AN EXISTENCE THEOREM FOR GENERALIZED DIRECT PRODUCTS WITH AMALGAMATED SUBGROUPS

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1. Generalized direct products with amalgamated subgroups were introduced by B. H. Neumann and Hanna Neumann in their joint paper **(4)**. In general, we call a given collection of groups with specified subgroups amalgamated an amalgam of groups; if all groups are abelian we speak of an abelian amalgam. The group freely generated by the amalgam is called the abelian free sum of the amalgam provided it contains the amalgam isomorphically. The free abelian sum need not exist. Hence one of the problems is to find necessary and sufficient conditions for its existence. In **(2**, § 16, p. 534**)** B. H. Neumann shows that this problem is connected with that of embedding an abelian amalgam in an abelian group. It was pointed out by M. F. Newman in his thesis (Manchester, 1960) that B. H. Neumann, at the place quoted, clearly refers to the unique maximal generalized direct product, i.e. the free sum of the amalgam, but omits to state the maximality condition. In fact, the abelian free sum of an amalgam exists if and only if a generalized direct product (the precise definition is given below) exists. This is, essentially, the contents of the following theorem on the embedding of an abelian amalgam given in **(2)** and **(3)**.

Let A be a given abelian amalgam and P' be the abelian group generated by the elements $a' = \theta'(a)$, where θ' is the homomorphism of the amalgam A into P', $a \in A$, and $a'b' = c'$ whenever $a' = \theta'(a)$, $b' = \theta'(b)$, $c' = \theta'(c)$, and $ab = c$.

THEOREM. *If θ is a homomorphism of the amalgam A into an abelian group P, then there exists a homomorphism ϕ of the group P' into P such that $\theta = \phi\theta'$. If θ is one-to-one, then θ' must be one-to-one. The amalgam is embeddable in an abelian group if and only if the homomorphism θ' is one-to-one.*

In this paper we shall adopt another approach to the problem. In fact, our result suggests a process that can be used to determine the existence of a generalized direct product of a given amalgam in a "step-by-step" manner. In deriving the result we introduce a second type of generalized direct product in which each amalgam consists of precisely two groups. We shall call this

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product a generalized direct product of second type. For brevity we shall denote the former by GD-product I and the latter by GD-product II.

In Section 2 we shall briefly recall the definitions of a GD-product I and other necessary terms. In Section 3 we shall introduce the concept of GD-product II and show by means of an example that it may not necessarily exist and consequently derive necessary and sufficient conditions for its existence. In Section 4 we shall make use of the results of Section 3 to derive necessary and sufficient conditions for the existence of a GD-product I. Finally in Section 5 we shall apply this result to different cases. In particular, we note that Theorem 4.2 enables us to prove the existence of a GD-product I and hence of the free abelian sum of any amalgam of cyclic groups satisfying the compatibility condition as defined in Section 2 by a much simpler group-theoretic argument instead of making use of arithmetical relations as in **(6)**.

2. DEFINITION 2.1. *A group P is a generalized direct product of the groups $G_1, G_2, \ldots, G_n$ amalgamating the subgroups H_{ij} and H_{ji} with $H_{ij} \subseteq G_i$ ($i \neq j$; $1 \leqslant i, j \leqslant n$), if the following conditions are satisfied:*

(i) *P is generated by the groups $G_1, G_2, \ldots, G_n$.*

(ii) *for every $i \neq j$ every element of G_i permutes with every element of G_j,*

(iii) *$G_i \cap G_j = H_{ij} = H_{ji}$.*

Suppose that we are given a set of groups $G_1, G_2, \ldots, G_n$ together with a set of subgroups $H_{ij} \subseteq G_i$ and a set of isomorphisms θ_{ij} ($1 \leqslant i, j \leqslant n$) such that θ_{ij} maps H_{ij} isomorphically onto H_{ji}. Our problem is to find necessary and sufficient conditions for the existence of a GD-product I, P, of this set of groups amalgamating the subgroups H_{ij} with H_{ji} under the isomorphisms θ_{ij}. It is easy to see that in order for P to exist the subgroups H_{ij} must be in the centres of G_i, and θ_{ij} must satisfy certain compatibility conditions.

