{S'}$ (as in (1)) corresponds to the right multiplication on the $\bar{\Gamma}$-factor. Thus, $\hat{p}\colon \hat{\mathscr{E}}_S \to S$ is a $\bar{\Gamma}$-torsor with the right action of $\bar{\Gamma}$ (cf. Definition C.16).

We further define a $\bar{\Gamma}$-equivariant morphism $\hat{\beta}\colon \hat{\mathscr{E}}_S \to \bar{X}_{\vec{P}}$ as follows. For any $\varphi \in S(R)$, map any element of $\hat{\mathscr{E}}^o_S(\varphi)$ to the pair $(\hat{E}_\varphi, \hat{\sigma}_\varphi)$, where $\hat{\sigma}_\varphi$ is the corresponding section of $\hat{E}_{\varphi_{|\hat{\Sigma}^*_R}}$. The sections $\hat{\sigma}_\varphi$ (for $\varphi \in S(R)$) give rise to the section $\hat{\sigma}_o$ of $\bar{\hat{p}}^*(\hat{E}_S)$ over $\hat{\Sigma}^* \times \hat{\mathscr{E}}_S$, where $\bar{\hat{p}}\colon \hat{\Sigma} \times \hat{\mathscr{E}}_S \to \hat{\Sigma} \times S$ is the morphism $\mathrm{Id}_{\hat{\Sigma}} \times \hat{p}$. We call $\hat{\sigma}_o$ the *A-equivariant tautological section*. Now, the equivalence class $[\hat{E}_\varphi, \hat{\sigma}_\varphi] \in \mathscr{F}^{A,\vec{\tau}}_{G,\hat{\Sigma}^*}(R)$ (cf. Definition 6.1.11) corresponds to an element in $\bar{X}_{\vec{P}}(R)$ (cf. Theorem 6.1.12). This gives us the desired morphism

$$\hat{\beta}\colon \hat{\mathscr{E}}_S \to \bar{X}_{\vec{P}}.$$

It is easy to see that $\hat{\beta}$ is $\bar{\Gamma}$-equivariant, where we switch the right action of $\bar{\Gamma}$ on $\hat{\mathscr{E}}_S$ to the left action by the standard procedure:

$$\gamma \cdot x = x \cdot \gamma^{-1}, \text{ for } \gamma \in \bar{\Gamma} \text{ and } x \in \hat{\mathscr{E}}_S.$$

The functor ζ takes $\hat{E}_S \in \mathbf{Bun}^{A,\vec{\tau}}_G(\hat{\Sigma})$ to the pair $(\hat{p}, \hat{\beta})$.

Conversely, we define a functor $\eta\colon [\bar{\Gamma}\backslash\bar{X}_{\vec{P}}] \to \mathbf{Bun}^{A,\vec{\tau}}_G(\hat{\Sigma})$ as follows.

Take a $\bar{\Gamma}$-torsor (with the left action of $\bar{\Gamma}$) $\hat{p}\colon \hat{\mathscr{E}}_S \to S$ (over a scheme $S \in \mathfrak{S}$) and a $\bar{\Gamma}$-equivariant morphism $\hat{\beta}\colon \hat{\mathscr{E}}_S \to \bar{X}_{\vec{P}}$. The identity morphism $\mathrm{Id}\colon \bar{X}_{\vec{P}} \to \bar{X}_{\vec{P}}$ gives rise (via Theorem 6.1.12) to an A-equivariant G-bundle $\mathfrak{U}(\vec{\tau})$ over $\hat{\Sigma} \times \bar{X}_{\vec{P}}$ (with A acting trivially on $\bar{X}_{\vec{P}}$) of local type $\vec{\tau}$ restricted to any $\hat{\Sigma} \times x$ (for $x \in \bar{X}_{\vec{P}}$) together with an A-equivariant section $\hat{\sigma}_{\bar{X}_{\vec{P}}}$ of

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$\mathfrak{U}(\vec{\tau})$ over $\hat{\Sigma}^* \times \bar{X}_{\vec{P}}$. Moreover, the pair $(\mathfrak{U}(\vec{\tau}), \hat{\sigma}_{\bar{X}_{\vec{P}}})$ is unique up to a unique isomorphism. (Even though Theorem 6.1.12 only guarantees a G-bundle over $\hat{\Sigma} \times \operatorname{Spec} R$, for $R \in \mathbf{Alg}$, but the uniqueness insures its extension to $\hat{\Sigma} \times \bar{X}_{\vec{P}}$.) We fix one such pair in its isomorphism class. By the same proof as that of Lemma 5.2.12, the G-bundle $\mathfrak{U}(\vec{\tau})$ acquires the structure of a $\bar{\Gamma}$-equivariant G-bundle over $\hat{\Sigma} \times \bar{X}_{\vec{P}}$ commuting with the A-equivariant structure. The $\bar{\Gamma}$-equivariant morphism $\hat{\beta} \colon \hat{\mathscr{E}}_S \to \bar{X}_{\vec{P}}$ gives rise to a $\bar{\Gamma}$-equivariant G-bundle $\bar{\hat{\beta}}^*(\mathfrak{U}(\vec{\tau}))$ over $\hat{\Sigma} \times \hat{\mathscr{E}}_S$ via pull-back through $\bar{\hat{\beta}} := \operatorname{Id}_{\hat{\Sigma}} \times \hat{\beta} \colon \hat{\Sigma} \times \hat{\mathscr{E}}_S \to \hat{\Sigma} \times \bar{X}_{\vec{P}}$. Since $\hat{p} \colon \hat{\mathscr{E}}_S \to S$ is a $\bar{\Gamma}$-torsor, the $\bar{\Gamma}$-equivariant bundle $\bar{\hat{\beta}}^*(\mathfrak{U}(\vec{\tau}))$ descends to give a G-bundle (denoted) $\hat{E}(\hat{p}, \hat{\beta}) \to \hat{\Sigma} \times S$ (cf. Lemma C.17). Since $\mathfrak{U}(\vec{\tau})$ is of local type $\vec{\tau}$, so is $\hat{E}(\hat{p}, \hat{\beta})$. Further, since $\mathfrak{U}(\vec{\tau})$ has an A-equivariant structure commuting with the $\bar{\Gamma}$-equivariant structure and $\bar{\hat{\beta}}$ is an $A \times \bar{\Gamma}$-equivariant morphism with A acting only on $\hat{\Sigma}$ (acting trivially on $\hat{\mathscr{E}}_S$ and $\bar{X}_{\vec{P}}$ and $\bar{\Gamma}$ acting trivially on $\hat{\Sigma}$), $\hat{E}(\hat{p}, \hat{\beta})$ is an A-equivariant G-bundle over $\hat{\Sigma} \times S$. This is our map $\eta \colon [\bar{\Gamma} \backslash \bar{X}_{\vec{P}}] \to \mathbf{Bun}_G^{A,\vec{\tau}}(\hat{\Sigma})$, taking $(\hat{p}, \hat{\beta}) \mapsto \hat{E}(\hat{p}, \hat{\beta})$.

The proof that $\eta \circ \zeta \simeq \operatorname{Id}_{\mathbf{Bun}_G^{A,\vec{\tau}}(\hat{\Sigma})}$ and $\zeta \circ \eta \simeq \operatorname{Id}_{[\bar{\Gamma} \backslash \bar{X}_{\vec{P}}]}$ is similar to the one given in the proof of Theorem 5.2.14 and hence is left to the reader. This proves the first part of the theorem.

The 'In particular' part of the theorem follows from the first part and Exercise 5.2.E.3 together with Theorem 5.1.5. □

Following Definition 6.1.14, let $\operatorname{Bun}_G^{A,\vec{\tau}}(\hat{\Sigma})$ be the set of isomorphism classes of A-equivariant G-bundles over $\hat{\Sigma}$ of local type $\vec{\tau}$. Similarly, $\operatorname{Parbun}_G(\Sigma, \vec{P})$ is as defined in Corollary 5.2.17.

Definition 6.1.16 Let G be a connected reductive group. An A-equivariant G-bundle $\hat{E}$ over $\hat{\Sigma}$ is called *A-semistable* (resp. *A-stable*) if condition (2) of Definition 6.1.4(b) is satisfied for any standard maximal parabolic subgroup Q_k of G and any A-equivariant section μ of $\hat{E}/Q_k \to \hat{\Sigma}$.

Similarly, following Definition 6.1.4(c), $\hat{E}$ is called *A-polystable* if it has an A-equivariant reduction $\hat{E}_L$ to a Levi subgroup L such that the L-bundle $\hat{E}_L$ is A-stable and for any character χ of L which is trivial restricted to the center of G, we have

$$\deg\left(\hat{E}_L \times^L \mathbb{C}_\chi\right) = 0.$$

Similar to the definition of semistable and stable vector bundles as in Definition 6.1.4(a), an A-equivariant vector bundle $\mathscr{V}$ over $\hat{\Sigma}$ is called *A-semistable* (resp. *A-stable*) if the inequality (1) of Definition 6.1.4(a) is satisfied for any A-stable subbundle $(0) \subsetneq \mathscr{W} \subsetneq \mathscr{V}$.

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Similar to Exercise 6.1.E.5, an A-equivariant vector bundle $\mathscr{V}$ is A-semistable (resp. A-stable) if and only if the corresponding frame bundle $F(\mathscr{V})$ is so.

An A-equivariant vector bundle $\mathscr{V}$ over $\hat{\Sigma}$ of rank r is called *A-polystable* if the associated frame bundle $F(\mathscr{V})$ is A-polystable as a GL_r-bundle.

By Exercise 6.1.E.16, an A-equivariant vector bundle $\mathscr{V}$ over $\hat{\Sigma}$ is A-polystable if and only if we can write

$$\mathscr{V} = \bigoplus_i \mathscr{V}_i,$$

where each $\mathscr{V}_i$ is an A-stable vector bundle all of which have the same slope.

Similar to Corollaries 5.2.15 and 5.2.17, we get the following result from Theorem 6.1.15.

Theorem 6.1.17 *With the notation and assumptions as in Theorem 6.1.15, we have a natural set-theoretic bijection*

$$\mathrm{Bun}_G^{A,\vec{\tau}}(\hat{\Sigma}) \simeq \mathrm{Parbun}_G(\Sigma, \vec{P}). \tag{1}$$

In fact, there is a similar set-theoretic natural bijection as (1) *with* $\hat{\Sigma}$ *replaced by* $\hat{\Sigma} \times S$ *(for any ind-scheme S as parameter space).*

Under the bijection (1), *A-semistable (resp. A-stable) G-bundles over $\hat{\Sigma}$ correspond to the parabolic semistable (resp. stable) bundles over Σ with respect to the markings $\vec{\tau}$ (cf. Definition 6.1.4(d)). In fact, a more precise result is true (cf. identity* (17) *in the proof).*

Proof The bijection (1) (resp. its extension to $\hat{\Sigma} \times S$) follows immediately from Theorem 6.1.15 by specializing the equivalence of the groupoid fibrations $\mathbf{Bun}_G^{A,\vec{\tau}}(\hat{\Sigma})$ and $\mathbf{Parbun}_G(\Sigma, \vec{P})$ over a point (resp. over S).

We now prove the assertion about the correspondence of semistable and stable bundles. Take any standard maximal parabolic subgroup Q_k ($1 \leq k \leq \ell$) of G. Let $\hat{E} \to \hat{\Sigma}$ be an A-equivariant G-bundle over $\hat{\Sigma}$ of local type $\vec{\tau}$ and let $E \to \Sigma$ be the corresponding quasi-parabolic G-bundle over Σ of type $\vec{P}$ given by the correspondence (1). Then, by definition (given in the proofs of Theorems 6.1.12 and 6.1.15 following the notation therein which we follow freely) as A-equivariant G-bundles (with the trivial A-action on E):

$$\hat{E}_{|_{\hat{\Sigma}^*}} := \pi^*(E_{|_{\Sigma^*}}). \tag{2}$$

From this we see that the pull-back of sections provides a bijective correspondence between the sections of $(E_{|_{\Sigma^*}})/Q_k$ and A-equivariant sections of $(\hat{E}_{|_{\hat{\Sigma}^*}})/Q_k$. Moreover, since $Y_k := G/Q_k$ is a projective variety and Σ is a curve, this correspondence extends to give a

bijective correspondence ψ between the sections of E/Q_k and A-equivariant sections of $\hat{E}/Q_k$. (3)

(In fact, this bijective correspondence holds for any parabolic subgroup Q of G.) Take a section θ of E/Q_k and let $\hat{\theta} = \psi(\theta)$ be the corresponding section of $\hat{E}/Q_k$. For any $1 \leq j \leq s$, we assert that there exists a section $\hat{\mu}_j$ of $\hat{E}_{|_{\hat{\mathbb{D}}_j}}$ satisfying the following two conditions (writing $\hat{\tau}_j := \operatorname{Exp}(2\pi i \tau_j)$):

$$\gamma_j \cdot \hat{\mu}_j = \hat{\mu}_j \cdot \hat{\tau}_j, \text{ cf. (1) of the proof of Theorem 6.1.12,} \tag{4}$$

and

$$\hat{\theta}_{|_{\hat{\mathbb{D}}_j}} = \hat{\mu}_j \cdot \bar{w}_j Q_k, \text{ for some } \bar{w}_j \in N(H), \tag{5}$$

where $N(H)$ is the normalizer of H in G. To prove the existence of such a $\hat{\mu}_j$, take any $\hat{\mu}'_j$ satisfying

$$\gamma_j \cdot \hat{\mu}'_j = \hat{\mu}'_j \cdot \hat{\tau}_j, \tag{6}$$

which is guaranteed by Theorem 6.1.9. Write

$$\hat{\theta}_{|_{\hat{\mathbb{D}}_j}} = \hat{\mu}'_j \cdot \bar{\delta}, \quad \text{for a morphism} \quad \bar{\delta} \colon \hat{\mathbb{D}}_j \to Y_k. \tag{7}$$

Since $\hat{\theta}$ is A-equivariant, we get from (6) and (7) that

$$\gamma_j \bar{\delta} = \hat{\tau}_j^{-1}\ \bar{\delta}; \quad \text{in particular,} \quad \bar{\delta}(\hat{p}_j) = \hat{\tau}_j\, \bar{\delta}(\hat{p}_j), \tag{8}$$

where γ_j acts on $\hat{\Sigma}$ and not on Y_k. But it is easy to see that

$$(Y_k)^{\hat{\tau}_j} := \left\{ g Q_k : \hat{\tau}_j g Q_k = g Q_k \right\} \quad \text{is given by} \quad (Y_k)^{\hat{\tau}_j} = \bigcup_{w \in W} L_j w Q_k,$$

where L_j is the Levi subgroup of P_j containing H. Thus, $\bar{\delta}(\hat{p}_j) = l_j \bar{w}_j Q_k$ for some $\bar{w}_j \in N(H)$ and $l_j \in L_j$. Thus, $U^-_{Q_k} \cdot Q_k$ being an open subset of Y_k, $\bar{\delta}$ lands as a map $\bar{\delta} \colon \hat{\mathbb{D}}_j \to l_j \bar{w}_j U^-_{Q_k} Q_k$, where $U^-_{Q_k}$ is the opposite unipotent radical of Q_k. Define $\delta \colon \hat{\mathbb{D}}_j \to l_j \bar{w}_j U^-_{Q_k} \subset G$ obtained from the isomorphism $U^-_{Q_k} \cdot Q_k/Q_k \subset Y_k \simeq U^-_{Q_k} \subset G$. Let $\delta_o \colon \hat{\mathbb{D}}_j \to U^-_{Q_k}$ be the map $\delta_o := (l_j \bar{w}_j)^{-1}\delta$. Then, by (8) (since $(\operatorname{Ad} l_j^{-1})\hat{\tau}_j^{-1} = \hat{\tau}_j^{-1}$),

$$\gamma_j \cdot \delta_o = \left(\bar{w}_j^{-1} \hat{\tau}_j^{-1} \bar{w}_j \right) \delta_o \left(\bar{w}_j^{-1} \hat{\tau}_j \bar{w}_j \right). \tag{9}$$

Now, consider a new section of $\hat{E}_{|\hat{\mathbb{D}}_j}$:

$$\hat{\mu}_j := \hat{\mu}'_j \cdot (l_j \bar{w}_j \delta_o \bar{w}_j^{-1}).$$

Then, by (6), (7) and (9),

$$\gamma_j \cdot \hat{\mu}_j = \hat{\mu}_j \cdot \hat{\tau}_j \quad \text{and} \quad \hat{\theta}_{|\hat{\mathbb{D}}_j} = \hat{\mu}_j \cdot \bar{w}_j Q_k.$$

This proves (4) and (5).

We next show that the relative position of the section θ of the bundle E/Q_k with respect to the quasi-parabolic structure on E at p_j is given by $W_{P_j} w_j W_{Q_k}$, where $w_j \in W$ is the image of $\bar{w}_j$ in W (cf. Definition 6.1.4 for the definition of the relative position). By (5),

$$\begin{aligned}
\hat{\theta}_{|\hat{\mathbb{D}}^*_j} &= \hat{\mu}^*_j \bar{w}_j Q_k \\
&= \hat{\sigma}_{|\hat{\mathbb{D}}^*_j} \hat{\beta}_j \bar{w}_j Q_k, \ \text{ by (2) of the proof of Theorem 6.1.12, where} \\
&\qquad \hat{\sigma} \ \text{ is an } A\text{-equivariant section of } \hat{E}_{|\hat{\Sigma}^*} \\
&= \hat{\sigma}_{|\hat{\mathbb{D}}^*_j} \beta_j (\hat{t}_j)^{-\bar{\tau}_j} \bar{w}_j Q_k, \quad \text{see (4) and (5) of the proof of Theorem 6.1.12,} \\
&\qquad \text{where } \beta_j \text{ is } \gamma_j\text{-invariant} \\
&= \hat{\sigma}_{|\hat{\mathbb{D}}^*_j} \beta_j \bar{w}_j Q_k. \qquad (10)
\end{aligned}$$

But $\hat{\sigma}_{|\hat{\mathbb{D}}^*_j} \cdot \beta_j$ descends (since $\hat{\sigma}$ is the pull-back of a section σ of $E_{|\Sigma^*}$) and extends to give a section μ_j of $E_{|\mathbb{D}_j}$ (cf. Proposition 5.2.4 and (14) of the proof of Theorem 6.1.12) and hence

$$\theta_{|\mathbb{D}_j} = \mu_j \bar{w}_j Q_k. \qquad (11)$$

Now, from the definition of the relative position as in Definition 6.1.4(d), since $\mu_j(p_j)P_j$ gives the quasi-parabolic structure on E at p_j (cf. Exercise 5.2.E.3, especially the equation (*) therein), we get from (11) that $W_{P_j} w_j W_{Q_k}$ is the relative position of θ at p_j.

We finally compute the degree of the line bundle $\mathscr{S} := \hat{\theta}^*(\hat{\mathscr{L}}) \otimes (\pi^* \theta^* \mathscr{L})^*$ over $\hat{\Sigma}$, where

$$\hat{\mathscr{L}} := \hat{E} \times^{Q_k} \mathbb{C}_{\omega_k} \quad \text{and} \quad \mathscr{L} := E \times^{Q_k} \mathbb{C}_{\omega_k}.$$

By Exercise 6.1.E.14, the section $\theta_{|\Sigma^*}: \Sigma^* \to E/Q_k$ lifts to a *holomorphic* section $\Theta: \Sigma^* \to E$, i.e., $\Theta \bmod Q_k = \theta_{|\Sigma^*}$. Moreover, let $\hat{\Theta}: \hat{\Sigma}^* \to \hat{E}$ be the A-equivariant *holomorphic* section given as $\pi^*\Theta$, which lifts $\hat{\theta}_{|\hat{\Sigma}^*}$. Then $\hat{\Theta}$ provides a trivialization $\hat{\Theta}_{\mathscr{S}}$ of $\mathscr{S}_{|\hat{\Sigma}^*}$.

For any $1 \leq j \leq s$, take a section $\hat{\mu}_j$ of $\hat{E}_{|\hat{\mathbb{D}}_j}$ satisfying (4) and (5) and write

$$\hat{\mu}_{j|\hat{\mathbb{D}}_j^*} = \hat{\Theta}_{|\hat{\mathbb{D}}_j^*} \cdot \hat{\beta}_j, \quad \text{for} \quad \hat{\beta}_j \in G((\hat{t}_j)),$$

$$\text{cf. (2) of the proof of Theorem 6.1.12 taking } \hat{\sigma} = \hat{\Theta}. \tag{12}$$

Let μ_j be a section of $E_{|\mathbb{D}_j}$ given above satisfying (11), again replacing $\hat{\sigma} = \hat{\Theta}$. Then, by definition,

$$(\pi^* \mu_j)_{|\hat{\mathbb{D}}_j^*} = \hat{\Theta}_{|\hat{\mathbb{D}}_j^*} \cdot \hat{\beta}_j \cdot (\hat{t}_j)^{\bar{\tau}_j}. \tag{13}$$

Let $\mathbb{1}_{\omega_k}$ be a nonzero vector of $\mathbb{C}_{\omega_k}$ and let $\hat{s}_o$ be the section $\hat{E} \to \hat{E} \times \mathbb{C}_{\omega_k}$, $x \mapsto (x, \mathbb{1}_{\omega_k})$, of the trivial line bundle $\hat{E} \times \mathbb{C}_{\omega_k} \to \hat{E}$ (which is viewed as the pull-back of the line bundle $\hat{\mathscr{L}}_{Q_k}(-\omega_k)$ over $\hat{E}/Q_k$ as in Definition 6.1.3(c) via the projection $\hat{q}\colon \hat{E} \to \hat{E}/Q_k$). Similarly, we define the section $s_o\colon E \to E \times \mathbb{C}_{\omega_k}$. From the identities (5) and (12) (by considering the sections $\hat{s}_o$ and s_o) we get that the line bundle $(\hat{\theta}^* \hat{\mathscr{L}})_{|\hat{\mathbb{D}}_j}$ has a section

$$\begin{aligned}
\hat{\delta}_j &:= \left[\hat{\mu}_j \cdot \bar{w}_j, \mathbb{1}_{\omega_k}\right], \\
&= \left[\hat{\Theta}_{|\hat{\mathbb{D}}_j^*} \cdot \hat{\beta}_j \cdot \bar{w}_j, \mathbb{1}_{\omega_k}\right] \quad \text{over} \quad \hat{\mathbb{D}}_j^* \\
&= \left[\hat{\Theta}_{|\hat{\mathbb{D}}_j^*}, \hat{\beta}_j \bar{w}_j \cdot \mathbb{1}_{\omega_k}\right],
\end{aligned} \tag{14}$$

since $\hat{\beta}_j \bar{w}_j$ has image in Q_k by the identity (10) taking $\hat{\sigma} = \hat{\Theta}$, where $[\hat{x}, \mathbb{1}_{\omega_k}] \in \hat{\mathscr{L}}$ denotes $(\hat{x}, \mathbb{1}_{\omega_k}) \bmod Q_k$, for $\hat{x} \in \hat{E}$. Similarly, using the identities (11) and (13), the line bundle $(\pi^* \theta^* \mathscr{L})_{|\hat{\mathbb{D}}_j}$ has a section

$$\begin{aligned}
\delta_j &:= \left[(\pi^* \mu_j) \cdot \bar{w}_j, \mathbb{1}_{\omega_k}\right] \\
&= \left[\hat{\Theta}_{|\hat{\mathbb{D}}_j^*} \cdot \hat{\beta}_j \cdot \bar{w}_j, \bar{w}_j^{-1} (\hat{t}_j)^{\bar{\tau}_j} \bar{w}_j \cdot \mathbb{1}_{\omega_k}\right] \quad \text{over} \quad \hat{\mathbb{D}}_j^* \\
&= \left[\hat{\Theta}_{|\hat{\mathbb{D}}_j^*} \cdot \hat{\beta}_j \cdot \bar{w}_j, (\hat{t}_j)^{\omega_k(\bar{w}_j^{-1} \bar{\tau}_j)} \cdot \mathbb{1}_{\omega_k}\right] \\
&= \left[\hat{\Theta}_{|\hat{\mathbb{D}}_j^*}, (\hat{t}_j)^{\omega_k(\bar{w}_j^{-1} \bar{\tau}_j)} \hat{\beta}_j \bar{w}_j \cdot \mathbb{1}_{\omega_k}\right].
\end{aligned} \tag{15}$$

From (14) and (15), we get that the line bundle $\mathscr{S}$ over $\hat{\Sigma}$ has section $\hat{\Theta}_{\mathscr{S}}$ over $\hat{\Sigma}^*$ and sections $(\hat{\mu}_j)_{\mathscr{S}}$ over $\hat{\mathbb{D}}_j$ satisfying the following equation over $\hat{\mathbb{D}}_j^*$:

$$(\hat{\mu}_j)_{\mathscr{S}} = \hat{\Theta}_{\mathscr{S}} \cdot (\hat{t}_j)^{-\omega_k(w_j^{-1} \bar{\tau}_j)}. \tag{16}$$

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Since $\hat{\Sigma} \to \Sigma$ is of degree $N = \# A$ and, for each $1 \leq j \leq s$, there are N/d_j isomorphic copies of $\hat{\mathbb{D}}_j$ over $\mathbb{D}_j$, we get from (16) and Exercise 6.1.E.3,

$$\begin{aligned}\deg(\hat{\theta}^* \hat{\mathscr{L}}) - N \deg(\theta^* \mathscr{L}) &= \deg \mathscr{S} \\ &= \sum_{j=1}^{s} \frac{N}{d_j} \omega_k (w_j^{-1} \bar{\tau}_j).\end{aligned}$$

Thus,

$$\deg(\hat{\theta}^* \hat{\mathscr{L}}) = N \left(\deg(\theta^* \mathscr{L}) + \sum_{j=1}^{s} \omega_k (w_j^{-1} \tau_j) \right), \quad \text{since} \quad \tau_j := \frac{\bar{\tau}_j}{d_j},$$

i.e.,

$$\deg \left(\hat{\theta}^* \hat{\mathscr{L}}_{Q_k}(-\omega_k) \right) = N \operatorname{Pardeg} \left(\theta^* \mathscr{L}_{Q_k}(-\omega_k) \right), \tag{17}$$

where Pardeg denotes the parabolic degree of the G-bundle E with respect to the section θ and the line bundle $\mathscr{L}_{Q_k}(-\omega_k)$ for the parabolic markings $\bar{\tau} = (\tau_1, \ldots, \tau_s)$ (cf. Definition 6.1.4(d)). This proves the theorem. □

6.1.E Exercises

In the following, Σ is a smooth irreducible projective curve.

(1) Let G be a connected affine algebraic group and let $n \geq 1$. Show that the affine algebraic group $G\left(\mathbb{C}[[t]]/\langle t^n \rangle\right)^+$ (cf. Exercise 1.3.E.10) is a unipotent (in particular, connected) group with Lie algebra $\mathfrak{g} \otimes (t\mathbb{C}[[t]]/\langle t^n \rangle)$, where $G\left(\mathbb{C}[[t]]/\langle t^n \rangle\right)^+$ is the kernel of the homomorphism $G\left(\mathbb{C}[[t]]/\langle t^n \rangle\right) \to G$ induced by the $\mathbb{C}$-algebra homomorphism $\mathbb{C}[[t]]/\langle t^n \rangle \to \mathbb{C}$, $t \mapsto 0$.

Use the above to show that for any $R \in$ **Alg** and $n \geq 0$, the canonical homomorphism

$$\pi_n^R : G\left(R[[t]]/\langle t^{n+1} \rangle\right)^+ \to G\left(R[[t]]/\langle t^n \rangle\right)^+$$

is surjective with kernel isomorphic (as a group) to the $\mathbb{C}$-vector space $R \otimes \left(\mathfrak{g} \otimes_{\mathbb{C}} \frac{t^n\mathbb{C}[[t]]}{t^{n+1}\mathbb{C}[[t]]}\right)$, where $G\left(R[[t]]/\langle t^n \rangle\right)^+$ is the kernel of the homomorphism $G\left(R[[t]]/\langle t^n \rangle\right) \to G(R)$ induced by $t \mapsto 0$.

Hint: By Exercises 1.3.E.10 and 1.3.E.6, $R \rightsquigarrow G\left(R[[t]]/\langle t^n \rangle\right)^+$ is a representable group functor, represented by an affine algebraic group with $\mathbb{C}$-points $G\left(\mathbb{C}[[t]]/\langle t^n \rangle\right)^+$.

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(2) Let A be a group and let $\{\pi_n : G_{n+1} \to G_n\}_{n\geq 1}$ be an inverse system of A-groups. Assume that each $\pi_n^A : G_{n+1}^A \to G_n^A$ is surjective, where G_n^A denotes the subgroup of A-equivariants in G_n. Let G be the inverse limit of G_n. Then G is canonically an A-group.

Prove that if $H^1(A, G_n) = 0$, for all n, then so is $H^1(A, G) = 0$.

