ARTICLES

RISK THEORY WITH THE GAMMA PROCESS

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Abstract

The aggregate claims process is modelled by a process with independent, stationary and nonnegative increments. Such a process is either compound Poisson or else a process with an infinite number of claims in each time interval, for example a gamma process. It is shown how classical risk theory, and in particular ruin theory, can be adapted to this model. A detailed analysis is given for the gamma process, for which tabulated values of the probability of ruin are provided.

Keywords

Aggregate claims; compound Poisson process; gamma process; infinite divisibility; risk theory; ruin probability; simulation; stable distributions; inverse Gaussian distribution.

1. Introduction

In classical collective risk theory, the aggregate claims process is assumed to be compound Poisson (Panjer and Willmot, 1984). Here we shall examine a more general model for the aggregate claims process: processes with independent, stationary and nonnegative increments. Such a process is either compound Poisson or else a process with an infinite number of claims in any time interval. The most prominent process with this intriguing property is the gamma process.

Since the process under consideration is either a compound Poisson process or a limit of compound Poisson processes, its properties can be derived from the basic properties of the compound Poisson process. The general results are derived in Section 2 (for the aggregate claims process) and Section 6 (for the probability of ruin). The gamma process is examined in detail in Sections 3, 4 and 5 (for the aggregate claims process) and Sections 7 and 8 (for the probability of ruin).

2. PROCESSES WITH INDEPENDENT, STATIONARY AND NONNEGATIVE INCREMENTS

Let $Q(x)$ be a nonnegative and nonincreasing function of x, $x > 0$, with the properties:

$$Q(x) \to 0 \quad \text{as} \quad x \to \infty$$

and

$$\int_0^\infty Q(x)\,dx < \infty. \tag{2.1}$$

Condition (2.1) can also be written as

$$\int_0^\infty x[-dQ(x)] < \infty,$$

which, if $q(x) = -Q'(x)$ exists, becomes

$$\int_0^\infty xq(x)\,dx < \infty.$$

Such a function $Q(x)$ defines an aggregate claims process $\{S(t)\}_{t\geq 0}$ in the following way. For each $x > 0$, let $N(t; x)$ denote the number of claims with an amount greater than x that occur before time t; let $S(t; x)$ be the sum of these claims. We assume that $\{N(t; x)\}_{t\geq 0}$ is a Poisson process with parameter $Q(x)$ and that $\{S(x; t)\}_{t\geq 0}$ is a compound Poisson process with Poisson parameter $Q(x)$ and individual claim amount distribution

$$P(y; x) = \begin{cases} 0 & y \leq x \\ \dfrac{Q(x)-Q(y)}{Q(x)} & y > x. \end{cases} \tag{2.2}$$

The process $\{S(t)\}$ is defined as the limit of the compound Poisson processes $\{S(t; x)\}$ as x tends to 0.

We write

$$Q(0) = \lim_{x\to 0} Q(x).$$

We need to distinguish two cases: $Q(0) < \infty$, and $Q(0) = \infty$. In the first case, $\{S(t)\}$ is a compound Poisson process with Poisson parameter $Q(0)$ and individual claim amount distribution

$$P(y) = 1 - \frac{Q(y)}{Q(0)}, \qquad y \geq 0. \tag{2.3}$$

This is the classical model for collective risk theory. Conversely, every compound Poisson process, given by Poisson parameter λ and individual claim amount distribution $P(y)$, is of this type if we set

$$Q(y) = \lambda[1-P(y)], \qquad y \geq 0. \tag{2.4}$$

In the second case, $\{S(t)\}$ is the limit of compound Poisson processes, but is not a compound Poisson process itself, because the expected number of claims per unit time, $Q(0)$, is infinite. Indeed, with probability one, the number of claims in any time interval is infinite. Nevertheless, $S(t)$ is finite, as the majority of the claims are very small in some sense. In both cases, $Q(y)$ is the expected number of claims per unit time with an amount exceeding y.