DEFINITION 2.2. *The set of isomorphisms θ_{ij} ($i \neq j$; $1 \leqslant i, j \leqslant n$) is compatible if:*

(i) *θ_{ij} is the inverse of θ_{ji},*

(ii) *θ_{ij} maps $H_{ij} \cap H_{ik}$ isomorphically onto $H_{ji} \cap H_{jk}$,*

(iii) *$\theta_{jk}\theta_{ij}$ coincides with θ_{ik} where it is defined, that is, if $h_{ij} \in H_{ij} \cap H_{ik}$, then $\theta_{jk}\theta_{ij}(h_{ij}) = \theta_{ik}(h_{ij})$. More generally $\theta_{pq} \ldots \theta_{jk}\theta_{ij}$ and $\theta_{uv} \ldots \theta_{rs}\theta_{ir}$ coincide for those elements for which they are both defined.*

We shall make use of the following notations:

$(G_1 \times G_2 \times \ldots \times G_n)_{H_{ij}=H_{ji}}$ will denote a group that is a generalized direct product of $G_1, G_2, \ldots, G_n$ amalgamating the subgroups H_{ij} and H_{ji} under the isomorphisms θ_{ij}.

$\theta|_H$ will mean θ restricted to the subgroup H.

We shall also make use of the following known result, which is analogous to Schreier's result on generalized free products with amalgamated subgroups.

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THEOREM. *The generalized direct product with one amalgamated subgroup always exists and is unique to within isomorphisms.*

In our definition of generalized direct products, we restrict ourselves to the case of amalgams with a finite collection of groups. The concept can actually be extended to infinite collections of groups. We shall restrict ourselves to a finite collection of groups. This does not impair the generality of the work since R. Bear has shown that an amalgam of an infinite collection of groups is embeddable in an abelian group if every subamalgam of a finite collection of groups is embeddable in an abelian group. Hence a generalized direct product of an amalgam of an infinite collection of groups exists if a generalized direct product of every subamalgam of a finite collection of groups exists.

3. DEFINITION 3.1. *A* GD-*product* II *of the groups G and G' amalgamating the subgroups $H_i \subseteq G$ and $H_i' \subseteq G'$ under the isomorphisms θ_i mapping H_i onto H_i' $(1 \leqslant i \leqslant n)$ exists if and only if there exists a group P containing two subgroups Q and Q' such that:*

(i) *P is generated by Q and Q';*
(ii) *every element of Q permutes with every element of Q';*
(iii) *there exist isomorphisms ϕ and ϕ' mapping G and G' onto Q and Q' respectively;*
(iv) *if K is the subgroup of G generated by the subgroups H_i $(1 \leqslant i \leqslant n)$ and K' the corresponding subgroup of G' generated by the subgroups H_i', then $\phi(K) = \phi'(K') = Q \cap Q'$;*
(v) *if $h_i \in H_i$, then $\phi(h_i) = \phi'\theta_i(h_i)$.*

As in the case of a GD-product I the necessary conditions for a GD-product II to exist are: (a) the subgroups H_i and H_i' must be contained in the centres of G and G' respectively; (b) the isomorphisms θ_i must satisfy certain compatibility conditions:

(i) θ_i maps $H_i \cap H_j$ isomorphically onto $H_i' \cap H_j'$;
(ii) θ_i coincides with θ_j where both are defined, that is, if $h \in H_i \cap H_j$, then $\theta_i(h) = \theta_j(h)$.

Again as in the case of a GD-product I these conditions are not sufficient to ensure the existence of a GD-product II. The following example will illustrate the non-existence of such a product although the above conditions are satisfied.

Example. Let G be the abelian group freely generated by the elements a, b, c, d, and let G_i be another abelian group freely generated by the elements a, u, v. Let $H_1 = \{a, bc^{-1}\}$, $H_1' = \{a, au\}$ where $\theta_1(a) = a$ and $\theta_1(bc^{-1}) = au$,

$$H_2 = \{a, cd^{-1}\}, \quad H_2' = \{a, u^{-1}v\} \quad \text{where} \quad \theta_2(a) = a \text{ and } \theta_2(cd^{-1}) = u^{-1}v,$$

$$H_3 = \{a, db^{-1}\}, \quad H_3' = \{a, v^{-1}\} \quad \text{where } \theta_3(a) = a \text{ and } \theta_3(db^{-1}) = v^{-1}.$$

It is clear that the θ_i's are compatible. But in forming a GD-product II, we

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need to identify bc^{-1} with au, cd^{-1} with $u^{-1}v$, and db^{-1} with v^{-1}. But such identification will give

$$1 = bc^{-1}\cdot cd^{-1}\cdot db^{-1} = au\cdot u^{-1}v\cdot v^{-1} = a.$$

Thus ϕ and ϕ' will not be isomorphisms. Hence no GD-product II of G and G' can exist.