(3) Let $\mathscr{L}$ be a line bundle over Σ with nowhere vanishing sections σ over $\Sigma \backslash p$ and μ over $\mathbb{D}_p$ (for a fixed $p \in \Sigma$), where $\mathbb{D}_p$ is the formal disc centered at p in Σ with a local parameter z. Write

$$\mu_{|_{\mathbb{D}_p^*}} = \sigma_{|_{\mathbb{D}_p^*}} \cdot \beta(z), \quad \text{for} \quad \beta(z) \in \mathbb{C}((z)),$$

where $\mathbb{D}_p^* := \operatorname{Spec} \mathbb{C}((z))$ is the punctured formal disc at p. Then show that $\deg \mathscr{L} = d$, where d is the unique integer such that $z^d \cdot \beta(z) \in \mathbb{C}[[z]]$ with nonzero constant term.

(4) For any connected reductive algebraic group G, show that the two alternative definitions of semistability/stability (cf. Definition 6.1.4(b)) are equivalent.

Moreover, show that if a G-bundle $E \to \Sigma$ is semistable (resp. stable), then for any standard parabolic subgroup P of G and any section μ of $E/P \to \Sigma$,

$$\deg \mu^* \left(\mathscr{L}_P(-\lambda)\right) \leq 0 \left(\text{resp. } \deg \mu^* \left(\mathscr{L}_P(-\lambda)\right) < 0\right),$$

for any nontrivial character λ of P which is trivial restricted to the connected center of G and is dominant (i.e., $\lambda(\alpha_i^\vee) \geq 0$ for all the simple coroots $\alpha_i^\vee$).

(5) Show that a vector bundle $\mathscr{V}$ over Σ is semistable (resp. stable) if and only if the associated frame bundle $F(\mathscr{V})$ (which is a principal GL_n-bundle over Σ, where $n := \operatorname{rank}(\mathscr{V})$) is semistable (resp. stable) in the sense of Definition 6.1.4(b).

Hint: A rank-r subbundle $\mathscr{W}$ of $\mathscr{V}$ is given by a P_r-subbundle $F(\mathscr{V})_{P_r} \subset F(\mathscr{V})$ (induced by a section μ of $F(\mathscr{V})/P_r \to \Sigma$) by taking the associated vector bundle $F(\mathscr{V})_{P_r} \times^{P_r} \mathbb{C}^r$ and conversely, where P_r is the maximal parabolic subgroup of GL_n stabilizing $\mathbb{C}^r \subset \mathbb{C}^n$ under the standard representation. Now,

$$\deg(\mathscr{W}) = \deg\left(\mu^* \mathscr{L}_{P_r}(-\omega_r)\right)$$

and $n\omega_r - r\omega_n$ is a character of P_r which vanishes on its center, where

$$\omega_r\left(\operatorname{diag}(t_1, \ldots, t_n)\right) := t_1 \ldots t_r.$$

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(6) Let $\mathscr{V}$ be a semistable vector bundle over Σ. Then show the following.

(a) For any nonzero $\mathscr{O}_\Sigma$-submodule $\mathscr{F}$ of $\mathscr{V}$,

$$\mu(\mathscr{F}) \leq \mu(\mathscr{V}),$$

where the slope $\mu(\mathscr{F})$ has the same definition as in Definition 6.1.4(a) for any $\mathscr{O}_\Sigma$-module $\mathscr{F}$.

(b) For any nonzero $\mathscr{O}_\Sigma$-module quotient $\mathscr{Q}$ of $\mathscr{V}$,

$$\mu(\mathscr{Q}) \geq \mu(\mathscr{V}).$$

(7) (a) Let $(\Sigma, \vec{p})$ be an s-pointed smooth irreducible projective curve, where $\vec{p} = \{p_1, \dots, p_s\}$ and let $\mathscr{V}$ be a rank-n vector bundle over Σ. A *parabolic structure* for $\mathscr{V}$ at p_i consists of a partial flag in the fiber:

$$V_i^1 \subsetneq V_i^2 \subsetneq \cdots \subsetneq V_i^{l_i} = V_{p_i},$$

together with a set of markings:

$$1 > \mu_i^1 > \mu_i^2 > \cdots > \mu_i^{l_i} \geq 0,$$

where $V_{p_i} := \mathscr{V}_{p_i}$. Such a $\mathscr{V}$ with a parabolic structure is called a *parabolic vector bundle*.

The *parabolic degree* of $\mathscr{V}$ (with the above parabolic structure) is defined to be

$$\operatorname{pardeg}(\mathscr{V}) := \deg \mathscr{V} + \sum_{1 \leq i \leq s} \sum_{1 \leq k \leq l_i} \dim(V_i^k / V_i^{k-1})\, \mu_i^k,$$

where we set $V_i^0 = (0)$. The *parabolic slope* of $\mathscr{V}$ (with the above parabolic structure) is defined to be

$$\mu_{\mathrm{par}}(\mathscr{V}) := \operatorname{pardeg}(\mathscr{V}) / \operatorname{rank}(\mathscr{V}).$$

The parabolic structure on $\mathscr{V}$ defines a parabolic structure on any subbundle $\mathscr{W}$ by defining a flag $\{W_i^d\}_{1 \leq d \leq m_i}$ in the fiber W_{p_i} by removing repeated terms in the filtration and renumbering them successively:

$$V_i^1 \cap W_{p_i} \subset V_i^2 \cap W_{p_i} \subset \cdots \subset V_i^{l_i} \cap W_{p_i} = W_{p_i}.$$

Further, we define the markings ν_i^d, $1 \leq d \leq m_i$ by setting

$$\nu_i^d := \mu_i^k,$$

where k is the smallest integer with $W_i^d \subset V_i^k$.

Finally, the parabolic bundle $\mathscr{V}$ is defined to be *parabolic semistable* (resp. *parabolic stable*) if for every proper nonzero subbundle $\mathscr{W}$, we have

$$\mu_{\text{par}}(\mathscr{W}) \leq \mu_{\text{par}}(\mathscr{V}) \ \left(\text{resp. } \mu_{\text{par}}(\mathscr{W}) < \mu_{\text{par}}(\mathscr{V})\right).$$

Now, prove that the above notion of parabolic semistability (resp. parabolic stability) corresponds precisely to the notion of parabolic semistability (resp. parabolic stability) for the corresponding frame bundle $F(\mathscr{V})$ as in Definition 6.1.4(d). Write down the precise parabolic subgroups P_i, the sections σ_i of $F(\mathscr{V})_{p_i}/P_i$ and the markings τ_i (cf. Definition 6.1.2) under this correspondence.

(b) Show that a parabolic semistable vector bundle $\mathscr{V}$ has a filtration by parabolic semistable subbundles

$$\mathscr{V} = \mathscr{V}_1 \supsetneq \mathscr{V}_2 \supsetneq \cdots \supsetneq \mathscr{V}_\ell = (0)$$

such that (under the canonical parabolic structure)

(b_1) $\mu_{\text{par}}(\mathscr{V}_i) = \mu_{\text{par}}(\mathscr{V}), \quad$ for all $1 \leq i \leq \ell - 1$, and

(b_2) $\mathscr{V}_i/\mathscr{V}_{i+1}$ is parabolic stable, and

(b_3) $\operatorname{gr} \mathscr{V} := \oplus_{i=1}^{\ell-1} \mathscr{V}_i/\mathscr{V}_{i+1}$ is independent (up to parabolic isomorphism) of the above filtration of $\mathscr{V}$ with properties (b_1) and (b_2).

(8) Let $f : G \to H$ be a surjective homomorphism between connected reductive groups such that the identity component of $\operatorname{Ker} f$ is a torus and let E be a G-bundle over Σ. Then, show that if $E(H)$ is stable (resp. semistable) then accordingly so is E.

(9) Let $\mathscr{V}$ be a vector bundle over Σ of degree d and rank r. Assume that $(d, r) = 1$. Then, show that $\mathscr{V}$ is semistable if and only if it is stable.

(10) Let $\mathscr{V}$ be a semistable vector bundle over Σ of degree 0. Then, show that any nonzero section of $\mathscr{V}$ is no-where zero.

Hint: A nonzero section gives rise to an injective $\mathscr{O}_\Sigma$-module map from $\mathscr{O}_\Sigma(D)$ to $\mathscr{V}$ for some effective divisor D.

(11) For a semisimple group G, show that any semistable G-bundle over $\mathbb{P}^1$ is trivial.

(12) Let E be a G-bundle over Σ for a connected reductive group G. Then E is semistable (resp. stable; polystable) if and only if $E(G/Z)$ is semistable (resp. stable; polystable), where Z is contained in the center of G.

Prove its analogue for the A-equivariant case.

Observe that Exercise 8 is a weaker version of this exercise.

(13) Let H be a connected affine algebraic group and Σ a smooth projective curve. Then show that any H-bundle over Σ is locally trivial in the Zariski topology.

Hint: Use the result that over a smooth affine curve any U-bundle is trivial, where U is a unipotent group. Moreover, prove that if H is reductive, then any H-bundle over Σ is Zariski locally trivial.

(14) Let H be a connected affine algebraic group and Σ a smooth projective curve. Then show that any holomorphic H-bundle over $\Sigma^* = \Sigma \backslash \{p_1, \dots, p_s\}$ (for $p_j \in \Sigma$ and $s \geq 1$) is *holomorphically* trivial.

Hint: Show that any holomorphic line bundle over Σ^* is holomorphically trivial by using the cohomology sequence corresponding to the sheaf exact sequence induced from $\mathscr{O}_{\text{hol}} \to \mathscr{O}^*_{\text{hol}}, f \mapsto e^{2\pi i f}$:

$$0 \to \mathbb{Z} \to \mathscr{O}_{\text{hol}} \to \mathscr{O}^*_{\text{hol}} \to 0.$$

(15) (a) A vector bundle $\mathscr{V}$ over Σ is polystable (cf. Definition 6.1.4) if and only if it is a direct sum of stable vector bundles all of which have the same slope.

(b) Let $\mathscr{V} = \mathscr{V}_1 \oplus \mathscr{V}_2$, where $\mathscr{V}_i$ are semistable vector bundles over Σ of the same slope μ. Then show that $\mathscr{V}$ is semistable.

Thus, a polystable vector bundle $\mathscr{V}$ is semistable.

Hint: Take any vector subbundle $\mathscr{W} \subset \mathscr{V}$ and consider its projection to $\mathscr{V}_1$. Now apply Exercise 6.1.E.6(a) or the construction $(*)$ in the proof of Lemma 6.3.22 to conclude that $\mu(\mathscr{W}) \leq \mu$.

(16) Following the notation in Definition 6.1.16, show that an A-equivariant vector bundle $\mathscr{V}$ over $\hat{\Sigma}$ is A-polystable if and only if we can write

$$\mathscr{V} = \oplus V_i,$$

where each V_i is an A-stable vector bundle all of which have the same slope.

6.2 Harder–Narasimhan Filtration for G-Bundles

In this section we assume that Σ is a smooth irreducible projective curve over $\mathbb{C}$ and G is a connected reductive group with a fixed Borel subgroup B and maximal torus $H \subset B$ with their Lie algebras $\mathfrak{g}$, $\mathfrak{b}$ and $\mathfrak{h}$, respectively. Let $Z(\mathfrak{g})$ $(\subset \mathfrak{h})$ be the center of $\mathfrak{g}$. By simple roots $\{\alpha_1, \dots, \alpha_\ell\}$ and fundamental weights $\{\omega_1, \ldots, \omega_\ell\}$, we mean the corresponding objects for the semisimple Lie algebra $\mathfrak{g}/Z(\mathfrak{g})$. In particular, $\omega_i, \alpha_i \in (\mathfrak{h}/Z(\mathfrak{g}))^*$.

Definition 6.2.1 Let $\pi : E \to \Sigma$ be a G-bundle. Then, a P-subbundle $E_P \subset E$ for a standard parabolic subgroup P of G (cf. Definition 5.1.1) is called a *Harder–Narasimhan reduction* (also called *HN reduction* for short or *canonical reduction*) if it satisfies the following two conditions.

(a) The associated L-bundle $E_P(L)$, obtained from the P-bundle E_P via the extension of the structure group $P \to P/U \simeq L$, is semistable, where L is the Levi subgroup of P containing H and U is the unipotent radical of P.

(b) For any nontrivial character λ of P such that $\lambda \in \bigoplus_{i=1}^{\ell} \mathbb{Z}_+\alpha_i$, where $\mathbb{Z}_+ := \mathbb{Z}_{\geq 0}$ (in particular, λ is trivial restricted to the identity component of the center of G),

$$\deg\ (E_P(\lambda)) > 0, \quad \text{where } E_P(\lambda) := E_P \times^P \mathbb{C}_\lambda.$$

By Theorem 6.2.3, such a reduction exists and is unique.

If we realize $E_P \subset E$ via a section μ of the bundle $E/P \to \Sigma$ (cf. Lemma 5.1.2), then the line bundle

$$E_P \times^P \mathbb{C}_\lambda \simeq \mu^*\left(\mathscr{L}_P(-\lambda)\right) \quad \text{(cf. Definition 6.1.3(c))}. \tag{1}$$

If E itself is semistable, then clearly it is an HN reduction.

Definition 6.2.2 For any G-bundle $E \to \Sigma$, define the integer

$$d_E = \min\left\{\deg \mu^*\left(T_v(E/Q)\right)\right\},$$

where Q runs over all the standard parabolic subgroups of G and μ runs over all the sections of $E/Q \to \Sigma$. (Here the relative tangent bundle $T_v(E/Q)$ is as defined in Definition 6.1.3(c).)

Since any $\mu^*(T_v(E/Q))$ is a quotient of the adjoint bundle

$$\operatorname{ad} E := E \times^G \mathfrak{g} \quad (G \text{ acting on } \mathfrak{g} \text{ via the adjoint action}),$$

deg $\mu^*(T_v(E/Q))$ is bounded from below by using the Riemann–Roch theorem for smooth curves (Hartshorne, 1977, Chap. IV, Theorem 1.3). Thus, d_E is indeed an integer.

Theorem 6.2.3 *Let P be a standard parabolic subgroup and let $E_P \subset E$ be a P-subbundle of a G-bundle $\pi : E \to \Sigma$ given by a section μ of $E/P \to \Sigma$ (via Lemma 5.1.2) satisfying the following conditions:*

(α) *deg $\mu^*\left(T_v(E/P)\right) = d_E$.*

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(β) *There does not exist any parabolic* $\tilde{P} \supsetneq P$ *with a section* $\tilde{\mu}$ *of* $E/\tilde{P} \to \Sigma$ *such that*

$$deg\ \tilde{\mu}^* \left(T_v(E/\tilde{P}) \right) = d_E.$$

Then E_P *is a HN reduction of* E. *Thus, a HN reduction of* E *exists. Moreover,* E_P *is the unique HN reduction.*

Further,

$$H^0 \left(\Sigma, E_P \times^P \mathfrak{g}/\mathfrak{p} \right) = 0. \tag{1}$$

Proof Let $E_P \subset E$ be a reduction satisfying conditions (α) and (β). We first prove that E_P satisfies condition (a) of Definition 6.2.1, i.e., $E_L := E_P(L)$ is semistable.

We choose $B_L := B \cap L$ as the Borel subgroup of L. Assume, for contradiction, that E_L is not semistable, i.e., by Definition 6.1.4(b) there exists a standard parabolic subgroup Q of L and a section σ of $E_L/Q \to \Sigma$ such that

$$\deg \sigma^* \left(T_v(E_L/Q) \right) < 0. \tag{2}$$

Consider the surjective group homomorphism $p_L \colon P \to P/U \simeq L$ and let $P_1 := p_L^{-1}(Q)$. Since $p_L(B) = B_L$, $P_1 \subset P$ is a standard parabolic subgroup of G. Since the homomorphism p_L induces an isomorphism:

$$\begin{array}{ccc} E_P/P_1 & \xrightarrow[\sim]{\hat{p}_L} & E_L/Q \\ & \searrow \quad \swarrow & \\ & \Sigma\,, & \end{array}$$

the section σ of E_L/Q induces a section μ_1 of $E_P/P_1 \subset E/P_1$. As earlier, we denote the Lie algebra of any group by the corresponding Gothic character. Then, we have the following exact sequence of P_1-modules:

$$0 \to \mathfrak{p}/\mathfrak{p}_1 \to \mathfrak{g}/\mathfrak{p}_1 \to \mathfrak{g}/\mathfrak{p} \to 0. \tag{3}$$

Since $\mathfrak{p}/\mathfrak{p}_1 \simeq \mathfrak{l}/\mathfrak{q}$ as Q-modules, the above exact sequence gives rise to the following exact sequence of vector bundles over Σ (cf. Definition 6.1.3(c)):

$$0 \to \sigma^* \left(T_v(E_L/Q) \right) \to \mu_1^* \left(T_v(E/P_1) \right) \to \mu^* \left(T_v(E/P) \right) \to 0. \tag{4}$$

Observe that $\pi \circ \mu_1 = \mu$, where $\pi \colon E/P_1 \to E/P$ is the projection. From (4), we get

$$\begin{aligned} \deg \mu_1^* \left(T_v(E/P_1) \right) &= \deg \sigma^* \left(T_v(E_L/Q) \right) + \deg \mu^* \left(T_v(E/P) \right) \\ &< \deg \mu^* \left(T_v(E/P) \right), \quad \text{by (2).} \end{aligned}$$

This contradicts the choice (α). Thus, E_L is semistable.

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We next show that E_P satisfies condition (b) of Definition 6.2.1.

For any nontrivial character λ of P such that $\lambda = \sum_{i=1}^{\ell} n_i\alpha_i$, with $n_i \in \mathbb{Z}_+$, we need to show that

$$\deg\left(E_P \times^P \mathbb{C}_\lambda\right) > 0. \tag{5}$$

Fix $\alpha_k \notin S_P$ (where S_P is the set of simple roots of L) and let $P_2 \supset P$ be the parabolic subgroup with $S_{P_2} = S_P \cup \{\alpha_k\}$ and let L_2 be its Levi component containing H again. (If $P = G$, there does not exist any nontrivial character λ of P such that $\lambda \in \sum_{i=1}^{\ell} \mathbb{Z}_+\alpha_i$.) Then the image of P under the homomorphism $p_{L_2}\colon P_2 \to L_2$ is a parabolic subgroup Q of L_2 giving rise to an isomorphism (as Q-modules, viewing Q as a subgroup of P under the embedding $L_2 \subset P_2$):

$$\mathfrak{p}_2/\mathfrak{p} \simeq \mathfrak{l}_2/\mathfrak{q}. \tag{6}$$

Similar to the exact sequence (4), we get the exact sequence

$$0 \to \sigma_2^*\left(T_v(E_{L_2}/Q)\right) \to \mu^*\left(T_v(E/P)\right) \to \mu_2^*\left(T_v(E/P_2)\right) \to 0, \tag{7}$$

where μ_2 is the section μ followed by the projection $E/P \to E/P_2$, $E_{L_2} := E_{P_2}(L_2)$ and σ_2 is the section of $E_{L_2}/Q \simeq E_{P_2}/P \subset E/P$ induced by μ. From the exact sequence (7), we get

$$\deg \mu^*\left(T_v(E/P)\right) = \deg \sigma_2^*\left(T_v(E_{L_2}/Q)\right) + \deg \mu_2^*\left(T_v(E/P_2)\right). \tag{8}$$

From the 'maximality' of P with the minimality of $\deg \mu^*\left(T_v(E/P)\right)$ as in (β), we get

$$\deg \sigma_2^*\left(T_v(E_{L_2}/Q)\right) < 0. \tag{9}$$

Now, by Definition 6.1.3(c),

$$T_v(E_{L_2}/Q) \simeq E_{L_2} \times^Q (\mathfrak{l}_2/\mathfrak{q}) \simeq E_{P_2} \times^P (\mathfrak{p}_2/\mathfrak{p}). \tag{10}$$

Clearly, $\wedge^{\mathrm{top}}(\mathfrak{l}_2/\mathfrak{q}) \simeq \wedge^{\mathrm{top}}(\mathfrak{p}_2/\mathfrak{p})$ is a P-module and the character

$$\mathrm{ch}_P\left(\wedge^{\mathrm{top}}(\mathfrak{p}_2/\mathfrak{p})\right) = -\theta_k, \text{ where } \theta_k = m_k\alpha_k + \sum_{\alpha_i \in S_P} m_i^k\alpha_i, \tag{11}$$

for some $m_k \geq 1$ and $m_i^k \in \mathbb{Z}_+$. Combining (9)–(11), we get

$$\deg \sigma_2^*\left(T_v(E_{L_2}/Q)\right) = -\deg\left(E_P \times^P \mathbb{C}_{\theta_k}\right) < 0.$$

Thus,

$$\deg\left(E_P \times^P \mathbb{C}_{d\theta_k}\right) > 0, \text{ for any } d > 0. \tag{12}$$

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From (12), we get that for some $N \gg 0$ and some $m_i \in \mathbb{Z}_+$,

$$\deg\left(E_P \times^P \mathbb{C}_\beta\right) > 0, \tag{13}$$

where $\lambda = \sum_{i=1}^{\ell} n_i \alpha_i$ is as in (5) (in particular, λ being a nontrivial character of P, $n_k > 0$ for some $\alpha_k \notin S_P$) and $\beta = N \sum_{\alpha_k \notin S_P} n_k \alpha_k + \sum_{\alpha_i \in S_P} m_i \alpha_i$. Now, $N\lambda$ and β are both characters of P (or equivalently characters of L) and clearly they coincide on $Z(\mathfrak{l})$, where $Z(\mathfrak{l})$ is the center of the Lie algebra $\mathfrak{l}$ of L. Of course, being characters of L, they both vanish on the commutator $[\mathfrak{l}, \mathfrak{l}]$. Hence, $N\lambda = \beta$ on $\mathfrak{l}$ and hence they coincide on L (L being connected). Thus, by (13), we get $\deg\left(E_P \times^P \mathbb{C}_{N\lambda}\right) > 0$, which gives $\deg\left(E_P \times^P \mathbb{C}_\lambda\right) > 0$. This proves (5), proving the first part of the theorem.

We now prove the uniqueness of the HN reduction.

Let $E_P \subset E$ and $E_{P'} \subset E$ be two HN reductions (for standard parabolic subgroups P and P') given by sections μ and μ' of $E/P \to \Sigma$ and $E/P' \to \Sigma$, respectively. The L-bundle E_L obtained from the extension of the structure group via $P \to P/U \simeq L$ is semistable by the definition of HN reduction, where $L \supset H$ (resp. U) is the Levi subgroup (resp. unipotent radical) of P. Consider the P-module filtration:

$$V_0 = 0 \subset V_1 \subset V_2 \subset \ldots \subset V_k = \mathfrak{g}/\mathfrak{p}$$

such that each $A_i := V_i/V_{i-1}$, $1 \le i \le k$, is an irreducible P-module. In particular, U acts trivially on each A_i (cf. Exercise 6.2.E.1).

Similarly, consider the P-module filtration

$$W_0 = 0 \subset W_1 \subset W_2 \subset \ldots \subset W_n = \mathfrak{u}$$

such that each $B_j := W_j/W_{j-1}$ is an irreducible P-module. Let $\mathscr{V}_i$, $\mathscr{A}_i$, $\mathscr{W}_j$ and $\mathscr{B}_j$ be the vector bundles over Σ associated to the P-bundle E_P by the P-modules V_i, A_i, W_j and B_j, respectively. For $1 \le i \le k$ and $1 \le j \le n$,

$$\mathscr{V}_i \subset \mu^*\left(T_v(E/P)\right) \quad \text{and} \quad \mathscr{W}_j \subset \operatorname{ad} E_P, \tag{14}$$

where $\operatorname{ad} E_P$ is the adjoint bundle $E_P \times^P \mathfrak{p} \to \Sigma$. We also let $\mathscr{B}_{n+1}$ be the vector bundle over Σ associated to $E_P \to \Sigma$ via the P-module $\mathfrak{p}/\mathfrak{u}$. Then, since $\mathfrak{l} \simeq \mathfrak{p}/\mathfrak{u}$, it is easy to see that

$$\mathscr{B}_{n+1} \simeq \operatorname{ad}(E_L). \tag{15}$$

Since each A_i and B_j (for $1 \le i \le k$ and $1 \le j \le n$) is an irreducible L-module, E_L is semistable and $\mathscr{B}_{n+1}$ is the associated adjoint bundle, by

Theorem 6.1.7, we get that each of vector bundles $\mathscr{A}_i$ $(1 \le i \le k)$ and $\mathscr{B}_j$ $(1 \le j \le n+1)$ is semistable (cf. Exercise 6.1.E.5). Clearly, for any $1 \le i \le k$ and any $1 \le j \le n$,

$$-\operatorname{ch}\left(\wedge^{\text{top}}(A_i)\right) \in \bigoplus_{p=1}^{\ell} \mathbb{Z}_+\alpha_p, \quad \operatorname{ch}\left(\wedge^{\text{top}}(B_j)\right) \in \bigoplus_{p=1}^{\ell} \mathbb{Z}_+\alpha_p, \tag{16}$$

and both of these are clearly nontrivial characters of P. Hence, by Definition 6.2.1(b), for any $1 \le i \le k$ and $1 \le j \le n$,

$$\deg \mathscr{A}_i < 0 \quad \text{and} \quad \deg \mathscr{B}_j > 0. \tag{17}$$

Moreover, since $\wedge^{\text{top}}(\mathfrak{l})$ is a trivial L-module,

$$\deg \mathscr{B}_{n+1} = 0. \tag{18}$$

In exactly the same way, we consider filtrations $V'_{i'}$ of $\mathfrak{g}/\mathfrak{p}'$ and $W'_{j'}$ of $\mathfrak{u}'$ giving rise to vector bundles $\mathscr{V}'_{i'}$, $\mathscr{A}'_{i'}$, $\mathscr{W}'_{j'}$ and $\mathscr{B}'_{j'}$ over Σ. Analogous to (17) and (18), we get, for all i' and j',

$$\deg \mathscr{A}'_{i'} < 0 \quad \text{and} \quad \deg \mathscr{B}'_{j'} \ge 0. \tag{19}$$

Moreover, the vector bundles $\mathscr{A}'_{i'}$ and $\mathscr{B}'_{j'}$ are semistable.

By the following lemma, there is no nonzero $\mathscr{O}_\Sigma$-linear map from any $\mathscr{B}_j$ to $\mathscr{A}'_{i'}$. Thus, working through the filtration $\mathscr{W}_j$ of $\operatorname{ad} E_P$ and the filtration $\mathscr{V}'_{i'}$ of $\mu'^*\left(T_v(E/P')\right)$, we get that

$$\operatorname{Hom}_{\mathscr{O}_\Sigma}\left(\operatorname{ad} E_P, \mu'^*\left(T_v(E/P')\right)\right) = 0. \tag{20}$$

The exact sequence of P-modules

$$0 \to \mathfrak{p} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{p}$$

gives rise to the exact sequence of vector bundles over Σ:

$$0 \to \operatorname{ad} E_P \to \operatorname{ad} E \to \mu^*\left(T_v(E/P)\right) \to 0,$$

and a similar sequence for the reduction (P', μ'). Thus, from (20), we get that

$$\operatorname{ad} E_P \subset \operatorname{ad} E_{P'}.$$

Similarly,

$$\operatorname{ad} E_{P'} \subset \operatorname{ad} E_P.$$

Thus,

$$\operatorname{ad} E_P = \operatorname{ad} E_{P'}, \quad \text{as subbundles of } \operatorname{ad} E. \tag{21}$$

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We assert that from (21) we get

$$E_P = E_{P'} \text{ as subbundles of } E. \tag{22}$$

Take $x \in \Sigma$ and $e_x \in E_x$ (the fiber over x) such that

$$(E_P)_x = e_x \cdot P \quad \text{and} \quad (E_{P'})_x = e_x \cdot g_x \cdot P', \quad \text{for some} \quad g_x \in G. \tag{23}$$

Then, by definition,

$$(\operatorname{ad} E_P)_x = [e_x, \mathfrak{p}] \quad \text{and} \quad (\operatorname{ad} E_{P'})_x = [e_x g_x, \mathfrak{p}'],$$

where $[e_x, \mathfrak{p}]$ is the set of equivalence classes of (e_x, Y) in $E \times^G \mathfrak{g}$ as Y ranges over $\mathfrak{p}$. Since

$$[e_x, \mathfrak{p}] = [e_x g_x, \mathfrak{p}'] = [e_x, (\operatorname{Ad} g_x) \cdot \mathfrak{p}'], \quad \text{by (21)}, \tag{24}$$

we get

$$\mathfrak{p} = (\operatorname{Ad} g_x) \cdot \mathfrak{p}', \quad \text{equivalently} \quad P = g_x P' g_x^{-1}.$$

But since P and P' are both standard parabolic subgroups, we get (cf. (Borel, 1991, Theorem 11.16 and Corollary 11.17)) that $g_x \in P$ and $P' = P$. Thus, from (23),

$$(E_P)_x = (E_{P'})_x \quad \text{for all} \quad x \in \Sigma,$$

proving that $E_P = E_{P'}$ and hence E_P is unique.