Since $\{S(t)\}$ is the limit of $\{S(t; x)\}$ as x tends to 0, we can use well-known results for the compound Poisson process to obtain results for the process $\{S(t)\}$. For example, it follows from

$$\begin{aligned} E[S(t; x)] &= tQ(x) \int_0^\infty [1-P(y; x)]\, dy \\ &= txQ(x)+t \int_x^\infty Q(y)\, dy \end{aligned}$$

that

$$E[S(t)] = t \int_0^\infty Q(y)\, dy = t \int_0^\infty y[-dQ(y)]. \tag{2.5}$$

To get the Laplace transform, we start with

$$\begin{aligned} E[e^{-zS(t;x)}] &= \exp\left\{ tQ(x) \left[\int_x^\infty e^{-zy}\, dP(y; x) - 1 \right] \right\} \\ &= \exp\left\{ t \int_x^\infty [e^{-zy}-1]\,[-dQ(y)] \right\}. \end{aligned}$$

Letting $x \to 0$, we obtain

$$E[e^{-zS(t)}] = \exp\left\{ t \int_0^\infty [e^{-zy}-1]\,[-dQ(y)] \right\}. \tag{2.6}$$

The process $\{S(t)\}$, defined by the function $Q(x)$, has independent, stationary and nonnegative increments, and $E[S(t)] < \infty$. The converse is true in the following sense. Every process $\{X(t)\}$ with these properties is of the form

$$X(t) = S(t)+bt,$$

where $\{S(t)\}$ is a process of the type presented above and b is a nonnegative constant. This is a consequence of the connection between processes with independent and stationary increments and infinitely divisible distributions, and the characterization of infinitely divisible distributions with nonnegative support (FELLER, 1971, p. 450, Theorem 2; p. 571, formula (4.7)).

3. THE GAMMA PROCESS

Assume that the function $Q(x)$ is differentiable and that $-Q'(x)$ is

$$q(x) = \frac{a}{x} e^{-bx}, \qquad x > 0, \tag{3.1}$$

where a and b are positive constants. Let $\{S(t)\}$ be the associated aggregate claims process. In a time interval of length t, the expected number of claims with an amount exceeding x is

$$tQ(x) = at \int_x^\infty \frac{e^{-by}}{y} dy.$$

Since $Q(0) = \infty$, there is an infinite number of claims in each time interval. By (2.5) the expected aggregate claims in a time interval of length t are

$$E[S(t)] = t \int_0^\infty yq(y)\,dy = at \int_0^\infty e^{-by}\,dy = \frac{at}{b}. \tag{3.2}$$

To obtain the distribution of $S(t)$, we compute its Laplace transform by (2.6):

$$\begin{aligned} E[e^{-zS(t)}] &= \exp\left\{ t \int_0^\infty [e^{-zy} - 1]\, q(y)\,dy \right\} \\ &= \exp\left\{ at \int_0^\infty \frac{e^{-(z+b)y} - e^{-by}}{y} dy \right\} \\ &= \left(\frac{b}{z+b} \right)^{at}. \end{aligned} \tag{3.3}$$

To verify the last step, consider the function

$$\varphi(z) = \int_0^\infty \frac{e^{-(z+b)y} - e^{-by}}{y} dy;$$

observe that $\varphi(0) = 0$ and $\varphi'(z) = -(z+b)^{-1}$. Formula (3.3) shows that the distribution of $S(t)$ is gamma, with shape parameter $\alpha_t = at$ and scale parameter $\beta_t = b$. Hence the process $\{S(t)\}$ is called a *gamma process*.

A gamma process with $a = b = 1$ is called a *standardized gamma process*. For an arbitrary gamma process with parameters a and b, we may set $t^* = at$ and $S^*(t^*) = bS(t)$. It follows from (3.3) that

$$(3.4) \qquad E[e^{-zS^*(t^*)}] = \left(\frac{1}{z+1}\right)^{t^*}.$$

Thus the transformed process $\{S^*(t^*)\}$ is a standardized gamma process.

The gamma process, given by (3.1), can be imbedded in a larger family of processes given by

$$(3.5) \qquad q(x) = ax^{\alpha-1}e^{-bx}, \qquad x > 0,$$

with $-1 < \alpha < \infty$. We note that

$$(3.6) \qquad \int_0^\infty yq(y)\,dy = a\int_0^\infty y^\alpha e^{-by}\,dy = \frac{a}{b^{\alpha+1}}\,\Gamma(\alpha+1)$$

is indeed finite.

For $\alpha > 0$,

$$(3.7) \qquad Q(0) = \int_0^\infty q(y)\,dy = \frac{a}{b^\alpha}\,\Gamma(\alpha)$$

is finite. Hence $\{S(t)\}$ is a compound Poisson process, with Poisson parameter λ given by (3.7) and claim amount density

$$(3.8) \qquad p(x) = \frac{q(x)}{\lambda} = \frac{b^\alpha}{\Gamma(\alpha)}\,x^{\alpha-1}e^{-bx}, \qquad x > 0,$$

which is a gamma density.