THEOREM 3.2. (Reduction Theorem). *A* GD-*product* II *of G and G' amalgamating H_i and H_i' $(1 \leqslant i \leqslant n)$ exists if and only if a* GD-*product* II *of K and K' amalgamating H_i and H_i' $(1 \leqslant i \leqslant n)$ exists.*

Proof. In one direction the theorem is trivial since the existence of a GD-product II of G and G' amalgamating H_i and H_i' automatically implies the existence of a GD-product II of K and K' amalgamating H_i and H_i'. To prove the converse, since a GD-product II of K and K' amalgamating H_i and H_i' $(1 \leqslant i \leqslant n)$ exists, there exists a group P satisfying the conditions of Definition 3.1. In fact, in this case we can take $\phi(K) = P = \phi'(K')$. Since the GD-product I amalgamating one subgroup always exists, we can form the GD-product I L of P and G amalgamating $\phi(K)$ and K under the isomorphism ϕ. Similarly, we form $L' = (G' \times P)_{K'=\phi'(K')}$. Since K and K' are contained in the centres of G and G' respectively, this implies that P is in the centres of L and L' respectively. Moreover, $L \cap L' = P$. Thus we can form the GD-product I, R, of L and L' amalgamating P under the identity isomorphism. We shall show that R is a GD-product II of G and G'. To prove this we note first that R is generated by G and G'. Thus we can take the identity isomorphism I to be the ϕ and ϕ' of Definition 3.1. We also note that in R every element of G permutes with every element of G'. Moreover, $G \cap P = P = G' \cap P$ or $P \subseteq G \cap G'$. But $G \cap G' \subseteq L \cap L' = P$. Hence $G \cap G' = P = I\phi(K) = I\phi'(K')$. Finally, the existence of P implies that $\phi(h_i) = \phi'\theta_i(h_i)$ for all $h_i \in H_i$, i.e., $I\phi(h_i) = I\phi'\theta_i(h_i)$. Hence a GD-product II of G and G' amalgamating H_i and H_i' under θ_i exists.

Since K and K' are contained in the centres of G and G' respectively, from Theorem 3.2, it follows that the problem of finding necessary and sufficient conditions for the existence of a GD-product II can be reduced to that of finding necessary and sufficient conditions for the existence of a GD-product II of two abelian groups generated respectively by the subgroups to be amalgamated. Hence, unless otherwise specified, we shall consider this case only.

THEOREM 3.3. *Let G and G' be two abelian groups generated respectively by their subgroups H_i and H_i' $(1 \leqslant i \leqslant n)$, such that for every i there exists an isomorphism θ_i that maps H_i onto H_i'. Then a* GD-*product* II *of G and G' amalgamating H_i and H_i' exists if and only if there exists an isomorphism θ of G onto G' such that $\theta|_{H_i} = \theta_i$ $(1 \leqslant i \leqslant n)$.*

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Proof. If a GD-product II of G and G' exists, then by Definition 3.1, putting $G = K$ and $G' = K'$, we have:

$$G \xrightarrow{\phi} \phi(G) \xrightarrow{I} \phi'(G') \xrightarrow{\phi'^{-1}} G',$$

where I is the identification map defined by $I\phi(h_i) = \phi'\theta_i(h_i)$ for all $h_i \in H_i$ $(1 \leqslant i \leqslant n)$. Let $\theta = \phi'^{-1}I\phi$. Then θ is the required isomorphism mapping G onto G' such that $\theta|_{H_i} = \theta_i$.

To prove the converse: Since θ is an isomorphism of G onto G' with $\theta|_{H_i} = \theta_i$, the GD-product I P of G and G' exists. It is clear that P is a GD-product II of G and G' amalgamating H_i and H_i'.

COROLLARY 3.4. $P \approx G \approx G'$.

THEOREM 3.5. *The* GD-*product* II *of G and G' amalgamating H_i and H_i' under the isomorphism θ_i $(1 \leqslant i \leqslant n)$ if it exists, is unique to within isomorphisms.*

The proof follows immediately from Theorem 3.3 and the fact that the GD-product I of two groups is unique to within isomorphisms.

4. We shall now derive necessary and sufficient conditions for the existence of a GD-product I of a given amalgam A consisting of the set of groups $G_1, G_2, \ldots, G_n$ and the subgroups H_{ij} together with the isomorphisms θ_{ij} $(1 \leqslant i, j \leqslant n)$, where the θ_{ij}'s are compatible.