To prove identity (1), from the filtration $\mathscr{V}_i$ of $E_P \times^P \mathfrak{g}/\mathfrak{p}$, it suffices to prove that $H^0(\Sigma, \mathscr{A}_i) = 0$ for all $1 \le i \le k$. But, as shown above, $\mathscr{A}_i$ are semistable vector bundles and further $\deg \mathscr{A}_i < 0$ (cf. (17)). Thus, $H^0(\Sigma, \mathscr{A}_i) = 0$ (e.g., by the next Lemma 6.2.4 applied to $\mathscr{E} = \mathscr{O}_\Sigma$ and $\mathscr{F} = \mathscr{A}_i$). This proves identity (1).

This proves the theorem modulo the next lemma. □

Lemma 6.2.4 *Let $\mathscr{E}$ and $\mathscr{F}$ be two semistable vector bundles over Σ such that*

$$\mu(\mathscr{E}) > \mu(\mathscr{F}). \tag{1}$$

Then

$$\operatorname{Hom}_{\mathscr{O}_\Sigma}(\mathscr{E}, \mathscr{F}) = 0.$$

Proof If possible, take a nonzero $f \in \operatorname{Hom}_{\mathscr{O}_\Sigma}(\mathscr{E}, \mathscr{F})$. Then by Exercise 6.1.E.6,

$$\mu(\mathscr{E}) \le \mu(f(\mathscr{E})) \le \mu(\mathscr{F}),$$

where $\mu(\mathscr{C}) := \frac{\deg \mathscr{C}}{\operatorname{rank} \mathscr{C}}$, for any $\mathscr{O}_\Sigma$-module $\mathscr{C}$. This is a contradiction to (1). This proves the lemma. □

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From the uniqueness of the HN reduction, we get the following.

Corollary 6.2.5 *Let $\hat{\Sigma}$ be a smooth irreducible projective curve with the action of a finite group A and let $\hat{E}$ be an A-equivariant G-bundle over $\hat{\Sigma}$. Then its HN reduction $\hat{E}_P$ remains stable under the A-action, i.e., $A \cdot \hat{E}_P \subset \hat{E}_P$.*

Let $f : G \hookrightarrow G'$ be an embedding between connected reductive algebraic groups. Choose a Borel subgroup B of G (resp. B' of G') and a maximal torus $H \subset B$ of G (resp. $H' \subset B'$ of G') such that

$$B' \cap G = B \quad \text{and} \quad H' \cap G = H.$$

We fix these choices.

Let E be a G-bundle and let $E' := E(G')$ be the associated (principal) G'-bundle. Thus, $E \subset E'$ can be thought of as a G-subbundle of E'. Let $E_P \subset E$ and $E'_{P'} \subset E'$ be the HN reductions to P and P', for standard parabolic subgroups P of G and P' of G'.

Theorem 6.2.6 *With the notation as above, assume the following.*

$$\textit{For any } g \in G' \textit{ such that if } P \subset gP'g^{-1}, \textit{ then } g \in P', \tag{1}$$

and

$$U(P' \cap G) \subset U(P'), \tag{2}$$

where $U(P')$ denotes the unipotent radical of P'. Then

$$E_P = E'_{P'} \cap E \quad \textit{as subsets of} \quad E'.$$

Proof Consider the filtrations $\mathscr{V}_i \subset \mu^*(T_v(E/P))$ and $\mathscr{W}_j \subset \operatorname{ad} E_P$ as in (14) of the proof of Theorem 6.2.3, where the P-subbundle $E_P \subset E$ is given by a section μ of $E/P \to \Sigma$. Similarly, consider the filtrations $\mathscr{V}'_{i'} \subset \mu'^*\left(T_v(E'/P')\right)$ and $\mathscr{W}'_{j'} \subset \operatorname{ad} E'_{P'}$. As in the proof of Theorem 6.2.3 (using Lemma 6.2.4), we conclude that (considered as subsets of $E \times^G \mathfrak{g}'$)

$$\operatorname{ad} E_P \subset \operatorname{ad} E'_{P'}.$$

So far we have not used any of the assumptions (1) and (2). By an argument towards the end of the proof of Theorem 6.2.3, we get (by using assumption (1)): $E_P \subset E'_{P'}$, which gives

$$E_P \subset E'_{P'} \cap E, \tag{3}$$

and

$$P \subset P' \cap G. \tag{4}$$

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Let $P_1 := P' \cap G$, which is a standard parabolic subgroup of G. From (3) and (4), we get

$$E_{P_1} := E_P \times^P P_1 = E'_{P'} \cap E. \tag{5}$$

Since the reduction E_P is a HN reduction, clearly E_{P_1} satisfies property (b) of HN reduction as in Definition 6.2.1.

So far, we have not used the assumption (2). Now, by assumption (2),

$$U(P_1) \subset U(P') \cap G.$$

Conversely, $U(P') \cap G$ being a normal unipotent subgroup of P_1, $U(P') \cap G \subset U(P_1)$. Thus,

$$U(P') \cap G = U(P_1). \tag{6}$$

The inclusions $P_1 \hookrightarrow P'$ and $U(P_1) \subset U(P')$ induce the commutative diagram

$$\begin{array}{ccc} P_1 & \longrightarrow & P_1/U(P_1) \\ \big\downarrow & & \big\downarrow \\ P' & \longrightarrow & P'/U(P'), \end{array}$$

where the right vertical map is injective by virtue of (6). This gives that

$$E_{P_1}(L_1) \hookrightarrow E'_{P'}(L'),$$

where L_1 (resp. L') is the Levi component of P_1 (resp. P') containing H (resp. H').

Now, since $E'_{P'}(L')$ is semistable (since $E'_{P'}$ is a HN reduction of E') so is its adjoint bundle $\operatorname{ad} E'_{P'}(L')$ (by Theorem 6.1.7). Moreover, $\operatorname{ad} E'_{P'}(L')$ has degree 0. Similarly, $\operatorname{ad} E_{P_1}(L_1)$ has degree 0. Thus, $\operatorname{ad} E_{P_1}(L_1)$ is a semistable vector bundle (by the definition of semistability of vector bundles as in Definition 6.1.4(a)). Thus, by Lemma 6.1.5, $E_{P_1}(L_1)$ is a semistable L_1-bundle. Hence, E_{P_1} satisfies property (a) of HN reduction as well. Thus, E_{P_1} is a HN reduction of E. From the uniqueness of HN reduction (cf. Theorem 6.2.3), we get that

$$P = P_1 \quad \text{and} \quad E_P = E_{P_1}.$$

Combining this with (5), we get

$$E_P = E'_{P'} \cap E, \quad \text{proving the theorem.}$$ □

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Corollary 6.2.7 *Let $G \hookrightarrow G'$ be an embedding of connected reductive groups and let E be a G-bundle over Σ. Then, we have the following:*

(a) If $E(G')$ is semistable, then so is E.

(b) If E is semistable and G is not contained in any proper (not necessarily standard) parabolic subgroup of G', then $E(G')$ is semistable.

Proof (a) Let $E_P \subset E$ be the HN reduction of E. Since $E(G')$ is semistable, this is the HN reduction of $E(G')$. Thus, from Theorem 6.2.6, $E_P = E$ proving (a).

(Observe that the conditions (1) and (2) of Theorem 6.2.6 are trivially satisfied since $U(G) = \{1\}$.)

(b) Let $E'_{P'} \subset E(G')$ be the HN reduction. By the assumption in (b), condition (1) of Theorem 6.2.6 is clearly satisfied. By the proof of Theorem 6.2.6 (specifically identity (4), which does not require assumption (2) of Theorem 6.2.6), we get

$$G \subset P' \cap G, \quad \text{which gives} \quad G \subset P'.$$

But since, by assumption, there is no proper parabolic subgroup of G' containing G, we get $P' = G'$ and hence $E(G')$ is semistable. This proves (b).

□

Remark 6.2.8 (1) The assumption in the (b)-part of Corollary 6.2.7 that there is no proper parabolic subgroup of G' containing G is, in general, required. Take, e.g., $G = H$, $G' = \mathrm{SL}_2(\mathbb{C})$, where H is the standard maximal torus of $\mathrm{SL}_2(\mathbb{C})$. Take any line bundle $\mathscr{L}$ over Σ of positive degree and let E be the corresponding G-bundle. Then $E' = E(G')$ corresponds to the frame bundle of the rank-2 vector bundle $\mathscr{L} \oplus \mathscr{L}^*$, which clearly is not semistable.

(2) Conditions (1) and (2) in Theorem 6.2.6 are missing in the corresponding theorem (Biswas and Holla, 2004, Theorem 5.1). Their proof has a gap which necessitated imposing conditions (1) and (2).

6.2.E Exercises

(1) Let H be an algebraic group and let V be an irreducible H-module. Show that the unipotent radical $U(H)$ of H acts trivially on V.

Hint: A unipotent group fixes a nonzero vector in any representation.

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(2) Let $\mathscr{V} \to \Sigma$ be a vector bundle. Then there is a unique filtration of $\mathscr{V}$ by subbundles:

$$0 = \mathscr{V}_0 \subsetneq \mathscr{V}_1 \subsetneq \cdots \subsetneq \mathscr{V}_n = \mathscr{V},$$

such that each $\mathscr{V}_i/\mathscr{V}_{i-1}$ is a semistable vector bundle and, moreover, for all $2 \leq i \leq n$,

$$\mu\left(\mathscr{V}_i/\mathscr{V}_{i-1}\right) < \mu\left(\mathscr{V}_{i-1}/\mathscr{V}_{i-2}\right),$$

cf. Definition 6.1.4(a) for the definition of μ.

This filtration is called the Harder–Narasimhan (for short HN) filtration of $\mathscr{V}$.

Let $F(\mathscr{V})$ be the frame bundle of $\mathscr{V}$ and let $F(\mathscr{V})_P$ be the HN reduction of $F(\mathscr{V})$, where P is a standard parabolic subgroup of GL_N, N being the rank of $\mathscr{V}$ (cf. Definition 6.2.1). Then, the filtration of $\mathbb{C}^N$ induced by the parabolic subgroup P gives rise to a filtration of the vector bundle $\mathscr{V} = F(\mathscr{V}) \times^{\mathrm{GL}_N} \mathbb{C}^N$ from the reduction $F(\mathscr{V})_P$. Show that this filtration is the unique HN-filtration of $\mathscr{V}$.

Conversely, show that the HN filtration of $\mathscr{V}$ gives rise to the HN reduction of the GL_N-bundle $F(\mathscr{V})$.

(3) Let $f : G \to H$ be a homomorphism between connected reductive algebraic groups such that the identity component of Ker f is a torus and let E be a G-bundle over Σ. Then, if $E(H)$ is semistable, so is E. (Compare this exercise with Exercise 6.1.E.8.)

Hint: Express f as the composite $G \to^{f} f(G) \hookrightarrow^{i} H$. By Corollary 6.2.7(a), $E(f(G))$ is semistable. Now use Exercise 6.1.E.8.

(4) Following the definition and assumptions as in Definition 6.1.16, show that an A-equivariant G-bundle $\hat{E}$ over $\hat{\Sigma}$ is A-semistable if and only if it is semistable.

Hint: Use Theorem 6.2.3.

6.3 A Topological Construction of Semistable G-Bundles (Result of Narasimhan–Seshadri and its Generalization)

Let G be a connected reductive group and let K be a maximal compact subgroup (which is an $\mathbb{R}$-analytic group unique up to a conjugation). Let Σ be a smooth projective irreducible curve of genus $g \geq 1$. *This will be our tacit assumption through this section unless stated otherwise.*

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Choose any base point $\infty \in \Sigma$. Then, as is well known (cf. (Spanier, 1966, Chap. 3, §8.12)), the fundamental group $\pi_1(\Sigma) = \pi_1(\Sigma, \infty)$ is isomorphic with

$$\pi_1(\Sigma) \simeq F(a_1, b_1, a_2, b_2, \dots, a_g, b_g)/\langle \Pi_{i=1}^{g} [a_i, b_i] \rangle, \qquad (*)$$

where F denotes the free group, $[a_i, b_i]$ is the commutator $a_i b_i a_i^{-1} b_i^{-1}$ and $\langle \ \rangle$ denotes the normal subgroup generated by the enclosed element(s). Moreover, under this isomorphism,

$$\pi_1(\Sigma \setminus p, \infty) \simeq F(a_1, b_1, a_2, b_2, \dots, a_g, b_g), \text{ for any } p \neq \infty \in \Sigma.$$

Definition 6.3.1 For any group homomorphism $\rho \colon \pi_1(\Sigma) \to G$, we get a holomorphic G-bundle E_ρ over Σ defined by extension of the structure group of the principal $\pi_1(\Sigma)$-bundle $q \colon \tilde{\Sigma} \to \Sigma$ to G via ρ, where $\tilde{\Sigma}$ is the simply-connected cover of Σ, i.e.,

$$E_\rho := \tilde{\Sigma} \times^{\pi_1(\Sigma)} G.$$

By the GAGA principle (Serre 1958, §6.3), E_ρ is an algebraic G-bundle over Σ (cf. Section 1.1).

If $\operatorname{Im} \rho \subset gKg^{-1}$ for some $g \in G$, then ρ is called a *unitary homomorphism* of $\pi_1(\Sigma)$ and E_ρ is called a *unitary G-bundle*. A representation of $\pi_1(\Sigma)$ in a finite-dimensional vector space V is called *unitary* if the corresponding homomorphism $\rho_o \colon \pi_1(\Sigma) \to \mathrm{GL}_V$ is unitary. Equivalently, V is unitary if it admits a positive-definite Hermitian form invariant under $\pi_1(\Sigma)$.

The homomorphism ρ is called *irreducible* if $\operatorname{Im} \rho$ is not contained in any proper (not necessarily standard) parabolic subgroup of G.

A vector bundle $\mathscr{V}$ over Σ is called *unitary* if there exists a finite-dimensional vector space V and a unitary representation $\rho_o \colon \pi_1(\Sigma) \to \mathrm{GL}_V$ such that $\mathscr{V} \simeq E_{\rho_o}(V) := E_{\rho_o} \times^{\mathrm{GL}(V)} V$.

Lemma 6.3.2 *Let $R_G(g)$ be the set of all the homomorphisms from $\pi_1(\Sigma)$ to G and let $R_K(g)$ be its subset consisting of unitary homomorphisms ρ with $\operatorname{Im} \rho \subset K$. Then $R_G(g)$ acquires an affine variety structure and $R_K(g)$ is a compact $\mathbb{R}$-analytic subset.*

Moreover, there exists a 'universal' $\mathbb{C}$-analytic G-bundle $\theta \colon \mathscr{E} \to \Sigma \times R_G(g)$ such that for any $\rho \in R_G(g)$, $\mathscr{E}_{|\Sigma \times \rho} \simeq E_\rho$.

Proof Consider the morphism of varieties:

$$\xi \colon (G \times G)^g \to [G, G], \ ((x_1, y_1), \dots, (x_g, y_g)) \mapsto [x_1, y_1] \dots [x_g, y_g].$$

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Then, by the identification (*) as at the beginning of this section, mapping $a_i \mapsto x_i, b_i \mapsto y_i$, we get

$$R_G(g) = \xi^{-1}(1).$$

Thus, $R_G(g)$ acquires a natural affine variety structure as the reduced scheme corresponding to the scheme-theoretic fiber over 1. Now, $R_K(g) := R_G(g) \cap K^{\times 2g}$ and $K \subset G$ is an $\mathbb{R}$-analytic subgroup. Hence, $R_K(g)$ has a natural $\mathbb{R}$-analytic space structure and $R_K(g)$ being a closed subset of $K^{\times 2g}$ is compact in the analytic topology.

We now construct the family $\mathscr{E}$ over $\Sigma \times R_G(g)$.

Consider the right holomorphic action of $\pi_1(\Sigma)$ on $\tilde{\Sigma} \times R_G(g) \times G$ by

$$(\tilde{x}, \rho, g) \cdot \gamma = (\tilde{x} \cdot \gamma, \rho, \rho(\gamma^{-1})g),$$

$$\text{for } \ \tilde{x} \in \tilde{\Sigma}, \rho \in R_G(g), g \in G \ \text{ and } \ \gamma \in \pi_1(\Sigma).$$

Since the action of $\pi_1(\Sigma)$ on $\tilde{\Sigma}$ is fixed point free and properly discontinuous, so is its action on $\tilde{\Sigma} \times R_G(g) \times G$. Thus, we get a $\mathbb{C}$-analytic space

$$\mathscr{E} = \left(\tilde{\Sigma} \times R_G(g) \times G\right) / \pi_1(\Sigma)$$

together with holomorphic projection

$$\theta : \mathscr{E} \to \Sigma \times R_G(g), [\tilde{x}, \rho, g] \mapsto (q(\tilde{x}), \rho),$$

where $[\tilde{x}, \rho, g]$ denotes the $\pi_1(\Sigma)$-orbit of $(\tilde{x}, \rho, g)$. Then, θ is a $\mathbb{C}$-analytic principal G-bundle under the right action of G on $\mathscr{E}$ via the right multiplication on the G-factor (local triviality of θ is easy to see since $q : \tilde{\Sigma} \to \Sigma$ is locally trivial). By construction,

$$\mathscr{E}_{|\Sigma \times \rho} \simeq E_\rho, \quad \text{for any} \quad \rho \in R_G(g). \qquad \square$$

Definition 6.3.3 Let $\hat{\Sigma}$ be a smooth irreducible projective curve with the faithful action of a finite group A and let $\Sigma := \hat{\Sigma}/A$ be the quotient (smooth) curve. *If the genus g of Σ is 0, we assume that there are at least three ramification points of $\hat{\Sigma} \to \Sigma$.* Let $\hat{q} : \tilde{\hat{\Sigma}} \to \hat{\Sigma}$ be the simply-connected cover of $\hat{\Sigma}$ and let π_1 be the fundamental group of $\hat{\Sigma}$ with respect to a fixed base point in $\hat{\Sigma}$. Then, there exists a subgroup $\pi \subset \mathrm{Aut}_{\mathrm{hol}}(\tilde{\hat{\Sigma}})$ such that π acts discontinuously on $\tilde{\hat{\Sigma}}$ and π_1 is a normal subgroup of π (π_1 acting of course properly discontinuously without fixed points on $\tilde{\hat{\Sigma}}$). Moreover, $\pi/\pi_1 \simeq A$ and they satisfy

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$$\begin{array}{ccc} \tilde{\hat{\tilde{\Sigma}}} & \xrightarrow{\hat{q}} & \hat{\Sigma} \\ \downarrow & & \downarrow \\ \tilde{\hat{\tilde{\Sigma}}}/\pi & \xrightarrow{\sim} & \hat{\Sigma}/A, \end{array}$$

where the two vertical maps are the canonical orbit maps and the bottom horizontal morphism is induced by

$$\tilde{\hat{\tilde{\Sigma}}}/\pi \simeq \left(\tilde{\hat{\tilde{\Sigma}}}/\pi_1\right)\Big/(\pi/\pi_1) \simeq \hat{\Sigma}/A.$$

In fact, π is the subgroup of $\mathrm{Aut}_{\mathrm{hol}}(\tilde{\hat{\tilde{\Sigma}}})$ consisting of those automorphisms σ of $\tilde{\hat{\tilde{\Sigma}}}$ which commute with the quotient $\bar{q}\colon \tilde{\hat{\tilde{\Sigma}}} \to \Sigma$, i.e., $\bar{q}\circ\sigma = \bar{q}$. Then, it is well known (e.g. (Serre, 1992, §6.4) or (Jones and Singerman, 1987, §5.10)) that π has a presentation

$$\pi \simeq F(a_1, b_1, a_2, b_2, \ldots, a_g, b_g, c_1, \ldots, c_s)/M,$$

where F is the free group and M is the normal subgroup generated by

$$\mu := \left(\Pi_{i=1}^{g}[a_i, b_i]\right)\cdot c_1\cdots c_s,\ c_1^{d_1}, \ldots, c_s^{d_s}.$$

Here d_j is the ramification index of p_j (i.e., the order of the isotropy subgroup of any preimage of p_j in $\hat{\Sigma}$, where $p_1, \ldots, p_s \in \Sigma$ are precisely the ramification points).

Let G be a connected reductive group. For any group homomorphism $\hat{\rho}\colon \pi \to G$ we get an A-equivariant G-bundles $\hat{E}_{\hat{\rho}}$ over $\hat{\Sigma}$ as follows:

$$\hat{E}_{\hat{\rho}} := \tilde{\hat{\tilde{\Sigma}}} \times^{\pi_1} G \to \hat{\Sigma},$$

where π_1 acts on G via the left multiplication through the representation $\hat{\rho}_{|\pi_1}$. The A-equivariant structure on $\hat{E}_{\hat{\rho}}$ is given by fixing an identification $A \simeq \pi/\pi_1$ and defining

$$\gamma\cdot[z,g] = [z\cdot\gamma^{-1}, \hat{\rho}(\gamma)g],\ \text{ for } \gamma\in\pi, z\in\tilde{\hat{\tilde{\Sigma}}} \text{ and } g\in G,$$

where $[z,g]\in\hat{E}_{\hat{\rho}}$ denotes the equivalence class of (z,g). This action clearly descends to give an A-equivariant structure on $\hat{E}_{\hat{\rho}}$.

We can clearly extend the definition of unitary (resp. irreducible) homomorphisms $\hat{\rho}\colon \pi \to G$. Similarly, we can define a unitary representation of π. If $\hat{\rho}$ is unitary, we call the corresponding A-equivariant G-bundle $\hat{E}_{\hat{\rho}}$ *A-unitary*. Similarly, an A-equivariant vector bundle $\hat{\mathscr{V}}$ over $\hat{\Sigma}$ is called *A-unitary* if there exists a finite-dimensional vector space V and a unitary representation $\hat{\rho}_o\colon \pi \to \mathrm{GL}_V$ such that $\hat{\mathscr{V}} \simeq \hat{E}_{\hat{\rho}_o}(V)$, as A-equivariant vector bundles.

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Proposition 6.3.4 *We follow the notation and assumptions as in the above Definition 6.3.3.*

(a) *For any unitary homomorphism $\hat{\rho}\colon \pi \to G$, the associated A-equivariant G-bundle $\hat{E}_{\hat{\rho}}$ is A-semistable.*
(b) *Further, such on $\hat{E}_{\hat{\rho}}$ is A-stable if and only if $\hat{\rho}$ is irreducible.*

Proof (a) Let $\hat{\rho}$ be a unitary homomorphism. To prove that $\hat{E}_{\hat{\rho}}$ is A-semistable, by Exercise 6.2.E.4 and Lemma 6.1.5, it suffices to show that the adjoint vector bundle ad $\hat{E}_\rho$ over $\hat{\Sigma}$ is semistable, where $\rho := \hat{\rho}_{|\pi_1}$. But ad $\hat{E}_\rho$ is the vector bundle associated to the $\mathrm{SL}_{\mathfrak{g}}$-bundle $\hat{E}_{\mathrm{Ad}\,\rho}$, where $\mathrm{Ad}\,\rho$ is the composite homomorphism

$$\pi_1 \xrightarrow{\rho} G \xrightarrow{\mathrm{Ad}} \mathrm{SL}_{\mathfrak{g}}.$$

Since ρ is unitary, so is $\mathrm{Ad}\,\rho$.

We now show that for a unitary homomorphism $\rho_o\colon \pi_1 \to \mathrm{SL}_V$ (for a finite-dimensional vector space V), the corresponding vector bundle $\mathscr{V} = \hat{E}_{\rho_o}(V) := \hat{E}_{\rho_o} \times^{\mathrm{SL}_V} V$ is semistable.

Let $\mathscr{W}$ be a vector subbundle of $\mathscr{V}$ of rank r. Since $\deg \mathscr{V} = 0$, we need to show that $\deg \mathscr{W} \leq 0$. Consider the line bundle

$$\wedge^r \mathscr{W} \subset \wedge^r \mathscr{V}.$$

Assume, if possible, that $\deg \mathscr{W} > 0$. Then, there exists a degree 0 line bundle $\mathscr{L}$ over $\hat{\Sigma}$ such that

$$H^0(\hat{\Sigma}, \mathscr{L} \otimes \wedge^r \mathscr{W}) \neq 0. \tag{1}$$

Since V is a unitary representation of π_1, so is $\wedge^r V$. Also, the line bundle $\mathscr{L}$ being of degree 0 comes from a unitary character χ (i.e., 1-dimensional unitary representation $\mathbb{C}_\chi$) of π_1 (cf. Exercise 6.3.E.3). Choose a positive-definite π_1-invariant Hermitian form on $\mathbb{C}_\chi \otimes \wedge^r V$ and decompose

$$\mathbb{C}_\chi \otimes \wedge^r V = V_o \oplus V_o^\perp,$$

where $V_o := [\mathbb{C}_\chi \otimes \wedge^r V]^{\pi_1}$ is the subspace of π_1-invariants and $V_o^\perp$ is its ortho-complement. Decompose the vector bundle accordingly as

$$\mathscr{L} \otimes \wedge^r \mathscr{V} = \mathscr{V}_o \oplus \mathscr{V}_o^\perp.$$

Of course, $\mathscr{V}_o$ is a trivial vector bundle. Consider the projections $p_1\colon \mathscr{L} \otimes \wedge^r \mathscr{V} \to \mathscr{V}_o$ and $p_2\colon \mathscr{L} \otimes \wedge^r \mathscr{V} \to \mathscr{V}_o^\perp$ and let i denote the inclusion $i\colon \mathscr{L} \otimes \wedge^r \mathscr{W} \hookrightarrow \mathscr{L} \otimes \wedge^r \mathscr{V}$. By the next Lemma 6.3.6,

$$H^0\left(\hat{\Sigma}, \mathscr{V}_o^\perp\right) = 0 \tag{2}$$

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and hence

$$p_2 \circ i = 0, \quad \text{by} \quad (1). \tag{3}$$

Further, since $\deg(\mathscr{L} \otimes \wedge^r \mathscr{W}) > 0$ and $\mathscr{V}_o$ is a trivial vector bundle,

$$p_1 \circ i = 0. \tag{4}$$

Combining (3) and (4), we get $i = 0$, which is a contradiction. Hence, $\deg \mathscr{W} \leq 0$, proving the (a)-part of the proposition.

(b) Assume that $\hat{\rho}: \pi \to G$ is an irreducible unitary homomorphism. We need to show that $\hat{E}_{\hat{\rho}}$ is an A-stable G-bundle.

Let Q_k be a standard maximal parabolic subgroup of G and let $\hat{\mu}$ be an A-equivariant section of $\hat{E}_{\hat{\rho}}/Q_k \to \hat{\Sigma}$. Then, following Definition 6.1.16, we need to show that

$$\deg \hat{\mu}^* \left(\hat{\mathscr{L}}_{Q_k}(-\bar{\omega}_k) \right) < 0, \tag{5}$$

where $d > 0$ is chosen so that $\bar{\omega}_k := d\omega_k$ is a character of Q_k and $\hat{\mathscr{L}}_{Q_k}(-\bar{\omega}_k) := \hat{E}_{\hat{\rho}} \times^{Q_k} \mathbb{C}_{\bar{\omega}_k}$ over $\hat{E}_{\hat{\rho}}/Q_k$. Since, by the (a)-part, $\hat{E}_{\hat{\rho}}$ is A-semistable, we get

$$\deg \hat{\mu}^* \left(\hat{\mathscr{L}}_{Q_k}(-\bar{\omega}_k) \right) \leq 0.$$

Assume, if possible, that

$$\deg(\hat{\mu}^* \hat{\mathscr{L}}_{Q_k}(-\bar{\omega}_k)) = 0. \tag{6}$$

In particular, $\hat{\mu}^* \hat{\mathscr{L}}_{Q_k}(-\bar{\omega}_k)$ is a A-unitary line bundle (cf. Exercise 6.3.E.3). Let $V(\bar{\omega}_k)$ be the irreducible representation of G with highest weight $\bar{\omega}_k$. Then, we get an embedding

$$j : G/Q_k \hookrightarrow \mathbb{P}(V(\bar{\omega}_k)), \quad gQ_k \mapsto [gv_+],$$

where v_+ is a nonzero highest-weight vector of $V(\bar{\omega}_k)$ and $[gv_+]$ denotes the line through gv_+. Let τ be the tautological line bundle over $\mathbb{P}(V(\bar{\omega}_k))$ restricted to G/Q_k. Then, since τ is the homogeneous line bundle over G/Q_k corresponding to the character $\bar{\omega}_k$ of Q_k, we get that, as A-equivariant line bundles over $\hat{E}_{\hat{\rho}}/Q_k$:

$$\hat{\mathscr{L}}_{Q_k}(-\bar{\omega}_k) \simeq \hat{E}_{\hat{\rho}} \times^{Q_k} \mathbb{C}v_+ \to \hat{E}_{\hat{\rho}}/Q_k.$$

Thus,

$$\hat{\mu}^* \hat{\mathscr{L}}_{Q_k}(-\bar{\omega}_k) \subset \hat{E}_{\hat{\rho}} \times^G C(G \cdot v_+) \subset \hat{E}_{\hat{\rho}}(V(\bar{\omega}_k)), \tag{7}$$

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where $C(G \cdot v_+)$ is the cone $\mathbb{C} \cdot (G \cdot v_+)$ inside $V(\bar{\omega}_k)$. Assuming (6), we get that $\hat{\mu}^*\big(\hat{\mathscr{L}}_{Q_k}(-\bar{\omega}_k)^*\big) \otimes \hat{E}_{\hat{\rho}}(V(\bar{\omega}_k))$ is a A-unitary vector bundle and, by (7),

$$H^0\left(\hat{\Sigma}, \hat{\mu}^*\left(\hat{\mathscr{L}}_{Q_k}(-\bar{\omega}_k)^*\right) \otimes \hat{E}_{\hat{\rho}}(V(\bar{\omega}_k))\right)^A \neq 0.$$

Thus, by the following Lemma 6.3.6 and the first inclusion in (7), we get that there exists a $g \in G$ such that the line $\mathbb{C}gv_+$ is stable under π. This gives, from the embedding $j \colon G/Q_k \hookrightarrow \mathbb{P}(V(\bar{\omega}_k))$, that $\mathrm{Im}(\hat{\rho}) \subset gQ_kg^{-1}$. This is a contradiction to the assumption that ρ is irreducible. Thus, (5) is satisfied and hence $\hat{E}_{\hat{\rho}}$ is A-stable.