For $-1 < \alpha \leq 0$, $Q(0) = \infty$. When $\alpha = 0$, we have the gamma process. To determine the probability density function $f(x, t)$ of $S(t)$ for $-1 < \alpha < 0$, we apply formula (2.6),

$$(3.9) \qquad E[e^{-zS(t)}] = e^{ta\varphi(z)},$$

with

$$(3.10) \qquad \varphi(z) = \int_0^\infty (e^{-zy}-1)\,y^{\alpha-1}e^{-by}\,dy.$$

From $\varphi(0) = 0$ and

$$(3.11) \qquad \varphi'(z) = -\int_0^\infty y^\alpha e^{-(z+b)y}\,dy = -\frac{\Gamma(\alpha+1)}{(z+b)^{\alpha+1}}$$

we obtain

$$(3.12) \qquad \varphi(z) = \frac{\Gamma(\alpha+1)}{\alpha}\left[\frac{1}{(z+b)^\alpha} - \frac{1}{b^\alpha}\right]$$

$$= \Gamma(\alpha)\,[(z+b)^{-\alpha} - b^{-\alpha}].$$

(Note that (3.12) is also valid for $\alpha > 0$; in this case it can derived by first expressing (3.10) as the difference of two convergent integrals.) For simplicity, assume $a = -1/\Gamma(\alpha)$ and $b = 1$. Write $\beta = -\alpha$. Then (3.9) becomes

$$E[e^{-zS(t)}] = \exp\{t[1-(1+z)^{\beta}]\}. \tag{3.13}$$

Recall the *stable distribution* of order β that is concentrated on the positive axis (FELLER, 1971, Sections XIII.6 and XIII.7). Let $g_{\beta}(x)$ denote its probability density function. Its Laplace transform is

$$\int_0^{\infty} e^{-zx} g_{\beta}(x)\, dx = e^{-z^{\beta}}.$$

Hence the Laplace transform of the function

$$t^{-1/\beta} g_{\beta}(t^{-1/\beta} x), \qquad x > 0,$$

is $\exp(-tz^{\beta})$. Finally, it follows from (3.13) that the probability density function of $S(t)$ is

$$f(x, t) = e^{t-x} t^{-1/\beta} g_{\beta}(t^{-1/\beta} x), \qquad x > 0. \tag{3.14}$$

For $\beta = 1/2$, a closed form expression for the stable density is available,

$$g_{1/2}(x) = \frac{1}{2\sqrt{\pi}\, x^{3/2}} \exp\left(-\frac{1}{4x}\right), \qquad x > 0, \tag{3.15}$$

and (3.14) becomes

$$f(x, t) = \frac{t}{2\sqrt{\pi}\, x^{3/2}} \exp\left[-\frac{(2x-t)^2}{4x}\right], \qquad x > 0, \tag{3.16}$$

which is the probability density of the *inverse Gaussian distribution*. A review on the inverse Gaussian distribution can be found in FOLKS and CHHIKARA (1978); WILLMOT (1987) has applied the inverse Gaussian distribution in modelling the claim number distribution, and GENDRON and CRÉPEAU (1989) and WILLMOT (1990) have modelled the individual claim amount distribution with the inverse Gaussian distribution.

4. PARAMETER ESTIMATION FOR THE GAMMA PROCESS

Let $\{S(t)\}$ be a gamma process with (at time $t = 0$) unknown parameters a and b. We claim that, if we can observe the process for a time interval of (arbitrarily short) length h, $h > 0$, the value of a can be obtained as a limit: For $0 < x < 1$, we define the random variable

$$A(x) = -\frac{N(h; x)}{h \ln(x)}; \tag{4.1}$$

then

$$\lim_{x \to 0} A(x) = a. \tag{4.2}$$

(We remark that a similar situation exists for the diffusion process with *a priori* unknown but constant infinitesmal drift μ and variance σ^2: If the sample path for an arbitrarily small time interval is known, σ^2 can be calculated.)