THEOREM 4.1. *A* GD-*product* I *P of the amalgam A exists if and only if a* GD-*product* I *G' of $G_1, \ldots, G_{n-1}$ amalgamating H_{ij} and H_{ji} $(i \neq j;\ 1 \leqslant i, j \leqslant n-1)$ exists in such a way that also the* GD-*product* II *Q of G' and G_n amalgamating H_{in} and H_{ni} $(1 \leqslant i \leqslant n-1)$ exists.*

Proof. If the GD-product I P exists, then certainly a GD-product I G' exists. Consider the subgroup K of P generated by H_{in} $(1 \leqslant i \leqslant n-1)$. We note that the subgroup K' of G' generated by H_{ni} $(1 \leqslant i \leqslant n-1)$ is the same as K with the identification $h_{ni} = \theta_{ni}(h_{ni})$, $h_{ni} \in H_{ni}$. Letting ϕ and ϕ' of Definition 3.1 be the identity maps, we see that all the conditions of the definition are satisfied. Combining this with Theorem 3.2, we find that the GD-product II Q of G_n and G' amalgamating H_{ni} and H_{in} $(1 \leqslant i \leqslant n-1)$ exists.

Conversely, if G' and Q exist, then Q is a GD-product I of A. For clearly Q is generated by $G_1, \ldots, G_n$ and every element of G_i permutes with every element of G_j for $i \neq j$. Also in Q, $G_i \cap G_n = H_{in}$ for all $1 \leqslant i \leqslant n-1$. Moreover, by Corollary 3.4 and Theorem 3.2, the subgroup of Q generated by $G_1, \ldots, G_{n-1}$ is isomorphic to G'. In fact, the isomorphism is the identity map, since ϕ and ϕ' of Definition 3.1 are identity maps in this case. Thus in Q we have $G_i \cap G_j = H_{ij}$ $(i \neq j; 1 \leqslant i, j \leqslant n-1)$. This completes the proof.

Combining Theorems 3.3 and 4.1, we obtain

THEOREM 4.2. *A* GD-*product* I *of the amalgam A exists if and only if a* GD-*product* I *of the subamalgam consisting of the groups $G_{\alpha_1}, \ldots, G_{\alpha_{n-1}}$ exists, and also there exists an isomorphism θ of the subgroup generated by $H_{\alpha_i \alpha_n}$ $(1 \leqslant i \leqslant n-1)$ onto the subgroup of G_{α_n} generated by $H_{\alpha_n \alpha_i}$ $(1 \leqslant i \leqslant n-1)$ such that $\theta|_{H_{\alpha_i \alpha_n}} = \theta_{\alpha_i \alpha_n}$ $(1 \leqslant i \leqslant n-1)$.*

Theorem 4.2 gives us a process of determining the existence of a GD-product I of a given amalgam by determining the existence of a GD-product I of its subamalgams and the corresponding extension of isomorphisms. It might be noted here that the process of extending isomorphisms is not an effective process. However, if the amalgam consists of groups characterized by some general properties such as an amalgam of cyclic groups, the theorem is very useful.

5. We shall apply the above results to different cases.

THEOREM 5.1. **(6)** *A* GD-*product* I *of an amalgam of three groups satisfying the compatibility conditions always exists.*

Proof. Let G_1, G_2, and G_3 be the groups with subgroups $H_{ij} \approx H_{ji}$ $(i \neq j;$ $1 \leqslant i, j \leqslant 3)$ to be amalgamated. We need only consider the case of G_i generated by H_{ij} $(i \neq j;\ 1 \leqslant j \leqslant 3)$. Let G' be the GD-product I of G_1 and G_2 amalgamating the subgroups H_{12} and H_{21}. Let K' be the subgroup of G' generated by H_{13} and H_{23}. Since θ_{13} and θ_{23} are compatible, it follows that $D' = H_{13} \cap H_{23}$ is isomorphic to $D = H_{31} \cap H_{32}$. Now every element x of K' can be expressed as $x = a_1 a_2 d$, where a_1, a_2, not elements of D', are elements of H_{13} and H_{23} respectively and $d \in D'$. Define ϕ on K such that

$$\phi(x) = \phi(a_1 a_2 d) = \theta_{13}(a_1 d)\theta_{23}(a_2 d)\theta_{13}(d^{-1}).$$