Conversely, assume that $\hat{\rho}$ is unitary and $\hat{E}_{\hat{\rho}}$ is A-stable. Then we need to show that $\hat{\rho}$ is irreducible. Assume, for contradiction, that $\hat{\rho}$ is not irreducible. Thus, $\mathrm{Im}\,\hat{\rho} \subset gQ_kg^{-1}$, for some $g \in G$ and a standard maximal parabolic subgroup Q_k of G. Since $\hat{E}_{\hat{\rho}} \simeq \hat{E}_{g^{-1}\hat{\rho}g}$ (cf. Exercise 6.3.E.1), we can assume that

$$\mathrm{Im}\,\hat{\rho} \subset Q_k.$$

Then

$$\hat{E}_{\hat{\rho}}/Q_k \simeq \hat{E}_{\hat{\rho}} \times^G G/Q_k \simeq \tilde{\hat{\Sigma}} \times^{\pi_1} G/Q_k \supset \tilde{\hat{\Sigma}} \times^{\pi_1} Q_k/Q_k \simeq \hat{\Sigma}.$$

This gives rise to an A-equivariant section $\hat{\mu}$ of $\hat{E}_{\hat{\rho}}/Q_k$ over $\hat{\Sigma}$. It is easy to see that (as A-equivariant line bundles over $\hat{\Sigma}$)

$$\hat{\mu}^*\left(\hat{\mathscr{L}}_{Q_k}(-\bar{\omega}_k)\right) \simeq \tilde{\hat{\Sigma}} \times^{\pi_1} \mathbb{C}_{\bar{\omega}_k}, \tag{8}$$

where π_1 acts on the 1-dimensional space $\mathbb{C}_{\bar{\omega}_k}$ via the character $\bar{\omega}_k$ of Q_k through the homomorphism $\hat{\rho}_{|\pi_1} \colon \pi_1 \to Q_k$. Since $\hat{\rho}$ is unitary, by (8), we have that $\hat{\mu}^*(\hat{\mathscr{L}}_{Q_k}(-\bar{\omega}_k))$ is a A-unitary line bundle. Further, $\hat{E}_{\hat{\rho}}$ being A-stable,

$$\deg \hat{\mu}^*(\hat{\mathscr{L}}_{Q_k}(\bar{\omega}_k)) > 0. \tag{9}$$

Thus, for $N \gg 0$, $H^0(\hat{\Sigma}, \hat{\mu}^*(\hat{\mathscr{L}}_{Q_k}(N\bar{\omega}_k))) \neq 0$ (cf. (Hartshorne, 1977, Chap. IV, Corollary 3.3)). Hence, by (8) and the next lemma for $A = (1)$, π_1 acts trivially on $\mathbb{C}_{N\bar{\omega}_k}$. In particular, by (8) again, $\hat{\mu}^*(\hat{\mathscr{L}}_{Q_k}(N\bar{\omega}_k))$ is (non-equivariantly) a trivial line bundle over $\hat{\Sigma}$, which gives $\deg \hat{\mu}^*(\hat{\mathscr{L}}_{Q_k}(N\bar{\omega}_k)) = 0$ and hence $\deg \hat{\mu}^*(\hat{\mathscr{L}}_{Q_k}(\bar{\omega}_k)) = 0$. This contradicts (9). This contradiction shows that for $\hat{E}_{\hat{\rho}}$ to be A-stable, $\hat{\rho}$ must be irreducible. This proves the proposition modulo the following Lemma 6.3.6. □

Remark 6.3.5 From the above proof we see that any unitary line bundle $\mathscr{L}$ over $\hat{\Sigma}$ has $\deg \mathscr{L} = 0$.

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Lemma 6.3.6 *We follow the notation and assumptions as in Definition 6.3.3. Let $\hat{\rho}_o : \pi \to \mathrm{GL}_V$ be a finite-dimensional complex representation such that* $\mathrm{Im}\,\hat{\rho}_o$ *leaves a positive-definite Hermitian form $\{\cdot,\cdot\}$ on V invariant. Then, there is a canonical isomorphism from the group cohomology*

$$H^0(\pi, V) \to H^0(\hat{\Sigma}, \mathscr{V}_{\hat{\rho}_o})^A,$$

where $\mathscr{V}_{\hat{\rho}_o}$ is the A-equivariant vector bundle $\hat{E}_{\hat{\rho}_o}(V)$ over $\hat{\Sigma}$.

Moreover, any section $s \in H^0(\hat{\Sigma}, \mathscr{V}_{\hat{\rho}_o})^A$ pulled back to $\tilde{\hat{\Sigma}}$ is of the form $\tilde{s}(\tilde{x}) = (\tilde{x}, v_o)$, for some fixed $v_o \in V^\pi$.

Proof Decompose $V = V_o \oplus V_o^\perp$, where $V_o := V^\pi$ is the subspace of π-invariants in V and $V_o^\perp$ is the ortho-complement of V_o in V (which is clearly a π-module). Then

$$H^0(\pi, V) \simeq V_o \quad \text{and} \quad \mathscr{V}_{\hat{\rho}_o} = \hat{E}_{\hat{\rho}_o}(V_o) \oplus \hat{E}_{\hat{\rho}_o}(V_o^\perp). \tag{1}$$

Of course, V_o being a trivial π-module and $\hat{\Sigma}$ being irreducible and projective,

$$H^0(\hat{\Sigma}, \hat{E}_{\hat{\rho}_o}(V_o))^A \simeq V_o. \tag{2}$$

So, to prove the lemma, by (1) and (2), it suffices to show that

$$H^0(\hat{\Sigma}, \hat{E}_{\hat{\rho}_o}(V_o^\perp))^A = 0. \tag{3}$$

Now, by the definition, for any π-module W, the associated A-equivariant vector bundle $\mathscr{W} := \tilde{\hat{\Sigma}} \times^{\pi_1} W$ has

$$H^0(\hat{\Sigma}, \mathscr{W})^A = \left\{\text{Hol. maps } f : \tilde{\hat{\Sigma}} \to W \text{ satisfying the following identity}\right\} \tag{4}$$

$$f(\tilde{x} \cdot \sigma) = \sigma^{-1} \cdot f(\tilde{x}), \text{ for all } \tilde{x} \in \tilde{\hat{\Sigma}} \text{ and } \sigma \in \pi. \tag{5}$$

If $\mathscr{W}$ is a A-unitary vector bundle, then, for any such f, we get

$$||f(\tilde{x} \cdot \sigma)|| = ||f(\tilde{x})||, \text{ for all } \tilde{x} \in \tilde{\hat{\Sigma}} \text{ and } \sigma \in \pi.$$

Thus, $||f(\tilde{x})||$ descends to a continuous function on Σ; in particular, it attains a maximum α say at $\tilde{x}_o \in \tilde{\hat{\Sigma}}$. Choosing an appropriate orthonormal basis of W, we can write (for $n = \dim W$)

$$f = (f_1, \dots, f_n), \quad f(\tilde{x}_o) = (\alpha, 0, \dots, 0).$$

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Now, $f_1 : \tilde{\hat{\tilde{\Sigma}}} \to \mathbb{C}$ being a holomorphic map, $\operatorname{Im} f_1$ is open (unless it is a constant). In particular, if f_1 is non-constant, there exists $\tilde{y} \in \tilde{\hat{\tilde{\Sigma}}}$ with $f(\tilde{y}) = (\alpha + \epsilon, f_2(\tilde{y}), \ldots, f_n(\tilde{y}))$, for some $\epsilon > 0$. This is a contradiction since $||f(\tilde{y})|| \leq \alpha$. Thus, f_1 is a constant giving

$$f(\tilde{x}) = (\alpha, f_2(\tilde{x}), \ldots, f_n(\tilde{x})), \quad \text{for all} \quad \tilde{x} \in \tilde{\hat{\tilde{\Sigma}}}.$$

But, since $||f(\tilde{x})|| \leq \alpha$, we get $f_2 = \ldots = f_n \equiv 0$, i.e.,

$$f(\tilde{x}) = (\alpha, 0, \ldots, 0). \tag{6}$$

Thus, by (4) and (5),

$$H^0(\hat{\Sigma}, \mathscr{W})^A \simeq W^\pi.$$

Since $(V_o^\perp)^\pi = 0$, we get (3). This proves the first part of the lemma. The second part follows from (4), (5) and (6). □

As a corollary of the proof of Lemma 6.3.6, we get the following with the same notation and assumptions as in Definition 6.3.3.

Corollary 6.3.7 *Let $\hat{\rho}$, $\hat{\rho}'$ be unitary homomorphisms $\pi \to G$. Then $\hat{E}_{\hat{\rho}} \simeq \hat{E}_{\hat{\rho}'}$ (as A-equivariant holomorphic G-bundles over $\hat{\Sigma}$) if and only if*

$$\hat{\rho}' = g\hat{\rho}g^{-1}, \quad \textit{for some} \quad g \in G.$$

Moreover, if $\hat{\rho}$ and $\hat{\rho}'$ both have images in a maximal compact subgroup K of G, then g (as above) can be taken to lie in K.

Proof If $\hat{\rho}'$ is conjugate of $\hat{\rho}$, then $\hat{E}_{\hat{\rho}} \simeq \hat{E}_{\hat{\rho}'}$ (see Exercise 6.3.E.1).

Conversely, assume that $\hat{E}_{\hat{\rho}} \overset{\varphi}{\simeq} \hat{E}_{\hat{\rho}'}$ as A-equivariant G-bundles. By conjugating $\hat{\rho}'$ by some $g \in G$, we can (and will) assume that $\operatorname{Im} \hat{\rho}$ and $\operatorname{Im} \hat{\rho}'$ both lie in the same maximal compact subgroup K of G. Similar to condition (5) in the proof of Lemma 6.3.6, we get that φ is induced from a holomorphic function $\overline{\varphi} : \tilde{\hat{\tilde{\Sigma}}} \to G$ satisfying

$$\overline{\varphi}(\tilde{x} \cdot \sigma)\hat{\rho}(\sigma^{-1}) = \hat{\rho}'(\sigma^{-1})\overline{\varphi}(\tilde{x}), \quad \text{for} \quad \tilde{x} \in \tilde{\hat{\tilde{\Sigma}}} \quad \text{and} \quad \sigma \in \pi, \tag{1}$$

in the sense that the map $(\tilde{x}, g) \mapsto (\tilde{x}, \overline{\varphi}(\tilde{x})g)$, for $\tilde{x} \in \tilde{\hat{\tilde{\Sigma}}}$, $g \in G$ descends to give the isomorphism φ.

Take a faithful representation $i : G \hookrightarrow \mathrm{GL}_V$ and realize $W := \operatorname{End} V$ as a G-module under the conjugation:

$$g \cdot A = i(g)A\, i(g)^{-1}, \quad \text{for} \quad g \in G \quad \text{and} \quad A \in \operatorname{End} V.$$

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Put the standard positive-definite Hermitian product on W:

$$\{A, B\} = \quad \text{trace} \quad AB^*,$$

where B^* is the adjoint of B taken with respect to a fixed K-invariant positive-definite Hermitian product on V. Clearly $\{,\}$ is K-invariant; in particular, it is invariant under both $\hat{\rho}$ and $\hat{\rho}'$ (this is where we use the assumption that Im $\hat{\rho}$, Im $\hat{\rho}' \subset K$).

From (1) we get, for any $\tilde{x} \in \tilde{\hat{\Sigma}}$ and $\sigma \in \pi$,

$$\begin{aligned} ||i(\overline{\varphi}(\tilde{x}\sigma))|| &= ||i(\hat{\rho}'(\sigma^{-1}))i(\overline{\varphi}(\tilde{x}))i(\hat{\rho}(\sigma))|| \\ &= ||i\overline{\varphi}(\tilde{x})||, \quad \text{since} \ \ \hat{\rho}(\sigma), \hat{\rho}'(\sigma^{-1}) \in K. \end{aligned}$$

Thus, $||i \circ \overline{\varphi}||$ descends to a continuous function on $\hat{\Sigma}$. By the same argument as in the proof of Lemma 6.3.6, we get that $\overline{\varphi}\colon \tilde{\hat{\Sigma}} \to G$ is a constant function, say $\overline{\varphi}(\tilde{\hat{\Sigma}}) = g_o \in G$. Thus, by (1),

$$\hat{\rho}(\sigma^{-1}) = g_o^{-1}\hat{\rho}'(\sigma^{-1})g_o \text{ , for all } \ \sigma \in \pi.$$

Hence, $\hat{\rho}$ and $\hat{\rho}'$ are conjugate.

The 'Moreover' assertion follows from (Helgason, 1978, Chap. VI, Theorem 1.1), proving the corollary. □

Lemma 6.3.8 *We follow the notation and assumptions as in Definition 6.3.3. Let $\rho\colon \pi_1 \to G$ be a unitary homomorphism such that the corresponding holomorphic G-bundle $E_\rho := \tilde{\hat{\Sigma}} \times^{\pi_1} G$ over $\hat{\Sigma}$ is holomorphically A-equivariant, where $\pi_1 := \pi_1(\hat{\Sigma})$. Then, ρ extends to a unitary homomorphism $\hat{\rho}\colon \pi \to G$ such that*

$$\hat{E}_{\hat{\rho}} \simeq E_\rho, \ \ \textit{as A-equivariant holomorphic G-bundles.} \tag{1}$$

Proof Any A-equivariant structure on E_ρ (using the pull-back of E_ρ to $\tilde{\hat{\Sigma}} \times G$) is given by

$$\sigma \cdot [\tilde{x}, g] = [\tilde{x}\sigma^{-1}, \varphi_\sigma(\tilde{x})g], \ \ \text{for } \sigma \in \pi, \tilde{x} \in \tilde{\hat{\Sigma}} \text{ and } g \in G, \tag{2}$$

where $\varphi_\sigma : \tilde{\hat{\Sigma}} \to G$ is a holomorphic map satisfying

(a) $\varphi_{\sigma_1\sigma_2}(\tilde{x}) = \varphi_{\sigma_1}(\tilde{x} \cdot \sigma_2^{-1})\varphi_{\sigma_2}(\tilde{x})$, for $\sigma_1, \sigma_2 \in \pi$ and $\tilde{x} \in \tilde{\hat{\Sigma}}$, and

(b) $\varphi_\sigma(\tilde{x}\mu^{-1})\rho(\mu) = \rho(\sigma\mu\sigma^{-1})\varphi_\sigma(\tilde{x})$, for $\mu \in \pi_1, \sigma \in \pi$ and $\tilde{x} \in \tilde{\hat{\Sigma}}$.

Moreover, since $A = \pi/\pi_1$; in particular, π_1 acts trivially on E_ρ. Thus,

(c) $\varphi_\mu(\tilde{x}) = \rho(\mu)$, for all $\mu \in \pi_1$ and $\tilde{x} \in \tilde{\hat{\Sigma}}$.

Since ρ is unitary, by using the same argument as in the proof of Corollary 6.3.7, we get (using (b) and comparing it with identity (1) of Corollary 6.3.7): $\varphi_\sigma : \hat{\widetilde{\Sigma}} \to G$ is a constant function with image denoted $\bar{\varphi}_\sigma \in G$. In particular, by (a),

$$\bar{\varphi}_{\sigma_1\sigma_2} = \bar{\varphi}_{\sigma_1}\bar{\varphi}_{\sigma_2}. \tag{3}$$

Further, by (c), we get

$$\bar{\varphi}_{|\pi_1} = \rho.$$

Thus, setting $\hat{\rho} = \bar{\varphi}: \pi \to G$ we get (1) from (2) and (3). Of course, $\hat{\rho}$ is unitary since π_1 is of finite index in π. □

Definition 6.3.9 (A construction of topological G-bundles) We take Σ and G as at the beginning of this section. For any continuous map $c: S^1 \to G$, construct a topological principal G-bundle F_c over Σ as follows. Fix a base point $p \in \Sigma$ and take an open disc D_p in Σ around p. Fix a homotopy equivalence $h: D_p^* \to S^1$, where $D_p^* = D_p \backslash \{p\}$, and let $\bar{c}: D_p^* \to G$ be the composite $c \circ h$. Let $\Sigma^* := \Sigma\backslash\{p\}$. Take the trivial G-bundles

$$D_p \times G \to D_p \quad \text{and} \quad \Sigma^* \times G \to \Sigma^*$$

and 'clutch' them via $\bar{c}$ to get a topological G-bundle F_c over Σ, i.e.,

$$F_c := (D_p \times G) \sqcup (\Sigma^* \times G)/\sim,$$

where

$$(x,g) \in D_p \times G \sim (x, \bar{c}(x)g) \in \Sigma^* \times G, \quad \text{for} \quad x \in D_p^* \quad \text{and} \quad g \in G. \tag{1}$$

The projection $F_c \to \Sigma$ is obtained by the projections to the first factor. It can be seen that the topological G-bundle F_c (up to an isomorphism) does not depend upon the choices of c in its homotopy class, p, D_p and h (cf. Exercise 6.3.E.2). Thus, we get the 'clutching' map

$$\eta: [S^1, G] \to \mathrm{Bun}_G^{\mathrm{top}}(\Sigma), \;\; [c] \mapsto F_c,$$

where $[S^1, G]$ is the set of (free) homotopy classes of maps from $S^1 \to G$ and $\mathrm{Bun}_G^{\mathrm{top}}(\Sigma)$ is the set of isomorphism classes of topological principal G-bundles over Σ.

Lemma 6.3.10 *The above map*

$$\eta: [S^1, G] \mapsto \mathrm{Bun}_G^{\mathrm{top}}(\Sigma)$$

is a bijection. Of course, $[S^1, G]$ is bijective with $\pi_1(G)$.

Proof Since D_p is contractible, any G-bundle over D_p is trivial (cf. (Steenrod, 1951, Corollary 11.6)). Further, since Σ^* is homotopic to a 1-complex and the classifying space BG has trivial fundamental group (G being connected), any G-bundle over Σ^* is trivial. From this, we see that η is surjective.

To prove the injectivity of η, take two continuous maps $c_1, c_2 \colon S^1 \to G$ and assume that there exists a G-bundle isomorphism:

$$\begin{array}{ccc} F_{c_1} & \xrightarrow[\sim]{\varphi} & F_{c_2} \\ & \searrow \quad \swarrow & \\ & \Sigma\,. & \end{array}$$

For any $i = 1, 2$, take the section μ_i of $F_{c_i | D_p}$ given by

$$\mu_i(z) = (z, 1) \in D_p \times G, \text{ for any } z \in D_p$$

and the section σ_i of $F_{c_i | \Sigma^*}$ given by

$$\sigma_i(z) = (z, 1) \in \Sigma^* \times G, \text{ for any } z \in \Sigma^*.$$

Thus, by identity (1) of Definition 6.3.9,

$$\mu_i = \sigma_i \cdot \bar{c}_i \text{ over } D_p^*. \tag{1}$$

Also, let μ_2' (resp. σ_2') be the section of $F_{c_2 | D_p}$ (resp. $F_{c_2 | \Sigma^*}$) given by

$$\mu_2' := \varphi \circ \mu_1 \text{ and } \sigma_2' := \varphi \circ \sigma_1.$$

Then, we get continuous functions $\alpha \colon D_p \to G$ and $\beta \colon \Sigma^* \to G$ such that

$$\mu_2 = \mu_2' \cdot \alpha \text{ over } D_p \text{ and } \sigma_2 = \sigma_2' \cdot \beta \text{ over } \Sigma^*. \tag{2}$$

From relations (1) and (2), we get

$$\bar{c}_1(z) = \beta(z) \bar{c}_2(z) \alpha(z)^{-1} \text{ for } z \in D_p^*. \tag{3}$$

Since α is defined as a continuous function on D_p,

$$\alpha \colon D_p^* \to G \text{ is homotopically trivial.} \tag{4}$$

Similarly, since β is defined as a continuous function on Σ^*, $\beta_{|D_p^*} \colon D_p^* \to G$ is also homotopically trivial, as the following argument shows.

Denote $\beta_{|D_p^*} = \bar{\beta}$. By (Spanier, 1966, Chap. III, §8), the induced map $i_* \colon \pi_1(D_p^*, \infty) \to \pi_1(\Sigma^*, \infty)$ from the inclusion $i \colon D_p^* \to \Sigma^*$, where ∞ is any base point in the boundary of D_p, takes a generator

$$\sigma \in \pi_1(D_p^*, \infty) \mapsto [a_1, b_1] \dots [a_g, b_g] \in \pi_1(\Sigma^*, \infty)$$

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(cf. the beginning of this Section 6.3 for the description of $\pi_1(\Sigma^*, \infty)$). Thus, the composite map

$$\bar{\beta}_* = \beta_* \circ i_* : \pi_1(D_p^*, \infty) \to \pi_1(G, \beta(\infty))$$

is trivial since $\pi_1(G, \beta(\infty))$ is an abelian group. From this we conclude that

$$\bar{\beta} : D_p^* \to G \text{ is homotopically trivial.} \tag{5}$$

Combining (3)–(5), we conclude that c_1 and c_2 are homotopic, proving the injectivity of η. □

Lemma 6.3.11 *Let genus g of Σ be ≥ 2 and let G be a connected semisimple group. Then, for any $F \in \mathrm{Bun}_G^{\mathrm{top}}(\Sigma)$, there exists an irreducible (unitary) homomorphism $\rho_F : \pi_1(\Sigma) \to K \subset G$ such that $E_{\rho_F} \simeq F$ as topological G-bundles, where E_{ρ_F} is as in Definition 6.3.1.*

Proof Let $\alpha : \tilde{K} \to K$ be the simply-connected cover of K. Fix any $c \in \operatorname{Ker} \alpha$. Then, of course, c is a central element of $\tilde{K}$ (hence c belongs to any maximal torus of $\tilde{K}$) and K being semisimple, $\operatorname{Ker} \alpha$ is finite. We assert that there exists a group homomorphism

$$\tilde{\rho}_c : \pi_1(\Sigma^*) \to \tilde{K}, \quad \text{such that} \quad \tilde{\rho}_c(\mu) = c,$$

where, as earlier, $\Sigma^* := \Sigma \setminus p$, $\infty \in \Sigma$ is any point lying on the boundary ∂D_p of a small disc D_p around p in Σ, $\pi_1(\Sigma^*)$ denotes $\pi_1(\Sigma^*, \infty)$ and μ is a generator of $\operatorname{Ker} \pi_1(\Sigma^*) \to \pi_1(\Sigma, \infty)$ (which is an infinite cyclic group generated by orientation preserving homeomorphism $(S^1, 1) \simeq (\partial D_p, \infty) \subset \Sigma^*$).

Recall, from the beginning of this Section 6.3, that we have generators $\{a_i, b_i\}_{1 \leq i \leq g} \subset \pi_1(\Sigma^*)$ such that

$$\pi_1(\Sigma^*) = F(a_1, b_1, a_2, b_2, \ldots, a_g, b_g\} \tag{1}$$

and $\mu = \Pi_{i=1}^g [a_i, b_i]$. Fix a maximal torus $\tilde{T} \subset \tilde{K}$ and take a Weyl group element $w \in W$ such that the map

$$c_w : \tilde{T} \to \tilde{T}, \quad \tilde{t} \mapsto w\tilde{t}w^{-1}\tilde{t}^{-1}$$

has finite kernel. (Such a $w \in W$ exists; e.g., we can take w to be a Coxeter element.) Since c_w has finite kernel, it is surjective. Take $\tilde{t}_o \in \tilde{T}$ such that

$$c_w(\tilde{t}_o) = c. \tag{2}$$

Now, consider the homomorphism (since $g \geq 2$ by assumption)

$$\tilde{\rho}_c : \pi_1(\Sigma^*) \to \tilde{K},\ a_1 \mapsto \dot{w},\ b_1 \mapsto \tilde{t}_o,\ a_2 \mapsto \tilde{t}',\ b_2 \mapsto \tilde{t}',\ a_i, b_i \mapsto 1 \text{ for } i > 2,$$

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where $\tilde{t}' \in \tilde{T}$ is any element such that $\tilde{T}$ is the smallest closed subgroup containing $\tilde{t}'$ and $\dot{w}$ is a representative of w in the normalizer $N_{\tilde{K}}(\tilde{T})$ of $\tilde{T}$ in $\tilde{K}$. From (2), we see that $\tilde{\rho}_c(\mu) = c$. Since $c \in \operatorname{Ker}\alpha$, $\alpha \circ \tilde{\rho}_c : \pi_1(\Sigma^*) \to K$ descends to a homomorphism $\rho_c : \pi_1(\Sigma) \to K \subset G$. We next claim that ρ_c is an irreducible homomorphism, i.e., $\operatorname{Im}\rho_c$ is not contained in any proper parabolic subgroup P of G.

Assume, if possible, that

$$\operatorname{Im}\rho_c \subset P. \tag{3}$$

By the definition of ρ_c,

$$\overline{\operatorname{Im}\rho_c} \supset \{T, \overline{w}\}, \text{ where } T := \alpha(\tilde{T}) \text{ and } \overline{w} := \alpha(\dot{w}).$$

If (3) were true,

$$\{T, \overline{w}\} \subset P \cap K.$$

In particular, T is a maximal torus of $P \cap K$. Moreover, P being a proper parabolic subgroup, there exists a nontrivial connected subgroup $Z \subset T$ centralizing $P \cap K$. Thus, $\overline{w}$ commutes with Z. From this we see that c_w has infinite kernel contradicting the choice of w. Thus, (3) is not possible, i.e., ρ_c is irreducible.

Take any

$$c \in \operatorname{Ker}\alpha \simeq \pi_1(K) \simeq \pi_1(G)$$

and let $\tilde{\rho}_c : \pi_1(\Sigma^*) \to \tilde{K} \subset \tilde{G}$ be as above, where $\tilde{G}$ is the simply-connected cover of G. This descends to give an irreducible (unitary) homomorphism $\rho_c : \pi_1(\Sigma) \to K \subset G$. Let E_{ρ_c} be the corresponding stable G-bundle over Σ (cf. Proposition 6.3.4 for $A = (1)$). Then

$$E_{\rho_c} \simeq \eta(c), \text{ as topological } G\text{-bundles,} \tag{4}$$

where c also denotes the corresponding element of the fundamental group $\pi_1(K) \simeq \pi_1(G)$ (cf. Exercise 6.3.E.4).

Thus, the lemma follows from Lemma 6.3.10. □

Continuing the assumption at the beginning of this Section 6.3, let G be a connected complex reductive group and let K be a maximal compact subgroup. Let Σ be a smooth irreducible projective curve of any genus $g \geq 1$ and let $\rho : \pi_1(\Sigma) \to K \subset G$ be a (unitary) homomorphism. Let E_ρ be the associated holomorphic G-bundle over Σ (cf. Definition 6.3.1). For any finite-dimensional complex representation $V_{\mathbb{C}}$ of G, we denote the associated vector bundle $E_\rho \times^G V_{\mathbb{C}}$ by $E_\rho(V_{\mathbb{C}})$. With this notation, we have the following proposition.

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Proposition 6.3.12 *Let V be a finite-dimensional real representation of K. Then the natural map (induced from the sheaf embedding $L(V_\rho) \hookrightarrow E_\rho(V_{\mathbb{C}})$)*

$$i : H^1\left(\Sigma, L(V_\rho)\right) \to H^1\left(\Sigma, E_\rho(V_{\mathbb{C}})\right)$$

is an isomorphism of real vector spaces, where $L(V_\rho)$ is the local system over Σ obtained from the representation V of $\pi_1(\Sigma)$ through ρ, $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ is the complexification of V which is canonically a G-module (obtained as the complexification of the K-module V) and $H^1\left(\Sigma, E_\rho(V_{\mathbb{C}})\right)$ is the coherent cohomology of the vector bundle $E_\rho(V_{\mathbb{C}})$ over Σ.