To prove (4.2), we write (4.1) as

$$A(x) = \frac{\displaystyle\int_x^{\infty} \frac{e^{-by}}{y} dy}{\displaystyle\int_x^1 \frac{dy}{y}} \cdot \frac{N(h; x)}{ah \displaystyle\int_x^{\infty} \frac{e^{-by}}{y} dy} \cdot a.$$

Applying L'Hôpital's rule, we see that the first ratio tends to 1 as x tends to 0. The second ratio is $N(h; x)/[hQ(x)]$; by the strong law of large numbers it converges to 1 (with probability one) as x tends to 0.

In the following we assume that the value of a is known, but that b is unknown. If the aggregate claims process has been observed to time t, $S(t)$ is a sufficient statistic, i.e., any additional information about the sample path is irrelevant for the estimation of b (De Groot, 1975, p. 304, #5). To illustrate this, let us treat the unknown b as a random variable Θ with prior probability density function $u(\theta)$, $\theta > 0$. Then the posterior density of Θ at time t, given the value of $S(t)$, is

$$u(\theta; t) = \frac{\theta^{at} e^{-\theta S(t)} u(\theta)}{\displaystyle\int_0^{\infty} r^{at} e^{-rS(t)} u(r)\, dr}.$$

Let us now assume that $u(\theta)$ is gamma, say,

$$u(\theta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}, \qquad \theta > 0,$$

with $\beta > 0$ and $\alpha > 1$. Then the posterior density is also gamma, with parameters

$$\alpha_t = \alpha + at$$

and

$$\beta_t = \beta + S(t).$$

At time $t = 0$, the expected aggregate claims per unit time are

$$E\left(\frac{a}{\Theta}\right) = a \int_0^{\infty} \frac{u(\theta)}{\theta} d\theta = a \frac{\beta}{\alpha - 1}.$$

Hence, with $S(t)$ known, the conditional expectation of the aggregate claims per unit time is

(4.3)
$$a\frac{\beta_t}{\alpha_t - 1} = a\frac{\beta + S(t)}{\alpha + at - 1}$$
$$= (1 - Z_t)\, a\frac{\beta}{\alpha - 1} + Z_t \frac{S(t)}{t},$$

where $Z_t = at/(at + \alpha - 1)$. Formula (4.3) corresponds to the well-known result for exact credibility in the gamma/gamma model.

5. SIMULATION OF THE GAMMA PROCESS

We can simulate a compound Poisson process by simulating the times and amounts of the claims. This straightforward approach is not applicable to the gamma process, since there are infinitely many claims in each time interval. We now present a method for simulating the gamma process.

Let $\{S(t)\}$ be the gamma process with parameters a and b. To simulate a sample path, we use the following result. For time $\tau > 0$, the conditional distribution of the ratio $S(\tau/2)/S(\tau)$, given $S(\tau)$, is symmetric beta with

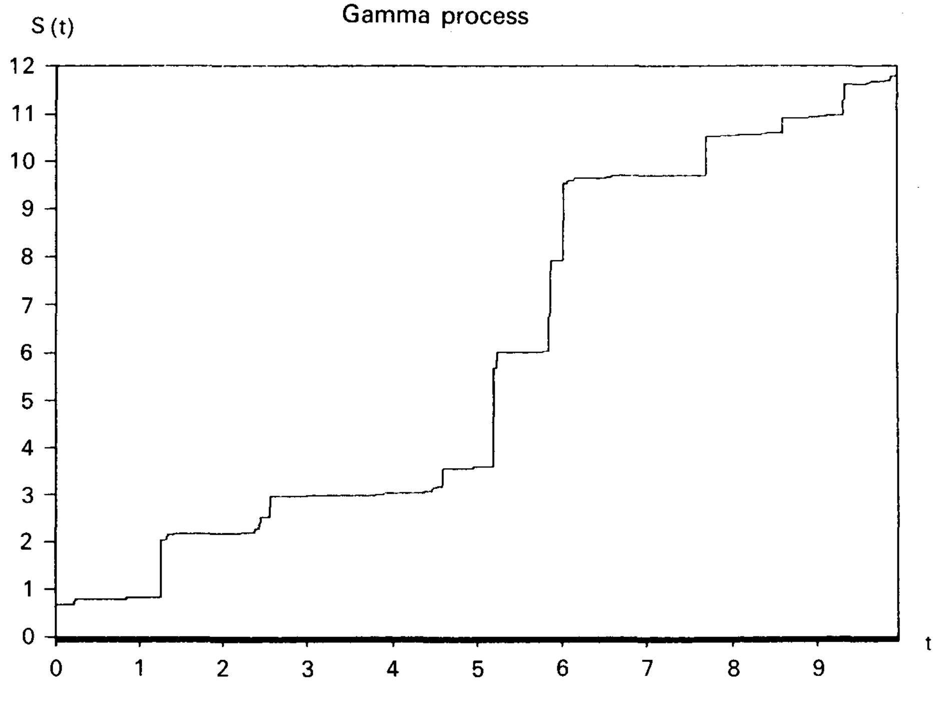

FIGURE 1.

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parameter $a\tau/2$ (DE GROOT, 1975, p. 244, #5). Thus, if we want to simulate a sample path for $S(t)$, $0 \leq t \leq T$, we can proceed as follows. First we simulate a value for $S(T)$, whose distribution is gamma with shape parameter aT and scale parameter b. Then we obtain $S(T/2)$ by simulating a value for $S(T/2)/S(T)$, which has a symmetric beta distribution with parameter $aT/2$. Next, we obtain $S(T/4)$ and $S(3\,T/4)$ by simulating the values of $S(T/4)/S(T/2)$ and $[S(3\,T/4)-S(T/2)]\ /\ [S(T)-S(T/2)]$, respectively, each of which has a symmetric beta distribution with parameter $aT/4$. Similarly, we can generate the values of $S(T/8)$, $S(3\,T/8)$, $S(5\,T/8)$, $S(7\,T/8)$, and so on.

We have simulated the standardized gamma process for various T. A sample path for $T = 10$ is shown in Figure 1.

6. RUIN THEORY

Let $\{S(t)\}$ be the aggregate claims process introduced in Section 2. In this section we present some ruin probability results for this process. In the next section, we specialize to the case that $\{S(t)\}$ is a gamma process.

Let the surplus of an insurance company at time t, $t \geq 0$, be

$$U(t) = u + ct - S(t). \tag{6.1}$$

Here u is a nonnegative number denoting the initial surplus and c is the rate at which the premiums are received. The relative security loading θ is defined by the equation

$$c = (1+\theta)\,E[S(1)] = (1+\theta)\int_0^\infty Q(x)\,dx. \tag{6.2}$$

We assume that $\theta > 0$. Let $\psi(u)$ denote the probability of ultimate ruin, i.e., the probability that the surplus becomes negative at some future time.

In view of formula (2.4), results for this model can be obtained via those for the compound Poisson model with the following recipe. We start with a formula for the case of the compound Poisson process with Poisson parameter λ and individual claim amount distribution $P(y)$. Then we substitute $Q(y)$ for $\lambda[1-P(y)]$ (or $q(y)$ for $\lambda p(y)$ if the derivatives exist) to obtain the corresponding formula for the more general model.

For example, in the compound Poisson model the probability of ruin satisfies the following defective renewal equation [e.g., BOWERS et al. (1986, p. 373, #12.11)]:

$$c\psi(u) = \lambda\int_0^u \psi(u-y)\,[1-P(y)]\,dy + \lambda\int_u^\infty [1-P(y)]\,dy, \qquad u \geq 0.$$

Substituting $Q(y)$ for $\lambda[1-P(y)]$, we get

$$c\psi(u) = \int_0^u \psi(u-y)\,Q(y)\,dy + \int_u^\infty Q(y)\,dy, \qquad u \geq 0. \tag{6.3}$$

For $u = 0$, this gives

$$\psi(0) = \frac{1}{c} \int_0^\infty Q(y)\, dy = \frac{1}{1+\theta}. \tag{6.4}$$

Let us now consider the maximal loss random variable

$$L = \max_{t \geq 0} \{S(t) - ct\}. \tag{6.5}$$

It is of interest since $1-\psi(u)$ is its distribution function. In the compound Poisson model, it is well known (BOWERS et al., 1986, Section 12.6) that L has a compound geometric distribution:

$$L = L_1 + L_2 + \ldots + L_N. \tag{6.6}$$

Here $N, L_1, L_2, \ldots$ are independent random variables, the L_i's are identically distributed with the probability density

$$h(x) = \frac{1-P(x)}{\int_0^\infty [1-P(y)]\, dy}, \qquad x > 0, \tag{6.7}$$

and N has a geometric distribution defined by

$$Pr(N = n) = \frac{\theta}{1+\theta}\left(\frac{\theta}{1+\theta}\right)^n, \qquad n = 0, 1, 2, \ldots. \tag{6.8}$$

If we multiply both numerator and denominator of (6.7) by λ, we see that (6.6) is valid for the general model, with

$$h(x) = \frac{Q(x)}{\int_0^\infty Q(y)\, dy}, \qquad x > 0. \tag{6.9}$$

These formulas can be used to determine numerical lower and upper bounds for the ruin probability; see Method 1 in DUFRESNE and GERBER (1989).