Further, since by assumption, the genus g of Σ is at least one, there is a natural isomorphism

$$j : H^1\left(\pi_1(\Sigma), V_\rho\right) \simeq H^1\left(\Sigma, L(V_\rho)\right). \tag{1}$$

Proof Choose a Hermitian metric on the complex curve Σ (which is automatically Kähler) and a K-invariant positive-definite Hermitian form on $V_{\mathbb{C}}$. By the Hodge decomposition applied to the local system $L(V_\rho \otimes_{\mathbb{R}} \mathbb{C})$ over Σ (cf. (Griffiths and Harris, 1978, Chapter 0, §7) where the corresponding result for the trivial local system is proved; the proof applies equally to the local systems), we get (using the Dolbeault isomorphism, cf. (Griffiths and Harris, 1978, Chapter 0, §3))

$$\begin{aligned} H^1\left(\Sigma, L(V_\rho \otimes_{\mathbb{R}} \mathbb{C})\right) &\simeq H^1\left(\Sigma, E_\rho(V_{\mathbb{C}})\right) \oplus H^0\left(\Sigma, \Omega^1(\Sigma) \otimes E_\rho(V_{\mathbb{C}})\right) \\ &\simeq H^1\left(\Sigma, E_\rho(V_{\mathbb{C}})\right) \oplus \overline{H^1\left(\Sigma, E_\rho(V^*_{\mathbb{C}})\right)}, \\ &\simeq H^1\left(\Sigma, E_\rho(V_{\mathbb{C}})\right) \oplus \overline{H^1\left(\Sigma, E_\rho(V_{\mathbb{C}})\right)}, \end{aligned} \tag{2}$$

since V has a K-invariant positive-definite form, where $\Omega^1(\Sigma)$ is the sheaf of holomorphic 1-forms on Σ and $\bar{M}$ for a $\mathbb{C}$-vector space M denotes the same space as M wherein the complex multiplication is twisted by the conjugation.

From this it is easy to see that the natural $\mathbb{R}$-linear map $i : H^1(\Sigma, L(V_\rho)) \to H^1(\Sigma, E_\rho(V_{\mathbb{C}}))$ is injective. By the isomorphism (2),

$$\begin{aligned} \dim_{\mathbb{R}} H^1(\Sigma, L(V_\rho)) = \dim_{\mathbb{C}} H^1\left(\Sigma, L(V_\rho \otimes_{\mathbb{R}} \mathbb{C})\right) &= 2 \dim_{\mathbb{C}} H^1\left(\Sigma, E_\rho(V_{\mathbb{C}})\right) \\ &= \dim_{\mathbb{R}} H^1\left(\Sigma, E_\rho(V_{\mathbb{C}})\right). \end{aligned} \tag{3}$$

From the injectivity of i and the equality of the dimensions as in (3), we get that i is an isomorphism.

The isomorphism (1) follows from Cartan and Eilenberg (1956, Chap. XVI, §9) by using the contractibility of the simply-connected cover $\tilde{\Sigma}$ of Σ (since $g \geq 1$). □

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Remark 6.3.13 For a complete proof of Proposition 6.3.12, we refer to Narasimhan and Seshadri (1964, Proposition 4.4). Moreover, in the setting of Definition 6.3.3, for a (unitary) homomorphism $\hat{\rho}\colon \pi \to K$, the map i of the above Proposition 6.3.12 is A-equivariant and hence we get an isomorphism:

$$H^1\left(\hat{\Sigma}, L(V_\rho)\right)^A \to H^1\left(\hat{\Sigma}, \hat{E}_{\hat{\rho}}(V_{\mathbb{C}})\right)^A, \text{ where } \rho = \hat{\rho}_{|\pi_1(\hat{\Sigma})}.$$

Corollary 6.3.14 *Let $\rho\colon \pi_1(\Sigma) \to K$ be a (unitary) homomorphism and let* ad ρ *be the corresponding adjoint representation of $\pi_1(\Sigma)$ in $\mathfrak{k} :=$* Lie$K$. *As earlier, we assume that the genus g of Σ is at least* 1. *Then*

$$\dim H^1(\pi_1(\Sigma), \operatorname{ad}\rho) = 2\dim H^0(\pi_1(\Sigma), \operatorname{ad}\rho) + 2(\dim K)(g-1). \quad (1)$$

Further, ρ is irreducible if and only if

$$\dim H^0(\pi_1(\Sigma), \operatorname{ad}\rho) = \dim \mathfrak{z}, \quad (2)$$

where $\mathfrak{z}$ is the center of $\mathfrak{k}$.

Thus, ρ is irreducible if and only if

$$\dim H^1(\pi_1(\Sigma), \operatorname{ad}\rho) = 2\left((\dim K)(g-1) + \dim \mathfrak{z}\right). \quad (3)$$

Proof By Proposition 6.3.12,

$$\dim_{\mathbb{R}} H^1(\pi_1(\Sigma), \operatorname{ad}\rho) = 2\dim_{\mathbb{C}} H^1\left(\Sigma, \operatorname{ad} E_\rho\right), \quad (4)$$

where ad $E_\rho = E_\rho \times^G \mathfrak{g}$.

By the Riemann–Roch theorem (cf. (Fulton, 1998, Example 15.2.1))

$$\dim_{\mathbb{C}} H^0\left(\Sigma, \operatorname{ad} E_\rho\right) = \dim_{\mathbb{C}} H^1\left(\Sigma, \operatorname{ad} E_\rho\right) + (\dim K)(1-g), \quad (5)$$

since the adjoint action of G on $\wedge^{\text{top}}(\mathfrak{g})$ is trivial.

Combining (4) and (5), we get

$$\begin{aligned}\dim_{\mathbb{R}} H^1(\pi_1(\Sigma), \operatorname{ad}\rho) &= 2(\dim K)(g-1) + 2\dim_{\mathbb{C}} H^0\left(\Sigma, \operatorname{ad} E_\rho\right)\\ &= 2(\dim K)(g-1) + 2\dim_{\mathbb{C}} H^0(\pi_1(\Sigma), (\operatorname{ad}\rho)_{\mathbb{C}}),\\ &\qquad \text{by Lemma 6.3.6 for } A = (1)\\ &= 2(\dim K)(g-1) + 2\dim_{\mathbb{R}} H^0(\pi_1(\Sigma), \operatorname{ad}\rho).\end{aligned}$$

This proves (1).

We next prove (2). If $\dim H^0(\pi_1(\Sigma), \operatorname{ad}\rho) > \dim \mathfrak{z}$, then there exists a non-central element $x \in \mathfrak{k}$ fixed by $\pi_1(\Sigma)$. Thus, $\operatorname{Im}\pi_1(\Sigma) \subset Z_K(x)$, where $Z_K(x)$ is the centralizer of x in K, which is a proper Levi subgroup of K (since x is non-central) contained in a parabolic subgroup of G. Thus, ρ is not irreducible.

Conversely, if ρ is not irreducible, then $\operatorname{Im}\rho \subset K \cap P$, where P is a proper parabolic subgroup of G. But $K \cap P$ being compact, $K \cap P \subset L_P$, for some Levi subgroup L_P of P. Recall that for any Levi subgroup L_P of a proper parabolic subgroup P, the centralizer $\mathfrak{z}_{\mathfrak{k}}(K \cap L_P)$ of $K \cap L_P$ in $\mathfrak{k}$ satisfies

$$\mathfrak{z}_{\mathfrak{k}}(K \cap L_P) \supsetneqq \mathfrak{z}.$$

Hence

$$\dim H^0(\pi_1(\Sigma), \operatorname{ad}\rho) > \dim \mathfrak{z},$$

contradicting (2). This shows that ρ is irreducible and hence (2) is proved. Combining (1) and (2) we, of course, get (3). □

Let K be a compact connected Lie group. For any integer $g \geq 1$, let F_g be the free group on the symbols $\{a_1, b_1, a_2, b_2, \ldots, a_g, b_g\}$. Define the map

$$\beta\colon K^{2g} \to [K,K],\ \big((h_1,k_1),(h_2,k_2),\ldots,(h_g,k_g)\big) \mapsto \Pi_{i=1}^{g}[h_i,k_i].$$

Any $\bar{\rho} = \big((h_1,k_1),\ldots,(h_g,k_g)\big) \in K^{2g}$ determines a group homomorphism $\tilde{\rho}\colon F_g \to K$ taking $a_i \mapsto h_i$ and $b_i \mapsto k_i$. If $\bar{\rho} \in \beta^{-1}(e)$, then the homomorphism $\tilde{\rho}$ descends to a group homomorphism $\rho\colon \pi_1(\Sigma) \to K$, where g is the genus of Σ (cf. equation (*) at the beginning of this section).

Proposition 6.3.15 *For any $\bar{\rho} \in \beta^{-1}(e)$,*

$$\operatorname{Ker}((d\beta)_{\bar{\rho}}) \simeq Z^1(\pi_1(\Sigma), \operatorname{ad}\rho), \tag{1}$$

where $Z^1(\pi_1(\Sigma), \operatorname{ad}\rho)$ denotes the space of 1-cocycles of $\pi_1(\Sigma)$ with coefficients in $\operatorname{ad}\rho = \mathfrak{k}$ (in the standard cochain complex as in Serre (1997, Chap. I, §5.1)).

Proof For any $\bar{\sigma} \in K^{2g}$, the tangent space $T_{\bar{\sigma}}(K^{2g})$ is identified with $T_{\bar{e}}(K^{2g}) = \mathfrak{k}^{2g}$ under the right multiplication by $\bar{\sigma}^{-1}$, where $\bar{e} := ((e,e),(e,e),\ldots,(e,e))$ and similarly the tangent space $T_k(K)$ is identified with $T_e(K) = \mathfrak{k}$. For any $\alpha \in F_g$, define the function

$$\Phi_\alpha\colon K^{2g} \to K \quad \text{by} \quad \Phi_\alpha(\bar{\sigma}) = \tilde{\sigma}(\alpha). \tag{2}$$

Then, for $\alpha_1, \alpha_2 \in F_g$, clearly

$$\Phi_{\alpha_1\alpha_2}(\bar{\sigma}) = \Phi_{\alpha_1}(\bar{\sigma})\Phi_{\alpha_2}(\bar{\sigma}), \quad \text{for any} \quad \bar{\sigma} \in K^{2g}. \tag{3}$$

For any $\bar{\rho} \in K^{2g}$ and $v \in T_{\bar{\rho}}(K^{2g})$, define the function

$$\mathfrak{F}_v\colon F_g \to \mathfrak{k} \quad \text{by} \quad \mathfrak{F}_v(\alpha) = (d\Phi_\alpha)_{\bar{\rho}}(v),$$

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where we have identified $T_{\Phi_\alpha(\bar\rho)}(K) \simeq \mathfrak{k}$ as above. Then we claim that

$$\mathfrak{F}_v \in Z^1(F_g, \operatorname{ad}\tilde\rho). \tag{4}$$

For $\alpha_1, \alpha_2 \in F_g$, by (3),

$$\begin{aligned}\mathfrak{F}_v(\alpha_1\alpha_2) &= d(\Phi_{\alpha_1} \cdot \Phi_{\alpha_2})_{\bar\rho}(v)\\ &= (d\Phi_{\alpha_1})_{\bar\rho}(v) + \left(\operatorname{Ad}\Phi_{\alpha_1}(\bar\rho)\right) \cdot \left((d\Phi_{\alpha_2})_{\bar\rho}(v)\right),\end{aligned} \tag{5}$$

where the last equality follows from the following equality for any $\bar\sigma \in K^{2g}$:

$$\begin{aligned}&\Phi_{\alpha_1}(\bar\sigma)\Phi_{\alpha_2}(\bar\sigma) \cdot \Phi_{\alpha_2}(\bar\rho)^{-1}\Phi_{\alpha_1}(\bar\rho)^{-1}\\ &\quad = \left(\Phi_{\alpha_1}(\bar\sigma)\Phi_{\alpha_1}(\bar\rho)^{-1}\right) \cdot \left(\Phi_{\alpha_1}(\bar\rho) \cdot \left(\Phi_{\alpha_2}(\bar\sigma) \cdot \Phi_{\alpha_2}(\bar\rho)^{-1}\right) \cdot \Phi_{\alpha_1}(\bar\rho)^{-1}\right).\end{aligned}$$

Rewritten, the identity (5), of course, is the identity

$$\mathfrak{F}_v(\alpha_1\alpha_2) = \mathfrak{F}_v(\alpha_1) + (\operatorname{Ad}\tilde\rho(\alpha_1)) \cdot (\mathfrak{F}_v(\alpha_2)),$$

which proves (4).

Now, by definition,

$$\beta(\bar\sigma) = \Phi_{\Pi_{i=1}^g [a_i, b_i]}(\bar\sigma),$$

and hence, for any $v \in T_{\bar\rho}(K^{2g})$,

$$(d\beta)_{\bar\rho}(v) = \mathfrak{F}_v\left(\Pi_{i=1}^g [a_i, b_i]\right).$$

Thus,

$$v \in \operatorname{Ker}(d\beta)_{\bar\rho} \iff \mathfrak{F}_v\left(\Pi_{l=1}^g [a_i, b_i]\right) = 0. \tag{6}$$

So far, in the proof, we took an arbitrary $\bar\rho \in K^{2g}$. But now we take $\bar\rho \in \operatorname{Ker}\beta$ so that $\operatorname{ad}\tilde\rho$ is a $\pi_1(\Sigma)$-module. In this case, by (6) and (*) at the beginning of the section, we get a linear map

$$\mathfrak{F}\colon \operatorname{Ker}(d\beta)_{\bar\rho} \to Z^1(\pi_1(\Sigma), \operatorname{ad}\rho),\ v \mapsto \mathfrak{F}_v. \tag{7}$$

We claim that $\mathfrak{F}$ is an isomorphism.

Take $v \in \operatorname{Ker}\mathfrak{F}$, i.e., $\mathfrak{F}_v \equiv 0$. In particular, $\mathfrak{F}_v(a_i) = \mathfrak{F}_v(b_i) = 0$. By the definition of $\mathfrak{F}_v$, this gives that for all the coordinate projections $\pi_j\colon K^{2g} \to K$, $1 \le j \le 2g$, $(d\pi_j)_{\bar\rho}(v) = 0$. This, of courses, forces $v = 0$, i.e., $\mathfrak{F}$ is injective.

To prove that $\mathfrak{F}$ is surjective, take $\delta \in Z^1(\pi_1(\Sigma), \operatorname{ad}\rho)$. Since δ is a cocycle and $\{a_i, b_i\}$ generate $\pi_1(\Sigma)$, δ is completely determined by its values $\delta(a_i), \delta(b_i) \in \operatorname{ad}\rho$. Consider the vector

$$v = \left((\delta(a_1), \delta(b_1)), \ldots, (\delta(a_g), \delta(b_g))\right) \in \mathfrak{k}^{2g}.$$

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Then, it is easy to see from the definition of $\mathfrak{F}_v$ that

$$\mathfrak{F}_v(a_i) = \delta(a_i) \quad \text{and} \quad \mathfrak{F}_v(b_i) = \delta(b_i), \quad \text{for all } 1 \le i \le g.$$

But since both of δ and $\mathfrak{F}_v$ are cocycles for F_g with coefficients in $\operatorname{ad}\tilde{\rho}$, and they coincide on a_i and b_i, we get that $\delta = \mathfrak{F}_v$. Moreover, since $\delta \in Z^1(\pi_1(\Sigma), \operatorname{ad}\rho)$ and $\delta = \mathfrak{F}_v$ as cocycles for F_g, by (6), we get that $v \in \operatorname{Ker}(d\beta)_{\bar{\rho}}$. This proves the surjectivity of $\mathfrak{F}$ as well. Hence $\mathfrak{F}$ is an isomorphism, proving the proposition. □

Combining Corollary 6.3.14 and Proposition 6.3.15, we get the following result.

Corollary 6.3.16 *Let $\beta\colon K^{2g} \to [K, K]$ be the map given above Proposition 6.3.15. Take $\bar{\rho} \in \beta^{-1}(e)$. Then, $(d\beta)_{\bar{\rho}}$ is of maximal rank (equal to* $\dim[K, K]$*) if and only if the corresponding representation $\rho\colon \pi_1(\Sigma) \to K$ is irreducible.*

Thus,

$$M_g(K) := \{\bar{\rho} \in \beta^{-1}(e) : \rho \text{ is irreducible}\}$$

is an $\mathbb{R}$-analytic (smooth) manifold of dimension $(2g-1)\dim K + \dim \mathfrak{z}$ *with the tangent space at any $\bar{\rho}$ identified with* $Z^1(\pi_1(\Sigma), \operatorname{ad}\rho)$.

Proof By Proposition 6.3.15,

$$\begin{aligned}
\operatorname{rank}(d\beta)_{\bar{\rho}} &= 2g \dim K - \dim Z^1(\pi_1(\Sigma), \operatorname{ad}\rho) \\
&= 2g \dim K - \dim H^1(\pi_1(\Sigma), \operatorname{ad}\rho) \\
&\quad + \dim H^0(\pi_1(\Sigma), \operatorname{ad}\rho) - \dim K \\
&= (2g-1)\dim K - \dim H^0(\pi_1(\Sigma), \operatorname{ad}\rho) - 2(\dim K)(g-1), \\
&\qquad \text{by identity (1) of Corollary 6.3.14} \\
&= \dim K - \dim H^0(\pi_1(\Sigma), \operatorname{ad}\rho). \qquad (1)
\end{aligned}$$

By identity (2) of Corollary 6.3.14, ρ is irreducible if and only if

$$\dim H^0(\pi_1(\Sigma), \operatorname{ad}\rho) = \dim \mathfrak{z}.$$

Thus, by (1),

$$\operatorname{rank}(d\beta)_{\bar{\rho}} = \dim[K, K] \text{ if and only if } \rho \text{ is irreducible.}$$ □

An infinitesimal deformation map for a family of fiber bundles is defined in Kodaira and Spencer (1958a, §7). We recall the definition for a family of G-bundles over Σ parameterized by a smooth variety.

Definition 6.3.17 Let $\mathscr{E} \to \Sigma \times T$ be a family of G-bundles over Σ parameterized by a smooth variety T. For $t_o \in T$, any tangent vactor $v \in T_{t_o}(T)$ is given by $\theta_v\colon \operatorname{Spec}\mathbb{C}(\epsilon) \to T$ such that its restriction to $\operatorname{Spec}\mathbb{C}$ (under $\epsilon \mapsto 0$) corresponds to the point t_o (cf. Definition B.7). Thus, pulling $\mathscr{E}$ via $\hat{\theta}_v := \mathrm{Id}_\Sigma \times \theta_v$, we get a G-bundle denoted $\mathscr{E}_v$ over $\Sigma(\epsilon)$, where $\Sigma(\epsilon) := \Sigma \times \operatorname{Spec}\mathbb{C}(\epsilon)$. Clearly,

$$\mathscr{E}_{v|\Sigma} = \mathscr{E}_{t_o}, \text{ where } \mathscr{E}_{t_o} := \mathscr{E}_{|\Sigma \times t_o}. \tag{1}$$

Take an affine Zariski open cover $\{U_i\}_i$ of Σ such that $\mathscr{E}_{v|U_i(\epsilon)}$ are trivial. (This is possible by Ramanathan (1983, Proposition 4.3) for $\mathscr{E}_{t_o}$ and affineness of U_i gives the result for $\mathscr{E}_v$.) Taking sections $s_i^\epsilon \in \Gamma(U_i(\epsilon), \mathscr{E}_v)$, we get transition functions

$$g_{ij}^\epsilon \colon (U_i \cap U_j)(\epsilon) \to G \text{ given by } s_i^\epsilon g_{ij}^\epsilon = s_j^\epsilon.$$

Thus, $\{g_{ij}^\epsilon\}$ satisfy the cocycle condition

$$g_{ij}^\epsilon g_{jk}^\epsilon = g_{ik}^\epsilon : (U_i \cap U_j \cap U_k)(\epsilon) \to G. \tag{2}$$

Moreover, by (1), $g_{ij} := g^\epsilon_{ij|U_i \cap U_j}$ provide transition functions for the bundle $\mathscr{E}_{t_o}$. Let $U_i \cap U_j = \operatorname{Spec}(R_{ij})$ for a $\mathbb{C}$-algebra R_{ij} (observe that $U_i \cap U_j$ is affine by Hartshorne (1977, Chap. II, Exercise 4.3)). Then, we can view $g_{ij}^\epsilon \in G(R_{ij}(\epsilon))$, where $R_{ij}(\epsilon) := R_{ij} \otimes \mathbb{C}(\epsilon)$. Consider the exact sequence of groups (cf. Lemma B.11 and Definition B.15(b)):

$$\mathfrak{g} \otimes R_{ij} \xrightarrow{\iota} G(R_{ij}(\epsilon)) \xrightarrow{\theta_{ij}} G(R_{ij}),$$

where $\mathfrak{g} := \operatorname{Lie} G$ and θ_{ij} is induced by taking $\epsilon \mapsto 0$. By definition, $\theta_{ij}(g_{ij}^\epsilon) = g_{ij}$. Write

$$g_{ij}^\epsilon = \iota(h_{ij}) \cdot g_{ij}, \text{ where } h_{ij} : U_i \cap U_j \to \mathfrak{g}$$

and g_{ij} is thought of as an element of $G(R_{ij}(\epsilon))$ under the embedding $G(R_{ij}) \hookrightarrow G(R_{ij}(\epsilon))$ induced by $R_{ij} \hookrightarrow R_{ij}(\epsilon)$. Thus, by (2) and Definition B.17, we get the cocycle condition

$$h_{ik} = h_{ij} + \operatorname{Ad}(g_{ij})(h_{jk}), \text{ as morphisms } U_i \cap U_j \cap U_k \to \mathfrak{g}.$$

Hence $\{h_{ij}\}$ give rise to an element $\bar{\mathscr{E}}_v$ of $H^1(\Sigma, \operatorname{ad}\mathscr{E}_{t_o})$ in the Čech realization of cohomology (Hartshorne, 1977, Chap. III, §4). It is easy to see that the element $\bar{\mathscr{E}}_v \in H^1(\Sigma, \operatorname{ad}\mathscr{E}_{t_o})$ does not depend upon the choice of the open cover $\{U_i\}$ or the sections s_i^ϵ.

The *Kodaira–Spencer infinitesimal deformation map* of the family $\mathscr{E}$ at t_o is defined by

$$\eta\colon T_{t_o}(T) \to H^1(\Sigma, \operatorname{ad}\mathscr{E}_{t_o}),\ \ \eta(v) := \bar{\mathscr{E}}_v.$$

Then, η is a $\mathbb{C}$-linear map.

We can extend the above definition for any $\mathbb{R}$-analytic family of holomorphic G-bundles over Σ parameterized by a smooth $\mathbb{R}$-analytic space T to get an $\mathbb{R}$-linear map (as above) $\eta\colon T_{t_o}(T) \to H^1(\Sigma, \operatorname{ad}\mathscr{E}_{t_o})$, for any $t_o \in T$.

Let $R_K^s(g)$ be the set of irreducible homomorphisms from $\pi_1(\Sigma)$ to K. By Corollary 6.3.16, $R_K^s(g)$ is an $\mathbb{R}$-analytic (smooth) manifold. By Lemma 6.3.2, $R_G(g)$ parameterizes a 'universal' $\mathbb{C}$-analytic family $\theta\colon \mathscr{E} \to \Sigma \times R_G(g)$ of holomorphic G-bundles over Σ. Let us consider its restriction $\theta_K^s\colon \mathscr{E}_K^s \to \Sigma \times R_K^s(g)$ to $\Sigma \times R_K^s(g)$ giving rise to an $\mathbb{R}$-analytic family of holomorphic G-bundles over Σ parameterized by $R_K^s(g)$. The following proposition determines its infinitesimal deformation map $\eta = \eta(\theta_K^s)$.

Proposition 6.3.18 *For any $\rho \in R_K^s(g)$, the infinitesimal deformation map*

$$\eta\colon T_\rho\left(R_K^s(g)\right) \to H^1(\Sigma, \operatorname{ad} E_\rho)$$

coincides with the composition of the maps

$$T_\rho\left(R_K^s(g)\right) \underset{\sim}{\overset{\mathfrak{F}}{\to}} Z^1\left(\pi_1(\Sigma), \operatorname{ad}\rho\right) \overset{q}{\to} H^1\left(\pi_1(\Sigma), \operatorname{ad}\rho\right)$$
$$\underset{\sim}{\overset{j}{\to}} H^1(\Sigma, L(\operatorname{ad}\rho)) \underset{\sim}{\overset{i}{\to}} H^1(\Sigma, \operatorname{ad} E_\rho),$$

where the isomorphism $\mathfrak{F}$ is as defined in the proof of Proposition 6.3.15 (identifying $R_K^s(g)$ canonically with $M_g(K)$), the map q is the standard projection, $H^1(\Sigma, L(\operatorname{ad}\rho))$ denotes the singular cohomology of Σ with coefficients in the local system $L(\operatorname{ad}\rho)$ and the isomorphisms j and i are as in Proposition 6.3.12.

In particular, η is surjective.

Proof We identify $R_K^s(g)$ as the subset $M_g(K)$ of K^{2g}, taking

$$\sigma \mapsto (\sigma(a_1), \sigma(b_1), \ldots, \sigma(a_g), \sigma(b_g)).$$

Take a small enough finite open cover $\{U_k\}$ of Σ such that we can find a holomorphic section s_k of the simply-connected cover $\pi\colon \tilde{\Sigma} \to \Sigma$ over U_k. Thus, whenever $U_k \cap U_l \neq \emptyset$, we get an element $\alpha_{k,l} \in \pi_1(\Sigma)$ such that

$$s_l = s_k \cdot \alpha_{k,l}.$$

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Clearly, $\alpha_{k,l}$ satisfy the cocycle condition

$$\alpha_{k,l} \cdot \alpha_{l,m} = \alpha_{k,m} \quad \text{whenever} \quad U_k \cap U_l \cap U_m \neq \emptyset. \tag{1}$$

Take a tangent vector $v \in T_\rho(R^s_K(g))$. By the definition of $\mathfrak{F}$, $\mathfrak{F}_v(\alpha) = (d\Phi_\alpha)_\rho(v)$, where $\Phi_\alpha\colon R^s_K(g) \to K$ is the function $\Phi_\alpha(\sigma) = \sigma(\alpha)$. Take the cohomology class $\bar{\delta} \in H^1(\pi_1(\Sigma), \operatorname{ad}\rho)$ represented by a cocycle $\delta \in Z^1(\pi_1(\Sigma), \operatorname{ad}\rho)$. Then $j(\bar{\delta})$ is the cohomology class given by the Čech 1-cocycle

$$(U_k, U_l) \mapsto \left[s_{k|U_k \cap U_l}, \delta(\alpha_{k,l})\right] \in H^0\left(U_k \cap U_l, L(\operatorname{ad}\rho) := \tilde{\Sigma} \times^{\pi_1(\Sigma)} \operatorname{ad}\rho\right),$$

and so is the composite $i \circ j$ (cf. (Hartshorne, 1977, Chap. III, Lemma 4.4)). Thus, the composite map $i \circ j \circ q \circ \mathfrak{F}$ takes v to the cohomology class of $\operatorname{ad} E_\rho$ determined by the Čech 1-cocycle

$$(U_k, U_l) \mapsto \left[s_{k|U_k \cap U_l}, \left(d\Phi_{\alpha_{k,l}}\right)_\rho(v)\right], \tag{2}$$

where, as above, $\Phi_{\alpha_{k,l}}$ denotes the function $R^s_K(g) \to K, \sigma \mapsto \sigma(\alpha_{k,l})$ and $(d\Phi_{\alpha_{k,l}})_\rho(v) \in T_e(K) = \mathfrak{k} \subset \mathfrak{g}$ identifying $T_{\rho(\alpha_{k,l})}(K)$ with $T_e(K)$ under the right translation.

By the definition of the infinitesimal deformation map η as above in Definition 6.3.17, it can be seen that the cohomology class of the above Čech 1-cocycle (2) coincides with $\eta(v)$ (cf. Exercise 6.3.E.12). This proves the proposition. □

The following definition is an analogue of Kodaira and Spencer (1958b, Definition 2).

Definition 6.3.19 Let $\mathscr{F} \to \Sigma \times T$ be an $\mathbb{R}$-analytic family of holomorphic G-bundles over Σ (parameterized by an $\mathbb{R}$-analytic space T). Then, this family is said to be ($\mathbb{R}$-analytically) *complete* at $t_o \in T$ if for any $\mathbb{R}$-analytic family $\mathscr{F}' \to \Sigma \times T'$ with $t'_o \in T'$ such that $\mathscr{F}'_{t'_o} \simeq \mathscr{F}_{t_o}$, there exists an open neighborhood $U_{t'_o} \subset T'$ of t'_o and an $\mathbb{R}$-analytic map $f\colon U_{t'_o} \to T$ such that $f(t'_o) = t_o$ and the family $\mathscr{F}'$ restricted to $\Sigma \times U_{t'_o}$ is isomorphic to the pull-back family $(\mathrm{Id} \times f)^*(\mathscr{F})$.