For the next result we assume that $p(x) = P'(x)$ and $q(x) = -Q'(x)$ exist. Let T denote the time of ruin. Put $X = U(T-)$, the surplus immediately before ruin, and $Y = |U(T)|$, the deficit at the time of ruin. We assume that $u = 0$. Given that ruin occurs, the joint probability density of X and Y in the compound Poisson case is

$$h(x, y) = \frac{p(x+y)}{\int_0^\infty [1-P(s)]\, ds}, \qquad x > 0, y > 0 \tag{6.10}$$

(DUFRESNE and GERBER, 1988). Thus, in the general model, the joint density of X and Y is

$$h(x, y) = \frac{q(x+y)}{\int_0^\infty Q(s)\,ds}, \qquad x > 0, y > 0. \tag{6.11}$$

We note that both (6.10) and (6.11) are symmetric in x and y. The probability density of $Z = X+Y$ (the amount of the claim that cases ruin) is

$$g(z) = \int_0^z h(x, z-x)\,dx = \frac{zq(z)}{\int_0^\infty Q(s)\,ds}, \qquad z > 0. \tag{6.12}$$

The conditional probability density of X given $Z = z$ (and $u = 0$) is

$$\frac{h(x, z-x)}{g(z)} = \frac{1}{z}, \qquad 0 < x < z.$$

This is the somewhat surprising result that the conditional distribution of X (given $Z = z$) is uniform between 0 and z.

We wish to remark that, if $Q(0) = \infty$, the notion of an individual claim amount distribution of the process $\{S(t)\}$ *per se* does not make sense. However, the conditional claim amount distribution, given certain information, may still exist. For example, (2.2) is the distribution of an individual claim amount given that it exceeds x. Likewise, $g(z)$ is the probability density function of the amount of the claim that causes ruin.

We now turn to Lundberg's asymptotic formula. The adjustment coefficient R is defined as the positive solution $r = R$ of the equation

$$\int_0^\infty (e^{ry}-1)\,[-dQ(y)] = cr. \tag{6.13}$$

(Note that some regularity conditions have to be imposed on $Q(y)$ to guarantee the existence of R.) It follows from (2.6) that, for all t,

$$E[e^{R[S(t)-ct]}] = 1. \tag{6.14}$$

Lundberg's famous asymptotic formula states that

$$\psi(u) \sim Ce^{-Ru} \qquad \text{for } u \to \infty. \tag{6.15}$$

In the compound Poisson case,

(6.16)
$$C = \frac{\theta\lambda \int_0^\infty y dP(y)}{\lambda \int_0^\infty y e^{Ry}\, dP(y) - c}$$

(SEAL, formula (4.64)), which is translated as

(6.17)
$$C = \frac{-\theta \int_0^\infty y dQ(y)}{-\int_0^\infty y e^{Ry}\, dQ(y) - c}.$$

7. RUIN THEORY FOR THE GAMMA PROCESS

We now consider the special case that $\{S(t)\}$ is a gamma process. As we pointed out in Section 3, any gamma process can be transformed into a standardized gamma process. Thus we assume that, for $x > 0$,

(7.1)
$$q(x) = \frac{e^{-x}}{x}$$

or

(7.2)
$$Q(x) = \int_x^\infty \frac{e^{-y}}{y}\, dy.$$

In ABRAMOWITZ and STEGUN (1964, p. 227), the exponential integral (7.2) is denoted as $E_1(x)$.

Since

$$\int_0^\infty Q(x)\, dx = \int_0^\infty x q(x)\, dx = \int_0^\infty e^{-x}\, dx = 1,$$

formula (6.2) becomes

(7.3)
$$1+\theta = c.$$

By (6.9) the common probability density function of the random variables $\{L_i\}$ is

(7.4)
$$h(x) = Q(x) = E_1(x), \qquad x > 0,$$

and their distribution function is

(7.5)
$$H(x) = \int_0^x h(y)\, dy = 1 - e^{-x} + xE_1(x), \qquad x \geq 0.$$

From (6.11) and (6.12) we obtain

$$h(x, y) = \frac{e^{-(x+y)}}{x+y} \tag{7.6}$$

and

$$g(z) = e^{-z}, \tag{7.7}$$

respectively. Formula (7.7) is especially interesting, as it says that (if $u = 0$) the amount of the claim that causes ruin is exponentially distributed.

Substituting (3.4) and (7.3) in (6.14) yields the equation

$$\frac{1}{1-r} = e^{r(1+\theta)}. \tag{7.8}$$

The adjustment coefficient R is the positive root of (7.8). It follows from (6.17) and (7.3) that the asymptotic constant C in Lundberg's formula is

$$C = \frac{\theta}{\dfrac{1}{1-R} - (1+\theta)} = \frac{\theta(1-R)}{R-\theta(1-R)}. \tag{7.9}$$

Remark: As pointed out in Section 3, the gamma process is the limit of a certain family of compound Poisson processes, each with a gamma claim amount distribution. For these WILLMOT (1988) has given an elegant method to evaluate the probability of ruin.

8. THE PROBABILITY OF RUIN FOR THE GAMMA PROCESS

As in the last section we assume that the aggregate claims process is the standardized gamma process. Since (7.5) gives an explicit expression for $H(x)$, we can apply the method of lower and upper bounds to calculate the probability of ruin (DUFRESNE and GERBER, 1989). We have calculated lower and upper bounds for $\psi(u)$ for different values of the initial surplus u (0, 1, 2, ..., 20) and the relative security loading θ (0.1, 0.2, 0.3, ..., 1.0), for intervals of discretisation with length 0.01 and 0.001. For $\theta = 0.5$ these bounds are displayed in Table 1. Thus the exact value of the probability of ruin is known with sufficient accuracy (4 decimals). Table 2 shows these values.

Illustration: Assume that the annual aggregate claims have an expectation $\mu = 100{,}000$ and a standard deviation $\sigma = 20{,}000$. The initial reserve is 48,000 and the annual premium (net of expenses) is 120,000. What is the probability of ultimate ruin?

TABLE 1

LOWER AND UPPER BOUNDS FOR THE PROBABILITY OF RUIN

$\theta = 0.5$

u	Lower bounds		Upper bounds	
0	0.666667	0.666667	0.666667	0.666667
1	0.321352	0.322741	0.323055	0.324488
2	0.175016	0.176268	0.176550	0.177839
3	0.096653	0.097604	0.097819	0.098798
4	0.053619	0.054288	0.054439	0.055129
5	0.029801	0.030250	0.030352	0.030817
6	0.016577	0.016870	0.016936	0.017240
7	0.009225	0.009412	0.009454	0.009649
8	0.005135	0.005252	0.005279	0.005401
9	0.002858	0.002931	0.002948	0.003024
10	0.001591	0.001636	0.001646	0.001693
11	0.000886	0.000913	0.000919	0.000948
12	0.000493	0.000510	0.000513	0.000531
13	0.000275	0.000284	0.000287	0.000297
14	0.000153	0.000159	0.000160	0.000166
15	0.000085	0.000089	0.000089	0.000093
16	0.000047	0.000049	0.000050	0.000052
17	0.000026	0.000028	0.000028	0.000029
18	0.000015	0.000015	0.000016	0.000016
19	0.000008	0.000009	0.000009	0.000009
20	0.000005	0.000005	0.000005	0.000005

0.001

0.01

length of the interval of discretisation

Solution: We assume that the premiums are received continuously and the aggregate claims process can be modelled by a gamma process with parameters a and b. Then $a/b = \mu = 100{,}000$ and $a/b^2 = \sigma^2 = (20{,}000)^2$. It follows that $b = \mu/\sigma^2 = 1/4{,}000$. In order to use Table 2 (which is for the standardized gamma process), we have to transform the initial reserve to $u = 48{,}000 \times b = 12$. The relative security loading $\theta = 0.2$ does not change. Looking up Table 2, we obtain the probability of ruin $\psi(12) = 0.018$.