The family $\mathscr{F}$ is called ($\mathbb{R}$-analytically) *complete* if it is complete at each $t \in T$.

We recall the following general result, the proof of which can be extracted from Ramanathan (1983, Remark 8.11) or Biswas and Ramanan (1994, Theorem 3.1) (also see the proof of Kodaira and Spencer (1958b, Theorem); and for vector bundles see the article by Nitsure (2009)). The result hinges upon the fact that $H^2(\Sigma, \operatorname{ad}\mathscr{F}_{t_o}) = 0$, since Σ is a curve.

Theorem 6.3.20 *Let $\mathscr{F} \to \Sigma \times T$ be an $\mathbb{R}$-analytic family of holomorphic G-bundles over Σ parameterized by an $\mathbb{R}$-analytic space T. Let $t_o \in T$ be a smooth point such that the infinitesimal deformation map*

$$T_{t_o}(T) \to H^1(\Sigma, \operatorname{ad} \mathscr{F}_{t_o})$$

is surjective. Then the family $\mathscr{F}$ is complete at t_o.

Conversely, if $\mathscr{F}$ is complete at t_o, then the above deformation map is surjective.

As a corollary of the above theorem and Proposition 6.3.18, we obtain the following result.

Corollary 6.3.21 *Let $\mathscr{F} \to \Sigma \times T$ be an $\mathbb{R}$-analytic family of holomorphic G-bundles over Σ. Then the subset*

$$T_o := \left\{ t \in T : \mathscr{F}_t \simeq E_\rho \text{ for some } \rho \in R^s_K(g) \right\}$$

is an open subset of T.

Proof Take $t_o \in T_o$ so that $\mathscr{F}_{t_o} \simeq E_\rho$ (for some $\rho \in R^s_K(g)$). By Theorem 6.3.20 and Proposition 6.3.18, the family $\theta^s_K : \mathscr{E}^s_K \to \Sigma \times R^s_K(g)$ (cf. the discussion above Proposition 6.3.18) is ($\mathbb{R}$-analytically) complete. Applying its completeness at ρ, we get that there exists an open subset $U_{t_o} \subset T$ containing t_o and an $\mathbb{R}$-analytic map $f : U_{t_o} \to R^s_K(g)$ such that the family $\mathscr{F}_{|\Sigma \times U_{t_o}}$ is isomorphic with the pull-back family $\bar{f}^*(\theta^s_K)$, where $\bar{f} := I_\Sigma \times f$. In particular, for any $t \in U_{t_o}$, $\mathscr{F}_t \simeq E_{\rho'}$ for some $\rho' \in R^s_K(g)$, i.e., $t \in T_o$. Thus, T_o is open in T proving the corollary. □

Lemma 6.3.22 *Let $f : \mathscr{V} \to \mathscr{W}$ be a nonzero $\mathscr{O}_\Sigma$-module homomorphism between two semistable vector bundles over Σ such that at least one of them is stable. Assume further that they both have the same rank and the same degree. Then f is an isomorphism.*

Proof We first recall the following general construction from Narasimhan and Seshadri (1965, §4).

For any nonzero $\mathscr{O}_\Sigma$-module homomorphism $f : \mathscr{E} \to \mathscr{F}$ between any two vector bundles (not necessarily of the same rank) over Σ, since the structure sheaf $\mathscr{O}_\Sigma$ is a sheaf of PIDs, f has the following canonical factorization (obtained from the following commutative diagram):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{E}_1 & \longrightarrow & \mathscr{E} & \xrightarrow[\pi]{} & \mathscr{E}_2 & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} & & \\
0 & \longleftarrow & \mathscr{F}_2 & \longleftarrow & \mathscr{F} & \xleftarrow[i]{} & \mathscr{F}_1 & \longleftarrow & 0
\end{array} \qquad (*)$$

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where $\mathscr{E}_i, \mathscr{F}_i (i = 1, 2)$ are vector bundles, the above rows are exact and f' is of maximal rank (i.e., $\mathscr{E}_2$ and $\mathscr{F}_1$ are of the same rank say r and the induced map $\wedge^r(\mathscr{E}_2) \to \wedge^r(\mathscr{F}_1)$ is nonzero). *Then, $\mathscr{F}_1$ (resp. $\mathscr{E}_1$) is called the vector subbundle of $\mathscr{F}$ (resp. $\mathscr{E}$) generated by the image (resp. kernel) of f.*

We now come to the proof of the lemma assuming that $\mathscr{W}$ is stable. If f is of maximal rank, since $\deg \mathscr{V} = \deg \mathscr{W}$, f is an isomorphism. This is because a nonzero section of a degree 0 line bundle over Σ is nowhere zero.

So, assume that f is not of maximal rank and consider the decomposition (*) for $f = i \circ f' \circ \pi : \mathscr{V} \xrightarrow{\pi} \mathscr{V}_2 \xrightarrow{f'} \mathscr{W}_1 \xrightarrow{i} \mathscr{W}$. Since $\deg(E \otimes F) = (\deg E)(\operatorname{rank} F) + (\deg F)(\operatorname{rank} E)$ for vector bundles E, F over Σ (as can easily be seen from the Chern character of $E \otimes F$),

$$0 = \deg\left(\mathscr{V}^* \otimes \mathscr{V}\right) = \deg\left(\mathscr{V}^* \otimes \mathscr{V}_1\right) + \deg\left(\mathscr{V}^* \otimes \mathscr{V}_2\right), \tag{1}$$

where $\mathscr{V}_1 := \operatorname{Ker} \pi$. Since $\mathscr{V}$ is semistable,

$$\deg\left(\mathscr{V}^* \otimes \mathscr{V}_1\right) = (\operatorname{rank} V) \cdot \deg(\mathscr{V}_1) - (\operatorname{rank} \mathscr{V}_1) \cdot (\deg \mathscr{V}) \le 0. \tag{2}$$

Thus, combining (1) and (2), we get

$$\deg(\mathscr{V}^* \otimes \mathscr{V}_2) \ge 0. \tag{3}$$

Since f' is of maximal rank, we get

$$\operatorname{rank} \mathscr{V}_2 = \operatorname{rank} \mathscr{W}_1 \quad \text{and} \quad \deg \mathscr{W}_1 \ge \deg \mathscr{V}_2. \tag{4}$$

Thus, $\mathscr{V}$ and $\mathscr{W}$ having the same rank by assumption,

$$\begin{aligned} \deg\left(\mathscr{W}^* \otimes \mathscr{W}_1\right) &= \deg\left(\mathscr{V}^* \otimes \mathscr{W}_1\right), \quad \text{since } \deg \mathscr{V} = \deg \mathscr{W} \text{ by assumption} \\ &\ge \deg\left(\mathscr{V}^* \otimes \mathscr{V}_2\right), \quad \text{by (4)} \\ &\ge 0, \quad \text{by (3)}. \end{aligned} \tag{5}$$

But $\mathscr{W}_1$ is a proper subbundle of the stable bundle $\mathscr{W}$ (since f is assumed to be not of maximal rank). Thus

$$\deg\left(\mathscr{W}^* \otimes \mathscr{W}_1\right) < 0. \tag{6}$$

Then (5) and (6) contradict each other, and hence f must be of maximal rank. But since $\deg \mathscr{V} = \deg \mathscr{W}$, f must be an isomorphism. This proves the lemma when $\mathscr{W}$ is stable.

The case when $\mathscr{V}$ is stable can be handled similarly. □

Definition 6.3.23 Let $\phi : G \to \mathrm{GL}_V$ be a finite-dimensional representation and let $V = \bigoplus_{i=1}^r V_i$ be its decomposition into irreducible components. Let $\phi_i : G \to \mathrm{GL}_{V_i}$ be the restriction of ϕ to V_i.

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Let

$$C := \{(z_1\phi_1(g), \dots, z_r\phi_r(g)) \in \mathrm{GL}_V : z_i \in \mathbb{C}^* \text{ and } g \in G\}.$$

Then C being the image of an algebraic group homomorphism $(\mathbb{C}^*)^r \times G \to \mathrm{GL}_V$, C is closed in GL_V. Let

$$\bar{C} = \text{closure of } C \text{ in } \operatorname{End} V.$$

Then, C (and hence $\bar{C}$) is stable under left and right multiplications by $\phi(g)$ for any $g \in G$.

Let E and E' be two G-bundles over Σ. Then their fiber product $F\colon E \underset{\Sigma}{\times} E' \to \Sigma$ is canonically a $G \times G$-bundle. Consider the vector bundle

$$\mathbf{Hom}\big(E(V), E'(V)\big) = F(\operatorname{End} V),$$

where $G \times G$ acts on $\operatorname{End} V$ via

$$(g,h) \cdot f = \phi(g) f \phi(h)^{-1}, \quad \text{for } g,h \in G \text{ and } f \in \operatorname{End} V. \tag{1}$$

The subsets $\operatorname{End} V_i$, C, $\bar{C} \subset \operatorname{End} V$ are clearly stable under the above action of $G \times G$ (where $\operatorname{End} V_i$ is a block of $\operatorname{End} V$ through the decomposition $V = \oplus V_i$). Thus, we get fiber subbundles $F(C) \subset F(\bar{C}) \subset F(\operatorname{End} V)$ and also the vector subbundle $F(\operatorname{End} V_i)$.

Proposition 6.3.24 *With the notation as above, let E be a stable G-bundle and E' a semistable G-bundle of the same topological type (i.e., they are topologically isomorphic). Let*

$$s = (s_1, \dots, s_r) \in H^0(\Sigma, F(\operatorname{End} V_1 \times \cdots \times \operatorname{End} V_r)) = \bigoplus_{i=1}^{r} H^0(\Sigma, F(\operatorname{End} V_i))$$

be such that $s(\Sigma) \subset F(\bar{C})$. Then, any s_i is either 0 or an isomorphism $E(V_i) \to E'(V_i)$.

Further, if each s_i is nonzero and if $\phi\colon G \to \mathrm{GL}_V$ is a faithful representation, then there exists $(z_1, \dots, z_r) \in (\mathbb{C}^*)^r$ such that the section $(z_1 s_1, \dots, z_r s_r)$ is induced from a G-bundle isomorphism $\bar{s}\colon E \to E'$.

Before we come to the proof of the proposition, we need the following two lemmas.

As earlier, we fix a maximal compact subgroup K of G and take $K' := [K,K]$ as a maximal compact subgroup of the commutator subgroup $G' := [G,G]$. Fix a maximal abelian subalgebra $\mathfrak{a}'$ of $\mathfrak{k}' := \operatorname{Lie} K'$. Then $\mathfrak{h}' := \mathfrak{a}' \oplus i\,\mathfrak{a}'$ is a Cartan subalgebra of $\mathfrak{g}' := \operatorname{Lie} G'$ and $\mathfrak{h} := \mathfrak{h}' \oplus z(\mathfrak{g})$ is a Cartan subalgebra of $\mathfrak{g} = \operatorname{Lie} G$, where $z(\mathfrak{g})$ is the center of $\mathfrak{g}$. Let $\Pi = \{\alpha_1, \dots, \alpha_\ell\} \subset \mathfrak{h}'^*$ be

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a system of simple roots of $\mathfrak{g}'$. Then they take real values on $i\mathfrak{a}'$. Let (M, ρ) be an irreducible representation of G with highest weight Λ (with respect to the above choice of simple roots). For any subset $\Phi \subset \Pi$, define the $\mathfrak{h}$-module projection

$$p_\Phi : M \to M^\Phi \subset M, \text{ where } M^\Phi := \oplus_{\lambda \in \left(\Lambda - \sum_{\alpha_i \in \Phi} \mathbb{Z}_+ \alpha_i\right)} M_\lambda, \; \mathbb{Z}_+ := \mathbb{Z}_{\geq 0}$$

and M_λ is the λth weight space of M. Recall that any weight λ also takes real values on $i\mathfrak{a}$, where $\mathfrak{a} := \mathfrak{a}' \oplus z(\mathfrak{k})$. Let

$$C_M := \{z\rho(g) : z \in \mathbb{C}^*, g \in G\} \subset \operatorname{End} M$$

and let $\bar{C}_M$ be its closure in $\operatorname{End} M$. With this notation, we have the following lemma (having fixed $\mathfrak{h}'$ as above).

Lemma 6.3.25 *For any $f \in \bar{C}_M$, there exists a system of simple roots $\Pi \subset \mathfrak{h}'^*$ (depending upon f) and a subset $\Phi \subset \Pi$ such that*

$$f = z\rho(g) p_\Phi \rho(g'), \text{ for some } g, g' \in G \text{ and } z \in \mathbb{C}. \tag{1}$$

Conversely, any element f of the form (1) *lies in $\bar{C}_M$.*
In particular,

$$\{Im\, f : f \in \bar{C}_M\} = \{gM^\Phi : g \in G, \Phi \subset \Pi \text{ and } \Pi \text{ ranges over systems of simple roots in } \mathfrak{h}'^*\}.$$

Proof Take a sequence

$$z_n \rho(g_n) \to f, \text{ for } z_n \in \mathbb{C}^* \text{ and } g_n \in G'.$$

(Observe that the center of G acts by a scalar on M due to Schur's lemma and hence we can choose $g_n \in G'$.) Decompose

$$g_n = k_n a_n k_n' \text{ with } k_n, k_n' \in K' \text{ and } a_n \in A',$$

where A' is the real subgroup of G' with Lie algebra $i\mathfrak{a}'$ (cf. (Knapp, 2002, Theorem 7.39)). Replacing g_n by a suitable subsequence, we can assume that $k_n \to k, k_n' \to k'$ and there exist h_n all belonging to the same Weyl chamber inside $i\mathfrak{a}'$ such that $a_n = \operatorname{Exp}(h_n)$. Let Π be the set of simple roots corresponding to this Weyl chamber. Thus, $\alpha_i(h_n) \geq 0$ for all $\alpha_i \in \Pi$ and all n. By passing to a further subsequence (if needed) and reordering the simple roots, let $0 \leq q \leq \ell$ be such that

$$\alpha_i(h_n) \to x_i, \text{ for } 1 \leq i \leq q, \text{ and } \alpha_i(h_n) \to \infty, \text{ for } q < i \leq \ell.$$

Take $h \in i\mathfrak{a}'$ such that

$$\alpha_i(h) = x_i, \text{ for } 1 \le i \le q$$
$$= 0, \text{ for } q < i \le \ell.$$

Then

$$f = \rho(k)\rho(\operatorname{Exp} h)\left(\lim_{n\to\infty} z_n\rho(a_n \operatorname{Exp}(-h))\right)\rho(k').$$

Thus

$$f = \rho(k)\rho(\operatorname{Exp} h) \lim_{n\to\infty} \left(z_n\rho(\operatorname{Exp}(\bar{h}_n))\right)\rho(k'), \tag{2}$$

where $\bar{h}_n := h_n - h \in i\mathfrak{a}'$ is such that

$$\alpha_i(\bar{h}_n) \to 0 \text{ for all } 1 \le i \le q \quad \text{and } \alpha_i(\bar{h}_n) \to \infty, \text{for } q < i \le \ell.$$

Let $\Phi := \{\alpha_1, \ldots, \alpha_q\}$. The operator $z_n\rho(\operatorname{Exp}(\bar{h}_n))$ restricted to any weight space M_λ is given by $z_n e^{\lambda(\bar{h}_n)}$. In particular, denoting

$$P_\lambda = \lim_{n\to\infty}\left(z_n\rho(\operatorname{Exp}(\bar{h}_n))_{|M_\lambda}\right), \tag{3}$$

we get

$$P_\lambda = \left(\lim_{n\to\infty} e^{(\lambda-\Lambda)(\bar{h}_n)}\right)P_\Lambda = P_\Lambda \cdot \mathrm{Id}_{M_\lambda}, \text{ if } \lambda \in \Lambda - \sum_{\alpha_i\in\Phi}\mathbb{Z}_+\alpha_i$$
$$= 0, \text{ otherwise}, \tag{4}$$

where, M_Λ being a 1-dimensional space, we think of P_Λ as a scalar. Combining (2)–(4), we get (1).

For the converse, choose $h_n \in i\mathfrak{a}$ such that

$$\alpha_i(h_n) = 0, \text{ for } \alpha_i \in \Phi \text{ and } \alpha_i(h_n) = n \text{ for } \alpha_i \notin \Phi.$$

Set $z_n = ze^{-\Lambda(h_n)}$. Then the sequence $z_n\rho(\operatorname{Exp}(h_n)) \to zp_\Phi$. By the $G \times G$-invariance of $\bar{C}_M$, we get that any f of the form (1) lies in $\bar{C}_M$. This proves the lemma. □

Definition 6.3.26 Let M and Φ be as above. Then a subspace $M' \subset M$ is said to be of *type* Φ if $M' = \rho(g)M^\Phi$ for some $g \in G$, where M^Φ is defined above Lemma 6.3.25.

Clearly, the set of subspaces of M of type Φ can be viewed as a G-stable subset of the Grassmannian $\mathrm{Gr}(d^\Phi, M)$ of d^Φ-dimensional subspaces of M, where $d^\Phi := \dim M^\Phi$.

For a G-bundle E over Σ, a vector subbundle of $E(M)$ is said to be of *type* Φ, if the fibers are subspaces of M of type Φ.

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Lemma 6.3.27 *Let E be a stable (resp. semistable) G-bundle over Σ and let M be an irreducible representation of G. Then, for any proper nonzero subbundle $\mathscr{W}$ of $E(M)$ of type Φ (for a subset $\Phi \subset \Pi$), we have*

$$\mu(\mathscr{W}) < \mu(E(M)) \quad (\text{resp. } \mu(\mathscr{W}) \leq \mu(E(M)))\,. \tag{1}$$

Proof Let P be the stabilizer of $M^\Phi \in \mathrm{Gr}(d^\Phi, M)$ in G. Then clearly P is a standard parabolic subgroup of G (with respect to the choice of simple roots Π). This gives an embedding

$$G/P \hookrightarrow \mathrm{Gr}(d^\Phi, M), \;\; gP \mapsto gM^\Phi. \tag{2}$$

The subbundle $\mathscr{W}$ gives rise to a section $\sigma_{\mathscr{W}}$ of $E\big(\mathbb{P}(\wedge^{d^\Phi} M)\big)$. Since $\mathscr{W}$ is of type Φ, $\sigma_{\mathscr{W}}$ lands inside $E(G/P)$ (under the embedding (2)), giving rise to a reduction of the structure group of the G-bundle E to P (cf. Lemma 5.1.2). Let χ_Φ (resp. χ) be the character of the action of P on $\wedge^{d^\Phi}(M^\Phi)$ (resp. $\wedge^r M$, where $r := \dim M$). Let Z be the center of G. Then, since Z acts on M via scalars, the character $\bar{\chi}_\Phi := \chi_\Phi^r \cdot \chi^{-d^\Phi}$ is trivial restricted to Z. Moreover, since the line $\wedge^{d^\Phi}(M^\Phi) \subset \wedge^{d^\Phi}(M)$ is P-stable, the character χ_Φ (and hence $\bar{\chi}_\Phi$) is dominant. If $P \cap G'$ were to act trivially on $\wedge^{d^\Phi}(M^\Phi)$, then the line $\wedge^{d^\Phi}(M^\Phi) \subset \wedge^{d^\Phi}(M)$ would be G-stable and hence M^Φ would be G-stable, i.e., $M^\Phi = 0$ or M. But since $\mathscr{W} \neq 0$ is a proper subbundle of $E(M)$, M^Φ cannot be (0) or M. Thus, $\bar{\chi}_\Phi$ is a nontrivial character of P. Hence, by Exercise 6.1.E.4, if E is stable (resp. semistable)

$$\deg \sigma_{\mathscr{W}}^* (\mathscr{L}_P(-\bar{\chi}_\Phi)) < 0 \;\; \big(\text{resp. } \deg \sigma_{\mathscr{W}}^* (\mathscr{L}_P(-\bar{\chi}_\Phi)) \leq 0\big)\,. \tag{3}$$

But

$$\deg \sigma_{\mathscr{W}}^* (\mathscr{L}_P(-\bar{\chi}_\Phi)) = r \deg \mathscr{W} - d^\Phi \deg E(M). \tag{4}$$

Combining (3) and (4), we get the lemma. □

Remark 6.3.28 By Theorem 6.1.7, if E is semistable, then so is $E(M)$. Thus, inequality (1) of Lemma 6.3.27 in the semistable case follows from the definition of semistable vector bundles.

We return now to the proof of Proposition 6.3.24.

Proof of Proposition 6.3.24 We fix an $1 \leq i \leq r$ and denote V_i by M. Let $s_i \neq 0$ and let $\mathscr{F}_1$ be the vector subbundle of $E'(M)$ generated by the image of s_i (cf. proof of Lemma 6.3.22). Thus, similar to the diagram (*) in the same, we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{E}_1 & \longrightarrow & E(M) & \xrightarrow[\pi]{} & \mathscr{E}_2 & \longrightarrow & 0 \\ &&&& \big\downarrow{\scriptstyle s_i} && \big\downarrow{\scriptstyle f'} && \\ 0 & \longleftarrow & \mathscr{F}_2 & \longleftarrow & E'(M) & \xleftarrow[i]{} & \mathscr{F}_1 & \longleftarrow & 0. \end{array} \qquad (\mathcal{D})$$

Let $d = \operatorname{rank} \mathscr{F}_1$ and let $\operatorname{Gr}(d, M)$ be the Grassmannian of d-dimensional subspaces of M. Then $\mathscr{F}_1$ can be thought of as a section of $E' \times^G \operatorname{Gr}(d, M)$. Let

$$\mathcal{A} := \{N \in \operatorname{Gr}(d, M) : N = \operatorname{Im} f_i, \text{ for some } f = (f_1, \ldots, f_r) \in \bar{C}\},$$

where $\bar{C}$ is as in Definition 6.3.23. Clearly, $\mathcal{A}$ is stable under the action of G on $\operatorname{Gr}(d, M)$ (since $\bar{C}$ is $G \times G$-stable). Moreover, by Lemma 6.3.25, the stabilizer of any $N \in \mathcal{A}$ is a parabolic subgroup (since so is for M^Φ). Thus, G-orbits in $\mathcal{A}$ are closed in $\operatorname{Gr}(d, M)$ and, by Lemma 6.3.25, $\mathcal{A}$ has finitely many G-orbits all of which are of the form $\{gM^\Phi\}_{g \in G}$, for some M^Φ. In particular, $\mathcal{A}$ is a closed subset of $\operatorname{Gr}(d, M)$, which is a finite disjoint union $\mathcal{A} = \sqcup \mathcal{A}_j$ of closed subsets, with each $\mathcal{A}_j$ (with reduced structure) isomorphic with G/P_j (for a parabolic subgroup P_j).

Since the fibers of $\mathscr{F}_1$ coincide with $\operatorname{Im} s_i$ on a dense open subset U of Σ (which of course is connected), we can think of $\operatorname{Im}(s_{i|U}) \in H^0\big(U, E'(\mathcal{A}_{j_o})\big)$ for some fixed j_o. But then $\mathscr{F}_1 \in H^0\big(\Sigma, E'(\mathcal{A}_{j_o})\big)$, $E'(\mathcal{A}_{j_o})$ being closed in $E'(\operatorname{Gr}(d, M))$. Therefore, by Lemma 6.3.27,

$$\mu(\mathscr{F}_1) \leq \mu(E'(M)), \tag{1}$$

since $s_i \neq 0$ and $\mathcal{A}_{j_o}$ consists of subspaces of M of fixed type Φ_{j_o}. Considering the dual of the diagram $(\mathcal{D})$, we get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{F}_2^* & \longrightarrow & E'(M^*) & \longrightarrow & \mathscr{F}_1^* & \longrightarrow & 0 \\ &&&& \big\downarrow{\scriptstyle s_i^*} && \big\downarrow && \\ 0 & \longleftarrow & \mathscr{E}_1^* & \longleftarrow & E(M^*) & \longleftarrow & \mathscr{E}_2^* & \longleftarrow & 0, \end{array} \qquad (\mathcal{D}_1)$$

where $s_i^* \colon E'(M^*) \to E(M^*)$ is the dual morphism. Similar to (1), using Lemma 6.3.27 again, since E is a stable G-bundle, we get

$$\mu(\mathscr{E}_2^*) < \mu(E(M^*)), \text{ if } d < \dim M. \tag{2}$$

But

$$\mu(E(M^*)) = -\mu(E(M)) = -\mu(E'(M)),$$

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since E and E' are of the same topological type (by assumption). Thus, (2) gives

$$\mu(E'(M)) < \mu(\mathscr{E}_2) \le \mu(\mathscr{F}_1), \tag{3}$$

where the last inequality follows since f' is of maximal rank. Now, (3) contradicts (1), proving that $d = \dim M$, i.e., $s_i : E(M) \to E'(M)$ is an isomorphism over a nonempty open subset U of Σ. Think of s_i as a section of the degree 0 line bundle $E(\wedge^d M)^* \otimes E'(\wedge^d M)$ over Σ which does not vanish over U and hence it must not vanish anywhere. Thus, s_i is an isomorphism. This proves the first part of the proposition.

We now prove the second part. By the first part, since each s_i is an isomorphism, and since C is closed in GL_V and $s(\Sigma) \subset F(\bar{C})$ (by assumption), we get that $s(\Sigma) \subset F(C)$, i.e., for any $x \in \Sigma$, there exists $\mathfrak{z}(x) = (z_1(x), \ldots, z_r(x)) \in (\mathbb{C}^*)^r$ such that

$$(z_1(x)s_1(x), \ldots, z_r(x)s_r(x)) \in F_x(\phi(G)),$$

where $\phi(G) \subset \operatorname{End} V$ is stable under $G \times G$-action on $\operatorname{End} V$ given by (1) of Definition 6.3.23. Let H be the closed subgroup of $(\mathbb{C}^*)^r$ defined by

$$H = \left\{(y_1, \ldots, y_r) \in (\mathbb{C}^*)^r : (y_1 \operatorname{Id}_{V_1}, \ldots, y_r \operatorname{Id}_{V_r}) \in \phi(G)\right\}.$$

Then, for any $\mathfrak{y} = (y_1, \ldots, y_r) \in (\mathbb{C}^*)^r$,

$$(z_1(x)y_1s_1(x), \ldots, z_r(x)y_rs_r(x)) \in F_x(\phi(G)) \iff \mathfrak{y} \in H.$$

Hence, the function

$$\hat{\mathfrak{z}} : \Sigma \to (\mathbb{C}^*)^r/H, \quad x \mapsto (z_1(x), \ldots, z_r(x)) \cdot H$$

is a well-defined morphism. But, since Σ is a projective variety and $(\mathbb{C}^*)^r/H$ is affine, the function $\hat{\mathfrak{z}}$ is a constant. Write

$$\hat{\mathfrak{z}}(x) = (z_1, \ldots, z_r) \cdot H, \quad \text{for any } x \in \Sigma,$$

where $(z_1, \ldots, z_r) \in (\mathbb{C}^*)^r$ is a fixed point. Thus, taking

$$\tilde{s} = (z_1s_1, \ldots, z_rs_r),$$

we get that $\tilde{s}(\Sigma) \subset F(\phi(G))$. Finally, since ϕ is an embedding, we get that $\tilde{s}$ is induced from a G-bundle isomorphism $\bar{s} : E \to E'$. This proves the proposition. □ □

We have the following general result.

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Lemma 6.3.29 *Let X and T be $\mathbb{C}$-analytic spaces with X compact and let $\{\mathscr{W}_i\}_{1\le i\le r}$ be $\mathbb{C}$-analytic vector bundles over $X \times T$. Let $\mathscr{W} = \bigoplus_{i=1}^r \mathscr{W}_i$ and let $\mathscr{C} \subset \mathscr{W}$ be a closed $\mathbb{C}$-analytic subset which is stable under the homothety action of $(\mathbb{C}^*)^r$ on $\mathscr{W}_1 \oplus \cdots \oplus \mathscr{W}_r$. Then*

(a) The set $S_{\mathscr{C}} := \bigcup_{t\in T} \left\{s^t \in H^0(X, \mathscr{W}_t) : s^t(X) \subset \mathscr{C}\right\}$ has a natural structure of a $\mathbb{C}$-analytic space such that the projection $S_{\mathscr{C}} \to T$ is holomorphic, where $\mathscr{W}_t := \mathscr{W}_{|X\times t}$.

Moreover, its subset

$$S'_{\mathscr{C}} := \bigcup_{t\in T} \big\{s^t \in S_{\mathscr{C}} : s^t = (s_1^t, \dots, s_r^t) \text{ has each } s_i^t \neq 0, \text{ where } s_i^t \in H^0(X, \mathscr{W}_{i,t})\big\}$$

is an open subset of $S_{\mathscr{C}}$. In fact, $S'_{\mathscr{C}}$ is the complement of a closed $\mathbb{C}$-analytic subset of $S_{\mathscr{C}}$.