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TABLE 2

THE PROBABILITY OF RUIN FOR THE STANDARDIZED GAMMA PROCESS

Relative security loading θ

u	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0.9091	0.8333	0.7692	0.7143	0.6667	0.6250	0.5882	0.5556	0.5263	0.5000
1	0.7395	0.5736	0.4613	0.3816	0.3229	0.2782	0.2434	0.2155	0.1929	0.1743
2	0.6184	0.4165	0.2990	0.2253	0.1764	0.1424	0.1178	0.0994	0.0854	0.0743
3	0.5182	0.3038	0.1952	0.1344	0.0977	0.0741	0.0582	0.0470	0.0388	0.0327
4	0.4345	0.2219	0.1277	0.0805	0.0544	0.0388	0.0289	0.0224	0.0178	0.0145
5	0.3643	0.1621	0.0836	0.0482	0.0303	0.0204	0.0144	0.0107	0.0082	0.0065
6	0.3054	0.1185	0.0548	0.0289	0.0169	0.0107	0.0072	0.0051	0.0038	0.0029
7	0.2561	0.0866	0.0359	0.0173	0.0094	0.0056	0.0036	0.0025	0.0018	0.0013
8	0.2148	0.0632	0.0235	0.0104	0.0053	0.0030	0.0018	0.0012	0.0008	0.0006
9	0.1801	0.0462	0.0154	0.0062	0.0029	0.0016	0.0009	0.0006	0.0004	0.0003
10	0.1510	0.0338	0.0101	0.0037	0.0016	0.0008	0.0005	0.0003	0.0002	0.0001
11	0.1266	0.0247	0.0066	0.0022	0.0009	0.0004	0.0002	0.0001	0.0001	0.0001
12	0.1062	0.0180	0.0043	0.0013	0.0005	0.0002	0.0001	0.0001		
13	0.0890	0.0132	0.0028	0.0008	0.0003	0.0001	0.0001			
14	0.0746	0.0096	0.0019	0.0005	0.0002	0.0001				
15	0.0626	0.0070	0.0012	0.0003	0.0001					
16	0.0525	0.0051	0.0008	0.0002						
17	0.0440	0.0038	0.0005	0.0001						
18	0.0369	0.0027	0.0003	0.0001						
19	0.0309	0.0020	0.0002							
20	0.0259	0.0015	0.0001							

REFERENCES

ABRAMOWITZ, M. and STEGUN, I. A. (1964) *Handbook of Mathematical Functions*. National Bureau of Standards, Washington, D.C. Reprinted by Dover, New York.

BOWERS, N. J., JR., GERBER, H. U., HICKMAN, J. C., JONES, D. A. and NESBITT, C. J. (1986) *Actuarial Mathematics*. Society of Actuaries, Itasca, Illinois.

DE GROOT, M. H. (1975) *Probability and Statistics*. Addison-Wesley, Reading, Massachusetts.

DUFRESNE, F. and GERBER, H. U. (1988) The surplus immediately before and at ruin, and the amount of the claim causing ruin. *Insurance: Mathematics and Economics* **7**, 193–199.

DUFRESNE, F. and GERBER, H. U. (1989) Three methods to calculate the probability of ruin. *ASTIN Bulletin* **19**, 71–90.

FELLER, W. (1971) *An Introduction to Probability Theory and its Applications*, Vol. 2 (2nd edn). Wiley, New York.

FOLKS, J. L. and CHHIKARA, R. S. (1978) The Inverse Gaussian Distribution and Its Statistical Application — A Review. *Journal of the Royal Statistical Society B* **40**, 263–275; Discussion 275–289.

GENDRON, M. and CRÉPEAU, H. (1989) On the Computation of the Aggregate Claim Distribution When Individual Claims Are Inverse Gaussian. *Insurance: Mathematics and Economics* **8**, 251–258.

PANJER, H. H. and WILLMOT, G. E. (1984) Models for the distribution of aggregate claims in risk theory. *Transactions of the Society of Actuaries* **36**, 399–477.

SEAL, H. L. (1969) *The Stochastic Theory of a Risk Business*. Wiley, New York.

WILLMOT, G. E. (1987) The Poisson-Inverse Gaussian Distribution as an Alternative to the Negative Binomial. *Scandinavian Actuarial Journal*, 113–127.

WILLMOT, G.E. (1988) Further use of Shiu's approach to the evaluation of ultimate ruin probabilities. *Insurance: Mathematics and Economics* **7**, 275–281.
WILLMOT, G.E. (1990) A Queueing Theoretic Approach to the Analysis of the Claims Payment Process. *Transactions of the Society of Actuaries* **42**.

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