(b) Consider the projectivization $\mathbb{P}(S'_{\mathscr{C}}) := \left\{[s^t] : t \in T \text{ and } s^t \in S'_{\mathscr{C}}\right\}$, where

$$[s^t] := \big([s_1^t], \dots, [s_r^t]\big) \quad \text{with } [s_i^t] \in \mathbb{P}\left(H^0(X, \mathscr{W}_{i,t})\right).$$

Then $\mathbb{P}(S'_{\mathscr{C}})$ has a natural structure of a $\mathbb{C}$-analytic space such that $S'_{\mathscr{C}} \to \mathbb{P}(S'_{\mathscr{C}})$ is a holomorphic submersion and $\mathbb{P}(S'_{\mathscr{C}}) \to T$ is a proper holomorphic map.

In particular, the set

$$\left\{t \in T : \exists\, s^t \in H^0(X, \mathscr{W}_t) \text{ with } s^t \in S'_{\mathscr{C}}\right\}$$

is a closed $\mathbb{C}$-analytic subset of T.

Proof (a) Let $\mathrm{Hol}(X, \mathscr{W})$ be the space of holomorphic maps from X to $\mathscr{W}$ with the topology of uniform convergence. Then $\mathrm{Hol}(X, \mathscr{W})$ has a natural structure of a $\mathbb{C}$-analytic space such that for any $\mathbb{C}$-analytic space Y, any map $Y \to \mathrm{Hol}(X, \mathscr{W})$ is holomorphic if and only if the corresponding map $Y \times X \to \mathscr{W}$ is holomorphic (cf. (Barlet and Magnússon, 2014, Chap. IV, §9.4)[1]; also see (Douady, 1966)). In particular, the evaluation map $\mathrm{Hol}(X, \mathscr{W}) \times X \to \mathscr{W}$ is holomorphic. Hence, the subspace $\mathrm{Hol}(X, \mathscr{C})$ is a (closed) $\mathbb{C}$-analytic subspace of $\mathrm{Hol}(X, \mathscr{W})$.

[1] We thank D. Barlet for this reference.

Consider the composite projections $\pi_X : \mathscr{W} \to X \times T \to X$ and $\pi_T : \mathscr{W} \to X \times T \to T$. Define

$$\mathrm{Hol}_o(X, \mathscr{W}) := \{\text{holomorphic } s \colon X \to \mathscr{W} : \pi_T \circ s \ \text{ is a constant}\}.$$

Then $\mathrm{Hol}_o(X, \mathscr{W})$ is a (closed) $\mathbb{C}$-analytic subspace of $\mathrm{Hol}(X, \mathscr{W})$, being the inverse image of the set of constant maps under the holomorphic map $\mathrm{Hol}(X, \mathscr{W}) \to \mathrm{Hol}(X, T)$, $s \mapsto \pi_T \circ s$ (cf. Exercise 6.3.E.5).

By definition,

$$S_{\mathscr{C}} = \{s \in \mathrm{Hol}_o(X, \mathscr{C}) : \pi_X \circ s = \mathrm{Id}_X\}. \tag{1}$$

Of course, the map $\hat{\pi}_X \colon \mathrm{Hol}_o(X, \mathscr{C}) \to \mathrm{Hol}(X, X)$ induced from π_X is holomorphic, since the corresponding map given by $\mathrm{Hol}(X, \mathscr{C}) \times X \to X$, $(s, x) \mapsto \pi_X(s(x))$, is holomorphic. Since $S_{\mathscr{C}} = (\hat{\pi}_X)^{-1}(\mathrm{Id}_X)$, $S_{\mathscr{C}}$ is a $\mathbb{C}$-analytic space.

The projection $S_{\mathscr{C}} \subset \mathrm{Hol}_o(X, \mathscr{W}) \to T$ is given by $s \mapsto \pi_T(s(x))$ for any (fixed) $x \in X$. Hence, it is holomorphic.

Considering the (closed) $\mathbb{C}$-analytic subset $S_{\mathscr{C}(i)}$ of $S_{\mathscr{C}}$ for $\mathscr{C}(i) = \mathscr{C} \cap \big(\mathscr{W}_1 \oplus \cdots \oplus \underline{0} \oplus \cdots \oplus \mathscr{W}_r\big)$, where $\underline{0}$ is the zero vector bundle over $X \times T$ placed in the ith slot, we get that $S_{\mathscr{C}} \backslash S'_{\mathscr{C}} = \bigcup_{i=1}^r S_{\mathscr{C}(i)}$ is a closed subset of $S_{\mathscr{C}}$ and hence $S'_{\mathscr{C}}$ is an open subset of $S_{\mathscr{C}}$. This proves the (a)-part of the lemma.

(b) The standard action of $(\mathbb{C}^*)^r$ on $S'_{\mathscr{C}}$ (by homothety in each factor $H^0(X, \mathscr{W}_{i,t})$) is, of course, fixed-point free. Moreover, it is holomorphic. This follows since

$$(\mathbb{C}^*)^r \times S_{\mathscr{W}} \times X \to \mathscr{W}, \ \big((z_1, \ldots, z_r), s^t, x\big) \mapsto \Sigma z_i s_i^t(x)$$

is holomorphic, where $S_{\mathscr{W}}$ is defined by (1) taking $\mathscr{C} = \mathscr{W}$ and $s^t = (s_1^t, \ldots, s_r^t)$ with $s_i^t \in H^0(X, \mathscr{W}_{i,t})$. Also, since $S'_{\mathscr{C}}$ consists of nonzero sections in each $\mathscr{W}_{i,t}$, the action of $(\mathbb{C}^*)^r$ on $S'_{\mathscr{C}}$ is proper. Hence, the orbit space $S'_{\mathscr{C}}/(\mathbb{C}^*)^r$ is a $\mathbb{C}$-analytic space and the quotient map $S'_{\mathscr{C}} \to S'_{\mathscr{C}}/(\mathbb{C}^*)^r$ is holomorphic submersion (cf. (Cartan, 1957)). In particular, the holomorphic map $S'_{\mathscr{C}} \to T$ which clearly descends to a map $S'_{\mathscr{C}}/(\mathbb{C}^*)^r \to T$ is holomorphic. Introduce a positive-definite continuous Hermitian form on the vector bundle $\mathscr{W}$. Then the subset

$$S'_{\mathscr{C}}(1) = \Big\{s^t = (s_1^t, \ldots, s_r^t) \in S'_{\mathscr{C}} \ \text{ with } \\ t \in T, \ s_i^t \in H^0(X, \mathscr{W}_{i,t}) \ \text{ and } \ ||s_i^t|| = 1\Big\},$$

where $||s_i^t|| := \sup_{x \in X} |s_i^t(x)|$, maps surjectively onto $S'_{\mathscr{C}}/(\mathbb{C}^*)^r$.

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By Montel's theorem (cf. (Rudin, 1966, Theorem 14.6)), the map $S'_{\mathscr{C}}(1) \to T$ is proper and hence so is the map $S'_{\mathscr{C}}/(\mathbb{C}^*)^r \to T$. Now, the (b)-part of the lemma follows since

$$S'_{\mathscr{C}}/(\mathbb{C}^*)^r \simeq \mathbb{P}(S'_{\mathscr{C}}).$$

The 'In particular' part of the lemma follows from Remmert's theorem asserting that the image of a proper holomorphic map (between $\mathbb{C}$-analytic spaces) is a (closed) $\mathbb{C}$-analytic subspace (cf. (Remmert, 1957)). □

As a consequence of Lemmas 6.3.2, 6.3.29 and Proposition 6.3.24, we get the following.

Proposition 6.3.30 *Let $\mathscr{F} \to \Sigma \times T$ be a $\mathbb{C}$-analytic family of stable G-bundles over Σ (parameterized by a $\mathbb{C}$-analytic space T). Then the subset*

$$T_u := \left\{t \in T : \mathscr{F}_t \simeq E_\rho \textit{ for some unitary representation } \rho \textit{ of } \pi_1(\Sigma) \textit{ in } G\right\}$$

is a closed subset of T.

Proof We can of course assume that T is connected so that each $\mathscr{F}_t$ is of the same topological type.

Recall the definition of the tautological family $\theta : \mathscr{E} \to \Sigma \times R_G(g)$ from Lemma 6.3.2. Consider the fiber product

$$\mathscr{H} := \mathscr{F} \underset{\Sigma}{\times} \mathscr{E} \to \Sigma \times T \times R_G(g) \text{ of}$$
$$\mathscr{F} \to \Sigma \times T \to \Sigma \text{ and } \mathscr{E} \to \Sigma \times R_G(g) \to \Sigma.$$

Then $\mathscr{H}$ is a family of $G \times G$-bundles over Σ parameterized by $T \times R_G(g)$ with fiber $(\mathscr{H})_{t,\rho} = \mathscr{F}_t \times E_\rho$. Choose a faithful representation $\phi : G \to \mathrm{GL}_V$ and consider the $G \times G$-stable subset $\bar{C} \subset \bigoplus_{i=1}^r \mathrm{End}\, V_i \subset \mathrm{End}\, V$ as in Definition 6.3.23. Applying Lemma 6.3.29 for $X = \Sigma$, T replaced by $T \times R_G(g)$, $\mathscr{W}_i = \mathscr{H}(\mathrm{End}\, V_i)$, $\mathscr{C} = \mathscr{H}(\bar{C})$, we get that the subset

$$F := \left\{(t,\rho) \in T \times R_G(g) : \exists\, s \in H^0\left(\Sigma, \mathscr{W}_{(t,\rho)}\right) \text{ with } s \in S'_{\mathscr{C}}\right\}$$

is a closed $\mathbb{C}$-analytic subset of $T \times R_G(g)$, where $S'_{\mathscr{C}}$ is as defined in Lemma 6.3.29. Hence, $F_K := F \cap (T \times R_K(g))$ is a closed $\mathbb{R}$-analytic subset of $T \times R_K(g)$, where $R_K(g)$ is as in Lemma 6.3.2. But the projection $p_T : T \times R_K(g) \to T$ is proper (since $R_K(g)$ is compact) and hence $p_T(F_K)$ is closed in T. Now, using Propositions 6.3.24 and 6.3.4(a) for $A = (1)$, we get that $T_u = p_T(F_K)$ and hence T_u is closed in T, proving the proposition. □

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Lemma 6.3.31 *Let $\mathscr{F} \to \Sigma \times T$ be a $\mathbb{C}$-analytic family of G-bundles over Σ (parameterized by a $\mathbb{C}$-analytic space T). Then the subset*

$$T_s := \{t \in T : \mathscr{F}_t \text{ is a stable } G\text{-bundle}\}$$

is an open subset which is the complement of a (closed) $\mathbb{C}$-analytic subset of T.

Proof Take a standard maximal parabolic subgroup Q_k of G and take the irreducible representation $V_k := V(d\omega_k)$ of G with highest weight $d\omega_k$, where ω_k is the kth fundamental weight required to vanish on the center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$ and $d\omega_k$ is a suitable positive multiple of ω_k so that it is a character of the maximal torus of G. Thus, we get an embedding $G/Q_k \hookrightarrow \mathbb{P}(V_k)$, $gQ_k \mapsto [gv_+]$, where v_+ is a highest-weight vector of V_k and $[gv_+]$ is the line through $g \cdot v_+$. Let $J = J_\Sigma$ be the Jacobian of Σ (i.e., the group of isomorphism classes of degree 0 line bundles over Σ) and $\mathcal{P} \to \Sigma \times J$ the *Poincaré line bundle* (cf. (Arbarello et al., 1985, Chap. IV, §2)). Define a $\mathbb{C}$-analytic family of vector bundles $\mathscr{F}_{\mathcal{P}}(V_k) \to \Sigma \times (J \times T)$ by

$$\mathscr{F}_{\mathcal{P}}(V_k)_{(j,t)} := j^* \otimes \mathscr{F}_t(V_k).$$

Consider the closed $\mathbb{C}$-analytic cone $\mathscr{C} \subset \mathscr{F}_{\mathcal{P}}(V_k)$ over $\pi^*(\mathscr{F}(G/Q_k)) \subset \pi^*(\mathbb{P}(\mathscr{F}(V_k))) = \mathbb{P}(\mathscr{F}_{\mathcal{P}}(V_k))$, where $\pi : \Sigma \times J \times T \to \Sigma \times T$ is the projection. Then, by Lemma 6.3.29, the subset

$$Z_k := \left\{(j,t) \in J \times T : \exists\, \mu \neq 0 \in H^0\left(\Sigma, \mathscr{F}_{\mathcal{P}}(V_k)_{(j,t)}\right),\ \mu(\Sigma) \subset \mathscr{C}\right\}$$

is a (closed) $\mathbb{C}$-analytic subset of $J \times T$. Let $\tilde{Z}_k \subset T$ be the image of Z_k under the projection $J \times T \to T$. Since J is compact, by a theorem of Remmert (cf. (Remmert, 1957)), $\tilde{Z}_k$ is a (closed) $\mathbb{C}$-analytic subset of T.

We next claim that

$$\tilde{Z}_k \subset T \backslash T_s. \tag{1}$$

Take $t \in \tilde{Z}_k$. Thus, there exists $j \in J$ such that there exists nonzero $\mu \in H^0(\Sigma, j^* \otimes \mathscr{F}_t(V_k))$ with $\mu(\Sigma) \subset \mathscr{C}$. Hence, μ gives rise to a section $\bar{\mu}$ of $\mathbb{P}(\mathscr{F}_t(V_k))$ over a nonempty Zariski open subset $U \subset \Sigma$ (where μ is nonzero) with the image contained in $\mathscr{F}_t(G/Q_k)$. Since Σ is a curve and the fibers of $\mathbb{P}(\mathscr{F}_t(V_k))$ are projective varieties, the section $\bar{\mu}$ extends holomorphically to the whole of Σ with the image contained in $\mathscr{F}_t(G/Q_k)$. The section $\bar{\mu}$ of course provides a reduction of the structure group of $\mathscr{F}_t$ to Q_k (cf. Lemma 5.1.2). Let $\tau \to \mathbb{P}(\mathscr{F}_t(V_k))$ be the tautological line bundle, the pull-back of which to $\mathscr{F}_t(G/Q_k)$ can easily be seen to be $\mathscr{L}_{Q_k}(-d\omega_k)$ (following the notation of Definition 6.1.3(c)).

Let $\bar{\mu}^*(\tau)$ be the pull-back line bundle over Σ. Then, the section μ thought of as a bundle morphism $j \to \mathscr{F}_t(V_k)$ has its image contained in the line bundle $\bar{\mu}^*(\tau)$. Thus, $j^* \otimes \bar{\mu}^*(\tau)$ has a nonzero section showing that $\deg \bar{\mu}^*(\tau) \geq 0$. Thus, $\mathscr{F}_t$ is not stable (cf. Definition 6.1.4(b)), i.e., $t \in T \backslash T_s$, proving (1).

Conversely, take $t \in T\backslash T_s$. Thus, there exists a standard maximal parabolic subgroup Q_k such that $\deg \bar{\mu}^* \mathscr{L}_{Q_k}(-d\omega_k) \geq 0$ for a section $\bar{\mu}$ of $\mathscr{F}_t(G/Q_k)$. Thus, there exists a $j \in J$ and a nonzero $\mathscr{O}_\Sigma$-module morphism $j \to \bar{\mu}^* \mathscr{L}_{Q_k}(-d\omega_k)$ over Σ such that the corresponding (nonzero) section

$$\sigma(j^* \otimes \bar{\mu}^* \mathscr{L}_{Q_k}(-d\omega_k)) \in H^0\left(\Sigma, \mathscr{F}_{\mathcal{P}}(V_k)_{(j,t)}\right)$$

has its image contained in $\mathscr{C}$. Hence, $t \in \tilde{Z}_k$, i.e.,

$$T\backslash T_s \subset \bigcup_k \tilde{Z}_k, \tag{2}$$

where $\{k\}$ parameterizes the standard maximal parabolic subgroups Q_k of G. Combining (1) and (2), we get the lemma. □

Lemma 6.3.32 *Let E_0 and E_1 be two holomorphic G-bundles of the same topological type over Σ. Then there exists a holomorphic family $\mathscr{E}$ of G-bundles parameterized by $\mathbb{C}$ such that*

$$\mathscr{E}_0 \simeq E_0 \quad \textit{and} \quad \mathscr{E}_1 \simeq E_1. \tag{1}$$

Further, if E_0 and E_1 are stable G-bundles, then such a holomorphic family $\mathscr{E}$ satisfying (1) *can be chosen over a nonempty connected open subset T of $\mathbb{C}$ containing* $\{0,1\}$ *such that $\mathbb{C}\backslash T$ is a (closed) $\mathbb{C}$-analytic subset of $\mathbb{C}$ and $\mathscr{E}_t$ is stable for each $E \in T$.*

Proof Let $E \to X$ be a C^∞ principal G-bundle over a holomorphic manifold X. Then a connection form ∇ over E (which is a $\mathfrak{g}$-valued C^∞ 1-form on E) induces a unique structure of holomorphic G-bundle on E satisfying Koszul (1960, Proposition 1, §6.4) if and only if the corresponding curvature form Ω satisfies $\Omega^{0,2} = 0$, where $\Omega^{0,2}$ is the component of Ω of type $(0,2)$ with respect to the holomorphic structure on X (cf. (Koszul, 1960, Proposition 3, §6.4)). Conversely, the structure of a holomorphic G-bundle on E gives rise to a (not necessarily unique) connection form ∇ on E with $\Omega^{0,2} = 0$ such that the corresponding holomorphic structure on E coincides with the original holomorphic structure (cf. (Koszul, 1960, §6.4)).

Taking $X = \Sigma$, since Σ is of complex dimension 1, the condition $\Omega^{0,2} = 0$ is automatically satisfied. Since the holomorphic bundles E_0 and E_1 are of the some topological type, we can assume that they correspond to (different)

https://doi.org/10.1017/9781108997003.008 Published online by Cambridge University Press

holomorphic structures on the same underlying C^∞ principal G-bundle E over Σ. Choose connection forms ∇_0 and ∇_1 on E which give rise to the holomorphic structures E_0 and E_1 respectively. Now, consider the C^∞ product G-bundle $E \times \mathbb{C} \to \Sigma \times \mathbb{C}$ and define the connection form ∇ on $E \times \mathbb{C}$ by $\nabla_z = z\nabla_1 + (1-z)\nabla_0$ for $z \in \mathbb{C}$, i.e.,

$$\nabla(w, v) = \nabla_z(w), \quad \text{for} \quad w \in T_e(E) \quad \text{and} \quad v \in T_z(\mathbb{C}).$$

Thus, the connection form $\nabla = z\pi_E^*(\nabla_1) + (1-z)\pi_E^*(\nabla_0)$, where $\pi_E \colon E \times \mathbb{C} \to E$ is the projection.

From the definition of the curvature: $\Omega = d\nabla + \frac{1}{2}[\nabla, \nabla]$, it is easy to see that $\Omega^{0,2} = 0$ for the above connection form ∇ on $E \times \mathbb{C}$. Thus, we get the structure of a holomorphic bundle on $\mathscr{E} := E \times \mathbb{C}$ such that the holomorphic structure restricted to $E \times 0$ (resp. $E \times 1$) is isomorphic with E_0 (resp. E_1). This proves the first part of the proposition.

The second part follows immediately from the first part and Lemma 6.3.31 by observing that the complement of a (closed) $\mathbb{C}$-analytic subset of $\mathbb{C}$ is automatically connected. □

Definition 6.3.33 A G-bundle E over Σ is said to be of *degree* 0 if for any character χ of G, the line bundle $E \times^G \mathbb{C}_\chi$ has degree 0, where $\mathbb{C}_\chi$ is the 1-dimensional representation of G given by the character χ. (This definition coincides with the definition of degree 0 vector bundles.)

With all these preparations, we are now ready to prove the following celebrated theorem.

Theorem 6.3.34 *Let G be a connected reductive group and Σ a smooth irreducible curve of genus $g \geq 2$. Let $E \to \Sigma$ be a stable G-bundle of degree* 0. *Then there exists a unique (up to conjugacy by G) irreducible unitary representation $\rho \colon \pi_1(\Sigma) \to G$ such that (as holomorphic bundles)*

$$E \simeq E_\rho.$$

Proof Let $Z_o := G/G'$, where G' is the commutator $[G, G]$. Then Z_o is a (connected) torus. Let $E(Z_o)$ be the bundle obtained from E by extension of the structure group $G \to Z_o$. We claim that $E(Z_o)$ is topologically trivial since E is of degree 0 (by assumption). To prove this, since Z_o is a torus, it suffices to observe that a degree 0 line bundle over Σ is topologically trivial.

The topological triviality of $E(Z_o)$ allows a topological reduction of the structure group of E to G', i.e., there is a topological G'-bundle E' which is isomorphic topologically with E under the extension of the structure group to G. By Lemma 6.3.11, since G' is semisimple, there exists an irreducible unitary representation $\rho_o\colon \pi_1(\Sigma) \to G'$ such that

$$E_{\rho_o} \simeq E' \text{ as topological } G'\text{-bundles}$$

and hence

$$E_{\rho_o}(G) \simeq E \text{ as topological } G\text{-bundles,}$$

where $E_{\rho_o}(G)$ denotes the extension of the structure group G' of E_{ρ_o} to G. Observe that ρ_o clearly remains irreducible considered as a homomorphism $\pi_1(\Sigma) \to G$.

Take a holomorphic family of stable G-bundles $\mathscr{E} \to \Sigma \times T$, such that T is a connected open subset of $\mathbb{C}$ containing $\{0, 1\}$ and $\mathscr{E}_0 \simeq E$, $\mathscr{E}_1 \simeq E_{\rho_o}(G)$ (cf. Lemma 6.3.32). Let

$$T_o := \{t \in T : \mathscr{E}_t \simeq E_\sigma, \\ \text{for some unitary irreducible representation } \sigma \text{ of } \pi_1(\Sigma) \text{ in } G\}.$$

Then, by Corollary 6.3.21, T_o is an open subset of T. Further, by Proposition 6.3.30, T_o is a closed subset of T. (Observe that if $\mathscr{E}_t \simeq E_\sigma$ for some unitary representation σ of $\pi_1(\Sigma)$, then σ is automatically irreducible by Proposition 6.3.4 for $A = (1)$ since each $\mathscr{E}_t$ is stable). Of course, T_o is nonempty since $1 \in T_o$. Thus, $T_o = T$. The uniqueness of ρ (up to conjugation by G) follows from Corollary 6.3.7 for $A = (1)$. This proves the theorem. □

Recall the definition of polystable bundles from Definition 6.1.4(c). Then we have the following generalization of Theorem 6.3.34.

Theorem 6.3.35 *Let G be a connected reductive group and let E be a holomorphic G-bundle over a smooth irreducible projective curve Σ of genus $g \geq 2$. Then E is polystable of degree 0 if and only if $E \simeq E_\rho$ (as holomorphic G-bundles) for a unitary representation $\rho\colon \pi_1(\Sigma) \to G$.*

Proof Assume first that E is polystable of degree 0. Then E admits a reduction E_L to a Levi subgroup L such that E_L is stable of degree 0 (as an L-bundle). (To prove this, observe that for any character χ of L, there exists a character χ' of G and a character χ'' of L trivial on the center of G such that

$$\chi^N = \chi'_{|L} \cdot \chi'' \quad \text{for some } N \gg 0.)$$

Thus, by Theorem 6.3.34,

$$E_L \simeq E_{\rho_L}, \tag{1}$$

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for an irreducible unitary representation $\rho_L : \pi_1(\Sigma) \to L$. Let ρ be the same representation thought of as $\pi_1(\Sigma) \to G$. Then, from (1), we get $E \simeq E_\rho$.

Conversely, take a unitary representation $\rho : \pi_1(\Sigma) \to K \subset G$. Then, if it is not irreducible, there exists a proper parabolic subgroup P of G and a Levi subgroup L_P of P such that

$$\operatorname{Im} \rho \subset L_P$$

(since $P \cap K$ is contained in a Levi subgroup of P). Continuing this way (inducting on the semisimple rank of G), we find a Levi subgroup L with $\operatorname{Im} \rho \subset L$ and $\rho_L : \pi_1(\Sigma) \to L$ is irreducible, where $\rho_L = \rho$. Thus, from Proposition 6.3.4 for $A = (1)$, E_{ρ_L} is a stable L-bundle. Further, since E_{ρ_L} has a discrete structure group (thereby a flat connection), by the Chern–Weil theory,

$$\deg \left(E_{\rho_L} \times^L \mathbb{C}_\chi \right) = 0, \text{ for any character } \chi \text{ of } L.$$

In particular, $E_{\rho_L}(G) = E_\rho$ is polystable of degree 0. This proves the theorem. □

Definition 6.3.36 Let Σ be a smooth irreducible projective curve, G a connected semisimple algebraic group and let $E \to \Sigma$ be a holomorphic G-bundle. Then a C^∞-connection ∇ on E is called

(a) *complex connection* if the corresponding holomorphic structure on E (cf. the proof of Lemma 6.3.32) coincides with the original holomorphic structure.

(b) *unitary connection* if there exists a C^∞-reduction $E_K \subset E$ of the structure group of E to a maximal compact subgroup K of G and ∇ is reducible to E_K (i.e., ∇ is obtained as the direct image of a C^∞-connection on E_K).

Observe that the condition of ∇ being unitary is equivalent to the requirement that the holonomy group of ∇ is relatively compact.

(c) *Einstein connection* if the curvature form of ∇ is identically zero.

(d) *Einstein–Hermitian connection* if it satisfies the above properties (a)–(c).

Observe that since E admits a C^∞-reduction $E_K \subset E$ of the structure group (G/K being contractible), E admits a unique complex unitary connection (cf. (Kobayashi and Nomizu, 1969, Theorem 10.1 on p. 178 and Remark on p. 185)).

Moreover, the existence of an Einstein–Hermitian connection on E is equivalent to the unitarity of E (as in Definition 6.3.1) using the Holonomy Theorem (Koszul, 1960, Chap. 4).

Recall the definition of A-unitary G-bundles and A-unitary vector bundles from Definition 6.3.3, the notation of which we will follow.

Lemma 6.3.37 *Let $\hat{E}$ be an A-equivariant G-bundle over $\hat{\Sigma}$, where G is a connected semisimple group. Then $\hat{E}$ is A-unitary if and only if $\hat{E}(\mathfrak{g})$ is A-unitary vector bundle.*

The lemma is clearly false if G were a torus.

Proof Of course, if $\hat{E}$ is A-unitary, then so is $\hat{E}(\mathfrak{g})$. Conversely, assume that $\hat{E}(\mathfrak{g})$ is A-unitary. Then we show that $\hat{E}$ is A-unitary.

The bracket $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, $x \otimes y \mapsto [x, y]$, being G-equivariant, induces an A-equivariant bundle morphism

$$\varphi \colon \hat{E}(\mathfrak{g} \otimes \mathfrak{g}) \to \hat{E}(\mathfrak{g})$$

between A-unitary bundles. By Lemma 6.3.6, the bracket map $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ must be π-equivariant, where the action $\hat{\rho}$ of π on $\mathfrak{g}$ comes from the assumption that $\hat{E}(\mathfrak{g})$ is A-unitary. Thus, the representation $\hat{\rho} \colon \pi \to \mathrm{Aut}(\mathfrak{g})$ has its image inside $G_F := \mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{g})$, where $\mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{g})$ is the group of Lie algebra automorphisms of $\mathfrak{g}$. Thus, $\hat{E}(G_F)$ is A-unitary.

Assume now that G is of adjoint type (i.e., its center is trivial). Then we have the exact sequence of groups

$$1 \to G \to G_F \to F \to 1, \tag{*}$$

where F is the (finite) group of outer automorphisms of $\mathfrak{g}$ (its finiteness follows from the Whitehead Lemma (Hilton and Stammbach, 1997, Chap. VII, Proposition 6.1)). Since $\hat{E}(G_F)(F)$ admits a canonical A-equivariant section (coming from the embedding $\hat{E} = \hat{E}(G) \hookrightarrow \hat{E}(G_F)$), $\hat{E}(G_F)(F)$ being an A-equivariant principal F-bundle, it is A-equivariantly trivial. Hence, the composite map

$$\pi \xrightarrow{\hat{\rho}} G_F \to F$$

is trivial (use Corollary 6.3.7), i.e., $\hat{\rho}(\pi) \subset G$, which proves that $\hat{E}$ is A-unitary (in the case G is an adjoint group).

We now prove the A-unitarity of $\hat{E}$ when G is an arbitrary connected semisimple group. Consider the exact sequence

$$1 \to Z \to G \to G_{\mathrm{ad}} \to 1,$$

where Z is the center of G and G_{ad} is the corresponding adjoint group. We have already established that $\hat{E}(G_{\mathrm{ad}})$ is A-unitary. Thus, the A-unitarity of $\hat{E}$ follows from the following lemma. □

Lemma 6.3.38 *We follow the notation as in the above Lemma 6.3.37. Let $G \to H$ be a surjective morphism of connected semisimple algebraic groups with finite kernel. Let $\hat{E} = \hat{E}(G)$ be an A-equivariant G-bundle over $\hat{\Sigma}$ such that $\hat{E}(H)$ is A-unitary. Then so is $\hat{E}$.*

Proof Since $\hat{E}(H)$ is A-unitary, in particular unitary, it admits a unique Einstein–Hermitian connection ∇_H (cf. Definition 6.3.36). Moreover, by its uniqueness, ∇_H is A-invariant. Let ∇_G be the connection induced from ∇_H on $\hat{E}$ (using the isomorphism of tangent spaces of G and H). Then it is easy to see that ∇_G is an A-invariant Einstein–Hermitian connection (cf. (Ramanathan and Subramanian, 1988, Lemma 2)). Thus the bundle $\hat{E}$ is given by a representation of the fundamental group

$$\pi_1(\hat{\Sigma}) \to K \quad \text{for a maximal compact subgroup } K.$$

Moreover, since $\hat{E}$ is an A-equivariant G-bundle, by Lemma 6.3.8 we get that $\hat{E}$ is A-unitary. □

Let $\hat{\Sigma}$ be an irreducible smooth projective curve with faithful action of a finite group A and let G be a connected reductive group. The following equivariant generalization of Theorem 6.1.7 holds by the same proof.

Lemma 6.3.39 *Let $f: G \to G'$ be a homomorphism between connected reductive groups such that $f(Z^o(G)) \subset Z^o(G')$, where $Z^o(G)$ denotes the identity component of the center of G. Then, if $\hat{E} \to \hat{\Sigma}$ is a A-semistable (resp. A-polystable) G-bundle, then so is $\hat{E}(G')$ obtained from $\hat{E}$ by extension of the structure group to G'.*

In particular, for any A-semistable (resp. A-polystable) G-bundle $\hat{E}$, $\operatorname{ad} \hat{E}$ *is an A-semistable (resp. A-polystable) vector bundle.*

Lemma 6.3.40 *Let $\hat{\Sigma}, G$ and A be as above but we assume that $\hat{\Sigma}$ has genus $\hat{g} \geq 2$. Let $\hat{E}$ be an A-polystable G-bundle over $\hat{\Sigma}$. Then, $\hat{E}$ is polystable.*

In particular, $\hat{E}$ is A-semistable.

Proof[2] Observe first that $\hat{E}$ is A-polystable (resp. polystable) if and only if $\hat{E}(G/Z)$ is A-polystable (resp. polystable), where Z is the center of G (cf. Exercise 6.1.E.12). Thus, we can assume that G is semisimple. By Lemma 6.3.39, $\operatorname{ad} \hat{E}$ is A-polystable vector bundle. By Exercise 6.1.E.16, we can write $\operatorname{ad} \hat{E} = \oplus_i \mathscr{V}_i$, where $\mathscr{V}_i$ are A-stable vector subbundles of $\operatorname{ad} \hat{E}$ all with the same slope; in particular, $\mathscr{V}_i$ are A-semistable and hence semistable by Exercise 6.2.E.4. Fix any $\mathscr{V} = \mathscr{V}_i$ and let $\mathscr{V}^o$ be the socle of $\mathscr{V}$, which is an

[2] We thank V. Balaji for this proof.

https://doi.org/10.1017/9781108997003.008 Published online by Cambridge University Press

A-equivariant vector subbundle of $\mathscr{V}$ such that the slope $\mu(\mathscr{V}^o) = \mu(\mathscr{V})$ (cf. (Mehta and Ramanathan, 1984, Definition 2.1 and Lemma 2.2)). If $\mathscr{V}$ were not polystable, then by the same reference and Exercise 6.1.E.15, $\mathscr{V}^o \subsetneq \mathscr{V}$ and since $\mu(\mathscr{V}^o) = \mu(\mathscr{V})$, it contradicts the A-stability of $\mathscr{V}$. Hence, $\mathscr{V}$ is polystable and hence so is ad $\hat{E}$. Clearly, ad $\hat{E}$ is of degree 0. Thus, by Theorem 6.3.35, ad $\hat{E}$ is a unitary vector bundle. Hence, by Lemma 6.3.37, $\hat{E}$ is a unitary G-bundle, and thus is polystable by Theorem 6.3.35.

The 'In particular' part of the lemma follows from Definition 6.1.4(c) and Exercise 6.2.E.4. □

We now come to the following equivariant generalization of Theorem 6.3.35.

Theorem 6.3.41 *Let $\hat{\Sigma}$ be an irreducible smooth projective curve with faithful action of a finite group A such that $\Sigma := \hat{\Sigma}/A$ has genus $g \geq 2$ and G a connected reductive group. Then an A-equivariant G-bundle $\hat{E}$ over $\hat{\Sigma}$ is A-unitary if and only it is A-polystable of degree* 0.

In particular, an A-equivariant G-bundle over $\hat{\Sigma}$ is A-polystable if and only it is polystable.

Proof Assume first that $\hat{E}$ is A-unitary, i.e., there is a unitary homomorphism $\hat{\rho}\colon \pi \to G$ (following the notation of Definition 6.3.3) with

$$\hat{E} \simeq \hat{E}_{\hat{\rho}}, \text{ as } A\text{-equivariant } G\text{-bundles.} \tag{1}$$

Then, as in the proof of Theorem 6.3.35, there exists a Levi subgroup L with $\operatorname{Im}\hat{\rho} \subset L$ and $\hat{\rho}_L\colon \pi \to L$ is irreducible, where $\hat{\rho}_L := \hat{\rho}$. Thus, the corresponding bundle $\hat{E}_{\hat{\rho}_L}$ is A-stable by Proposition 6.3.4(b). Moreover, for any character χ of L,

$$\deg\left(\hat{E}_{\hat{\rho}_L} \times^L \mathbb{C}_\chi\right) = 0,$$

since $\hat{E}_{\hat{\rho}_L}$ has discrete structure group. Thus $\hat{E}$ is A-polystable of degree 0 by (1).

Conversely, assume that $\hat{E}$ is A-polystable of degree 0. Then, by Lemma 6.3.40, $\hat{E}$ is polystable (of degree 0). (Observe that $\hat{\Sigma}$ has genus $\hat{g} \geq 2$ by (Hartshorne, 1977, Chap. IV, Example 2.5.4) since $g \geq 2$ by assumption.) Now, by Theorem 6.3.35, as G-bundles,

$$\hat{E} \simeq E_\rho, \text{ for a unitary homomorphism } \rho\colon \pi_1(\hat{\Sigma}) \to G.$$

But, since $\hat{E}$ is an A-equivariant G-bundle, by Lemma 6.3.8, we get that ρ lifts to a unitary homomorphism $\hat{\rho}\colon \pi \to G$ such that

$$\hat{E} \simeq \hat{E}_{\hat{\rho}}, \text{ as } A\text{-equivariant } G\text{-bundles.}$$

Thus, $\hat{E}$ is A-unitary, proving the first part of the theorem.

We now prove the 'In particular' part. Of course, by Lemma 6.3.40, if $\hat{E}$ is A-polystable then it is polystable. For the converse part, using Exercise 6.1.E.12, we can assume that G is semisimple. Now, if $\hat{E}$ is polystable, then by Theorem 6.3.35 and Lemma 6.3.8, $\hat{E}$ is A-unitary. Thus, by the first part of the theorem, $\hat{E}$ is A-polystable. This proves the theorem. □

Let $\hat{\Sigma}$ and A be as in Theorem 6.3.41 (in particular, Σ has genus ≥ 2) and G a connected (not necessarily simply-connected) semisimple group. Using Theorem 6.3.41 and Lemma 6.3.37, we get the following generalization of Lemma 6.3.37.

Proposition 6.3.42 *Let $\hat{E}$ be an A-equivariant G-bundle over $\hat{\Sigma}$ and let $\theta : G \to \mathrm{GL}_V$ be a representation with finite kernel. Then the vector bundle $\hat{E}(V)$ is A-unitary if and only if $\hat{E}$ is A-unitary.*

Proof Clearly, if $\hat{E}$ is A-unitary, then so is $\hat{E}(V)$.

Conversely, assume that $\hat{E}(V)$ is A-unitary. Then so is $\hat{E}(W)$ for any GL_V-module W. In particular, for $W := V^* \otimes V = \mathrm{End}_V$, $\hat{E}(W)$ is A-unitary. Consider the G-module embedding

$$d\theta : \mathfrak{g} \hookrightarrow \mathrm{End}_V .$$

(Observe that $d\theta$ is injective since θ has finite kernel.) Take a G-submodule M of End_V such that

$$\mathrm{End}_V \simeq \mathfrak{g} \oplus M, \quad \text{as} \quad G\text{-modules.} \tag{1}$$

The bundle $\hat{E}(W)$ breaks up as a direct sum of A-equivariant bundles:

$$\hat{E}(W) = \hat{E}(\mathfrak{g}) \oplus \hat{E}(M), \tag{2}$$

obtained from the decomposition (1). Since $\hat{E}(W)$ is A-unitary (since so is $\hat{E}(V)$), by Theorem 6.3.41 for GL_W, $\hat{E}(W)$ is A-polystable of degree 0. Decompose

$$\hat{E}(W) = \bigoplus_{i=1}^{k} V_i,$$

where each V_i is an A-stable vector bundle of degree 0 (cf. Exercise 6.1.E.16). Let $\pi : \hat{E}(W) \to \hat{E}(\mathfrak{g})$ be the projection obtained from the decomposition (2) and choose the smallest subset $S \subset \{1, \ldots, k\}$ such that $\pi_{|V_S} : V_S \to \hat{E}(\mathfrak{g})$ is surjective, where $V_S := \bigoplus_{i \in S} V_i$. We claim that $\pi_{|V_S}$ is an isomorphism.

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Let K_S be the kernel of $\pi_{|V_S}$. Then clearly K_S is an A-equivariant vector bundle of degree 0. For any $i \in S$, let $\pi_i : K_S \to V_i$ be the projection on the ith factor. Then either $\pi_i \equiv 0$ or π_i is surjective since $\deg(K_S) = \deg V_i = 0$ and V_i is A-stable (cf. Exercise 6.3.E.11). (Observe that K_S is A-semistable since it is a degree 0 subbundle of an A-semistable vector bundle V_S of degree 0.) We next show that $\pi_i \equiv 0$ for all $i \in S$. For, if not, assume that $\pi_i \neq 0$ for some i and hence it is surjective. Thus, for any $y \in V_i$ we can choose $x \in K_S$ such that $\pi_i(x) = y$. Decompose (obtained from the decomposition $V_S = \bigoplus_{i \in S} V_i$):

$$x = \sum_{j \in S} x_j, \quad \text{with} \quad x_j \in V_j \quad \text{so that} \quad x_i = y.$$

Hence,

$$0 = \pi(x) = \pi(y) + \sum_{\substack{j \neq i \\ j \in S}} \pi(x_j).$$

This gives

$$\pi(V_i) \subset \pi \left(\bigoplus_{\substack{j \in S \\ j \neq i}} V_j \right).$$

This contradicts the minimality of S, proving that $\pi_i \equiv 0$ for all $i \in S$, i.e., $K_S = (0)$. This proves that $\hat{E}(\mathfrak{g}) \simeq \bigoplus_{i \in S} V_i$ and hence $\hat{E}(\mathfrak{g})$ is A-polystable of degree 0. Thus, by Theorem 6.3.41 for $G = \mathrm{GL}_{\mathfrak{g}}$, $\hat{E}(\mathfrak{g})$ is A-unitary. Thus, the proposition follows from Lemma 6.3.37. □

Remark 6.3.43 For any adjoint simple group G not of type $\mathrm{PGL}(n)$, there exist semistable but not stable G-bundles of any topological type. For a proof of a more general result see Ramanathan (1975, Proposition 7.8).

We end the chapter with the following result.

Lemma 6.3.44 *Let G be a connected reductive group and let $\pi : E \to \Sigma$ be a semistable G-bundle. Then, for any $p \in \Sigma$, the restriction map*

$$\mathrm{Aut}(E) \to \mathrm{Aut}(E_p)$$

is injective, where $\mathrm{Aut}(E)$ *denotes the group of automorphisms of the bundle E inducing the identity over the base and $E_p := E_{|p}$.*

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Proof Let Z be the center of G. Then we can think of Z as a central subgroup of $\mathrm{Aut}(E)$ by taking the embedding

$$\delta\colon Z \hookrightarrow \mathrm{Aut}(E),\ \ \delta(g)(e) = e \cdot g,\ \text{ for } e \in E, g \in Z.$$

We have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Aut}(E)/Z & \xrightarrow{\ \theta\ } & \mathrm{Aut}(\mathrm{ad}\, E) \\ \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_2} \\ \Pi_{x\in\Sigma}\ (\mathrm{Aut}(E_x)/Z) & \xrightarrow{\ \hat{\theta}\ } & \Pi_{x\in\Sigma}\ (\mathrm{Aut}(\mathrm{ad}\, E_x))\ . \end{array}$$

Clearly, i_2 and $\hat{\theta}$ are injective. Moreover, i_1 is also injective since any $\varphi \in \mathrm{Ker}\, i_1$ gives rise to a morphism $\bar{\varphi}\colon \Sigma \to Z$, which must be a constant. Hence, θ is injective. Now, consider the analogue of the above diagram for the fixed point $p \in \Sigma$:

$$\begin{array}{ccc} \mathrm{Aut}(E)/Z & \xrightarrow{\ \theta\ } & \mathrm{Aut}(\mathrm{ad}\, E) \\ \downarrow{\scriptstyle i_1(p)} & & \downarrow{\scriptstyle i_2(p)} \\ \mathrm{Aut}(E_p)/Z & \xrightarrow{\ \hat{\theta}(p)\ } & \mathrm{Aut}(\mathrm{ad}\, E_p)\, . \end{array}$$

Observe that $\mathrm{ad}\, E$ is a degree 0 vector bundle over Σ. We next claim that for any degree 0 semistable vector bundle $\mathscr{V}$ over Σ, the map $\bar{i}_2(p)\colon\ \mathrm{End}\,\mathscr{V} \to \mathrm{End}\,\mathscr{V}_p$ is injective, where $\mathrm{End}\,\mathscr{V}$ denotes the set of $\mathscr{O}_\Sigma$-module endomorphisms of $\mathscr{V}$. The injectivity of $\bar{i}_2(p)$ follows from the vanishing

$$H^0\left(\Sigma, \mathscr{O}_\Sigma(-p) \otimes \mathbf{End}\,\mathscr{V}\right) = 0, \tag{1}$$

where **End** is the corresponding sheaf and (1) follows from Lemma 6.2.4.

Taking $\mathscr{V} = \mathrm{ad}\, E$ and using Lemma 6.1.5, we get the injectivity of $i_2(p)$. Hence, from the second commutative diagram, we get the injectivity of $i_1(p)$. The injectivity of $i_1(p)$ implies the injectivity of $\mathrm{Aut}(E) \to \mathrm{Aut}(E_p)$, proving the lemma. □

6.3.E Exercises

In the following Σ is a smooth projective irreducible curve of any genus and G is a connected reductive group.

(1) Let $\rho\colon \pi_1(\Sigma) \to G$ be a homomorphism, where G is any algebraic group. Then, for any $g \in G$, show that the associated G-bundles E_ρ and $E_{g\rho g^{-1}}$ are isomorphic as algebraic G-bundles.

Following the notation as in Definition 6.3.3, prove the same result for $\hat{\rho}\colon \pi \to G$, i.e., $\hat{E}_{\hat{\rho}} \simeq \hat{E}_{g\hat{\rho}}g^{-1}$ as A-equivariant G-bundles.

(2) Following the notation in Definition 6.3.9, show that the topological G-bundle F_c (up to an isomorphism) does not depend upon the choices of c in its homotopy class, p, D_p and h.

Hint: Follow the argument as in Steenrod (1951, §18).

(3) Show that a degree 0 line bundle $\mathscr{L}$ over Σ comes from a unitary character χ (i.e., a 1-dimensional unitary representation $\mathbb{C}_\chi$) of $\pi_1(\Sigma)$.

Moreover, following the notation as in Definition 6.3.3, if $\mathscr{L}$ is an A-equivariant line bundle of degree 0, then show that it is A-unitary.

Hint: The universal bundle $\mathscr{E}_{|\Sigma\times R_K(g)}$ of Lemma 6.3.2 for $G = \mathrm{GL}_1$ (so that $K = S^1$) gives rise to an $\mathbb{R}$-analytic group homomorphism

$$\beta : R_K(g) = (S^1)^{2g} \to \mathrm{Jac}(\Sigma),$$

where $\mathrm{Jac}(\Sigma)$ is the Jacobian variety of Σ consisting of the set of isomorphism classes of degree 0 line bundles over Σ. Show that the above map is injective and hence surjective from the dimensional consideration. For the equivariant version, use the first part together with Lemma 6.3.8.

(4) Prove identity (4) in the proof of Lemma 6.3.11.

(5) Let X, Y be $\mathbb{C}$-analytic spaces such that X is compact. Let $\pi : Y \to T$ be a holomorphic map. Then show that the inverse image of the set of constant maps under the holomorphic map

$$\mathrm{Hol}(X,Y) \to \mathrm{Hol}(X,T), \quad f \mapsto \pi \circ f$$

is a closed $\mathbb{C}$-analytic subspace, where $\mathrm{Hol}(X,Y)$ has a natural $\mathbb{C}$-analytic structure as in the proof of Lemma 6.3.29.

(6) Let $\mathscr{V}$ be a stable vector bundle over Σ. Then show that $\mathscr{V}$ is simple, i.e., $H^0(\Sigma, \mathbf{End}\mathscr{V}) = \mathbb{C}$, where $\mathbf{End}\mathscr{V}$ denotes the sheaf of $\mathscr{O}_\Sigma$-module endomorphisms of $\mathscr{V}$.

Hint: Let $A := H^0(\Sigma, \mathbf{End}\mathscr{V})$ be the endomorphism algebra. Now, use Lemma 6.3.22 and finite-dimensionality of A.

(7) Let E be a G-bundle over Σ. Show that $H^0(\Sigma, \operatorname{ad} E) \simeq \mathrm{Lie}(\operatorname{Aut} E)$.

Hint:

$$\operatorname{Aut} E \simeq \{\varphi : E \to G : \varphi(eg) = g^{-1}\varphi(e)g \;\forall e \in E, g \in G\}$$

https://doi.org/10.1017/9781108997003.008 Published online by Cambridge University Press

and similarly

$$H^0(\Sigma, \mathrm{ad}\, E) \simeq \{f : E \to \mathfrak{g} : \varphi(eg) = \mathrm{Ad}(g^{-1}) \cdot (\varphi(e))\, \forall e \in E, g \in G\}.$$

(8) Let $\mathscr{V}_1$ and $\mathscr{V}_2$ be two stable vector bundles over Σ such that $\mu(\mathscr{V}_1) = \mu(\mathscr{V}_2)$. Show that any nonzero $\mathscr{O}_\Sigma$-module map $\varphi\colon \mathscr{V}_1 \to \mathscr{V}_2$ is an isomorphism.

(9) Prove Lemma 6.3.31 with T_s replaced by $T_{ss} := \{t \in T : \mathscr{F}_t \text{ is a semistable } G\text{-bundle}\}$.

Hint: Follow the proof of Lemma 6.3.31.

(10) Give an example of a stable G-bundle E such that ad E is *not* stable, where G is a simple group.

Hint: For Σ of genus ≥ 2, take a representation $\rho\colon \pi_1(\Sigma) \to \mathrm{SO}_n(\mathbb{R})$ with dense image (cf. Lemma 7.2.9). This representation remains irreducible considered as a homomorphism $\hat{\rho}\colon \pi_1(\Sigma) \to \mathrm{SL}_n(\mathbb{C})$. Thus, $E_{\hat{\rho}}$ is stable $\mathrm{SL}_n(\mathbb{C})$-bundle. However, show that ad $E_{\hat{\rho}}$ is not stable.

(11) Let $\mathscr{V}$ and $\mathscr{W}$ be two A-equivariant vector bundles over $\hat{\Sigma}$ of degree 0. Assume further that $\mathscr{V}$ (resp. $\mathscr{W}$) is A-stable (resp. A-semistable). Show that any nonzero A-equivariant $\mathscr{O}_{\hat{\Sigma}}$-linear map $f\colon \mathscr{W} \to \mathscr{V}$ is surjective.

Hint: Use the canonical factorization of f as given in the proof of Lemma 6.3.22.

(12) Following Proposition 6.3.18 and its proof, prove that the composite map $i \circ j \circ q \circ \mathfrak{F}$ coincides with the deformation map η.

6.C Comments

Mumford defined the notion of semistable and stable vector bundles over a smooth projective curve Σ as in Definition 6.1.4(a). Its extension to any G-bundles over Σ for a connected reductive group G as in Definition 6.1.4(b) is due to Ramanathan (1975). Definition 6.1.4(c) of polystability for any G-bundle is taken from Ramanan and Ramanathan (1984, Definition 3.16) (though they call it 'quasi-stable'). This extends the earlier definition of polystability for vector bundles, that is why we prefer to call it 'polystable.' The notion of parabolic structure on vector bundles and their semistablity (and stability) as in Exercise 6.1.E.7 is due to Mehta and Seshadri (1980) (also see (Seshadri, 1977) for an announcement of some of the results). Its extension to any G-bundles over Σ is due to Bhosle and Ramanathan (1989), though we have taken our Definition 6.1.4(d) from Teleman and Woodward

(2001, Definition 2.2). It might be mentioned that the paper by Bhosle and Ramanathan (1989) has a serious error in their association of $E(\rho, \tau)$ to a representation ρ in their §2.[3] Lemma 6.1.5 appears in Ramanathan (1996, Corollary 3.18).

A systematic study of A-equivariant vector bundles on $\hat{\Sigma}$ (where $\hat{\Sigma}$ is a smooth projective curve and A is a finite group acting faithfully on $\hat{\Sigma}$) was begun in Narasimhan and Seshadri (1965), wherein many of the results from Narasimhan and Seshadri (1964) were extended to an A-equivariant setting. One of the classical results (Narasimhan and Seshadri, 1965, Corollary 2, §12) (Theorem 6.3.35 for vector bundles) is derived from an analogous unitarity result in the A-equivariant setting (Narasimhan and Seshadri, 1965, Theorem 2, §12). Study of A-equivariant vector bundles on $\hat{\Sigma}$ was continued and expanded in Seshadri (2011).

We have taken Theorem 6.1.9 and its proof from Teleman and Woodward (2001) (though we have provided more complete details). Theorem 6.1.17 is due to Mehta and Seshadri (1980) for vector bundles (also see (Grothendieck, 1956–57), (Seshadri, 2011), (Boden, 1991), (Furuta and Steer, 1992) and (Biswas, 1997)). Theorems 6.1.15 and 6.1.17 for general G are taken from Teleman and Woodward (2001, Theorem 2.3) and Balaji and Seshadri (2015, Proposition 3.1.1, Theorems 5.3.1 and 6.3.5) (Theorem 6.1.17 is also proved in Balaji, Biswas and Nagaraj (2001, Theorem 4.3)). In fact, in Balaji and Seshadri (2015), the restriction $\theta(\tau_j) < 1$ plays no role by using Bruhat–Tits group schemes. Exercise 6.1.E.4 is taken from Ramanathan (1975, Lemma 2.1 and Remark 2.2), Exercise 6.1.E.5 is taken from Ramanathan (1975, Lemma 3.3) and Exercise 6.1.E.7(a) is taken from Bhosle and Ramanathan (1989, §1). Exercise 6.1.E.7(b) is taken from Mehta and Seshadri (1980, Remark 1.16). Exercise 6.1.E.8 is taken from Ramanathan (1975, Proposition 7.1). For Exercise 6.1.E.12 see Ramanathan and Subramanian (1988, Proposition 1) and Ramanathan (1975, Proposition 7.1). Exercise 6.1.E.13 is taken from Kumar, Narasimhan and Ramanathan (1994, Lemma 3.6). Some of these results on parabolic bundles have been extended to G-bundles over an arbitrary smooth projective variety over $\mathbb{C}$ by Balaji, Biswas and Nagaraj (2001).

Harder–Narasimhan (for short HN) filtration of vector bundles over Σ is due to Harder and Narasimhan (1975). Its extension for any G-bundles over Σ was announced by Ramanathan (1979). However, he did not publish its proof. Then, Atiyah and Bott (1982) provided an analogue of the HN filtration (or reduction) for G-bundles over Σ by looking at the original HN filtration of the corresponding adjoint bundle. Behrend (1995) proved the existence and

[3] I thank V. Balaji for pointing this out.

uniqueness of the HN reduction (in any characteristic) of G-bundles over Σ by using a 'complementary polyhedron.' A more bundle-theoretic proof of the existence and uniqueness of the HN reduction of G-bundles over Σ was given by Biswas and Holla (2004) and we have followed their proof in Section 6.2 (Theorem 6.2.3). Identity (1) of Theorem 6.2.3 is taken from Kumar and Narasimhan (1997, Lemma 3.6). Existence and uniqueness of the HN reduction of G-bundles over compact Kähler manifolds was established by Anchouche, Azad and Biswas (2002). The HN reduction of G-bundles over Σ in suitably positive characteristics was also studied by Mehta and Subramanian (2002) and Biswas and Holla (2004). Theorem 6.2.6 and Corollary 6.2.7 are taken from Biswas and Holla (2004), though (as mentioned in Remark 6.2.8) their proof has a gap which required us to put additional hypotheses (1) and (2) in Theorem 6.2.6. Exercise 6.2.E.4 is taken from Balaji, Biswas and Nagaraj (2001, Proposition 4.1).

Several of the results in Section 6.3 (including Lemma 6.3.2, Proposition 6.3.4 in the non-equivariant case, Corollary 6.3.7 in the non-equivariant case, Lemma 6.3.10, Lemma 6.3.11, Corollary 6.3.21, Proposition 6.3.24, Lemma 6.3.25, Lemma 6.3.27 and Lemma 6.3.29 are taken from Ramanathan (1975, 1996). Proposition 6.3.4 and Corollary 6.3.7 in the case of equivariant vector bundles as well as Lemma 6.3.6 in the equivariant case is proved in Seshadri (2011, Proposition 10 (Chap. II), Corollary (Chap. I), Proposition 1 (Chap. I)) (see also (Bhosle and Ramanathan, 1989, Propositions 2.1 and 2.2) for the parabolic analogue of Proposition 6.3.4 and Corollary 6.3.7).

Lemma 6.3.29 is attributed to R.R. Simha in Ramanathan (1975). Lemma 6.3.6 in the non-equivariant case is taken from Narasimhan and Seshadri (1964, Proposition 4.1), though the proof given here is a slight modification of their proof with help from Michael Taylor. Proposition 6.3.12, Corollary 6.3.14, Proposition 6.3.15, Corollary 6.3.16 and Proposition 6.3.18 are taken from Narasimhan and Seshadri (1964). Even though they prove their results for $G = \mathrm{GL}_n$, virtually the same proof works for any G. Proposition 6.3.12 is proved by them, more generally, for any compact, connected, Kähler manifold. Lemma 6.3.22 is taken from Narasimhan and Seshadri (1965, Proposition 4.3). Proposition 6.3.30, Lemmas 6.3.31 and 6.3.32 are due to Ramanathan (1975, §7, §4). Theorem 6.3.34 is a slight variant of Ramanathan (1975, Theorem 7.1). Its extension to Theorem 6.3.35 is straightforward. For $G = \mathrm{GL}_n$, this is a classical result due to Narasimhan and Seshadri (1965, §12, Corollary 2). Exercise 6.3.E.4 is taken from Ramanathan (1975, Proposition 6.1 and Remark 6.2). Exercise 6.3.E.6 is taken from Narasimhan and Seshadri (1965, §4). Exercise 6.3.E.9 is asserted in Ramanathan (1996, proof of Lemma 5.9.1). The analogue of most of the results in Section 6.3 for vector bundles is due to Narasimhan and Seshadri (1964) and Narasimhan and Seshadri (1965).

Corollary 6.3.7 for vector bundles is mentioned in Weil (1938). Theorem 6.3.41 for vector bundles is due to Seshadri (2011, Theorem 4 (Chap. II)). Lemma 6.3.37 is taken from Atiyah and Bott (1982, Lemma 10.12) though part of its proof via Lemma 6.3.38 is taken from Ramanathan and Subramanian (1988, Proposition 1). Proposition 6.3.42 is taken from Balaji, Biswas and Nagaraj (2001, §5).

There is an alternative proof of the Narasimhan–Seshadri theorem for stable vector bundles over Σ using the differential geometry of connections on holomorphic bundles (cf. (Donaldson, 1983)). For its extension to any reductive G and the base Σ replaced by any complex projective manifold, see Ramanathan and Subramanian (1988, Theorem 1).

We have restricted the discussion of parabolic G-bundles to the case when G is a simply-connected simple group. Its generalization to any connected reductive group G (under some restrictions on parabolic weights) can be found in Faltings (1993, §V) and in Balaji and Seshadri (2015, §8.2).