To show that ϕ is well defined, we let $y = b_1 b_2 h$ be another element of K' with $b_1 \in H_{13}$, $b_2 \in H_{23}$, $b_1, b_2 \notin D'$ and $h \in D'$. Then

$$\begin{aligned}\phi(x)\phi(y) &= \phi(a_1 a_2 d)\phi(b_1 b_2 h),\\ &= \theta_{13}(a_1 d)\theta_{23}(a_2 d)\theta_{13}(d^{-1})\theta_{13}(b_1 h)\theta_{23}(b_2 h)\theta_{13}(h^{-1}),\\ &= \theta_{13}(a_1 d b_1 h)\theta_{23}(a_2 d b_2 h)\theta_{13}(d^{-1} h^{-1}),\\ &= \phi(a_1 b_1 a_2 b_2 dh),\\ &= \phi(xy).\end{aligned}$$

It is not difficult to see that the map ϕ is also one-to-one and onto. Moreover, $\phi|_{H_{13}} = \theta_{13}$ and $\phi|_{H_{23}} = \theta_{23}$. Hence, by Theorem 4.2, a GD-product I of G_1, G_2, and G_3 exists.

COROLLARY 5.2. *The* GD-*product* II *of G and G' amalgamating two subgroups satisfying the compatibility conditions always exists.*

LEMMA 5.3. *Let G be a cyclic group generated by the element a, and let K and S be subgroups of G such that K and S are generated by a^r and a^s, $(r, s) = 1$. If K is generated by the subgroups $H_1, H_2, \ldots, H_n$, then the subgroup $D = K \cap S$ is generated by the subgroups $H_i \cap S$ $(1 \leqslant i \leqslant n)$.*

Proof. Let H_i be generated by the elements a^{α_i}. Since $a^{\alpha_i} \in K$, this implies $\alpha_i = k_i r$ for some integer k_i. Moreover, since K is generated by the subgroups H_i, this implies $(\alpha_1, \ldots, \alpha_n) = r$, where $(\alpha_1, \ldots, \alpha_n)$ denotes the greatest common divisor of $\alpha_1, \ldots, \alpha_n$. But this implies $(k_1, \ldots, k_n) = 1$. Let $H_i \cap S = \{a^{\beta_i}\}$, where $\beta_i = [\alpha_i, s]$ with $[\alpha_i, s]$ denoting the least common multiple of α_i and s. Clearly $rs|\beta_i$ for $1 \leqslant i \leqslant n$. We shall show that $(\beta_i, \ldots, \beta_n) = rs$. Let $(\alpha_i, s) = \gamma_i$. Then $\alpha_i = \gamma_i \delta_i$ and $s = d_i \gamma_i$ for some integers δ_i and d_i. This implies $\beta_i = \delta_i \gamma_i d_i = \delta_i s$. Since $r|\beta_i$, we have $\delta_i = \epsilon_i r$ for some integer ϵ_i. Thus $\alpha_i = \epsilon_i \gamma_i r = k_i r$. Therefore $k_i = \epsilon_i \gamma_i$. Hence $(\epsilon_1 \gamma_1, \ldots, \epsilon_n \gamma_n) = 1$. Also $\beta_i = \epsilon_i r \gamma_i d_i = \epsilon_i rs$. This implies $(\beta_1, \ldots, \beta_n) = rs$ if and only if $(\epsilon_1, \ldots, \epsilon_n) = 1$. But this is so, for if $(\epsilon_1, \ldots, \epsilon_n) \neq 1$, then $(\epsilon_1, \gamma_1, \ldots, \epsilon_n \gamma_n) \neq 1$. This proves the lemma.

THEOREM 5.4. **(5)** *A* GD-*product* I *of any amalgam of cyclic groups satisfying the compatibility conditions always exists.*

Proof. According to the remark in Section 2, we need only show the existence of a GD-product I of an amalgam consisting of only a finite number of cyclic groups. Moreover, because of the compatibility conditions, we need only consider two cases: (1) an amalgam of infinite cyclic groups; (2) an amalgam of finite cyclic groups.

The proof for both cases is the same. We shall begin by induction. Let G_i $(1 \leqslant i \leqslant n)$ be a set of cyclic groups generated by a_i and let H_{ij} $(i \neq j;$ $1 \leqslant i, j \leqslant n)$ be the set of subgroups to be amalgamated. Since the GD-product I of two groups always exists, therefore, by induction we can assume that a GD-product I of the groups $G_1, G_2, \ldots, G_{n-1}$ amalgamating the subgroups H_{ij} $(i \neq j;$ $1 \leqslant i, j \leqslant n-1)$ exists. Let P be the maximal such GD-product I that is the abelian free sum of the amalgam of $G_1, \ldots, G_{n-1}$. We shall show that the GD-product II of P and G_n amalgamating the subgroups H_{in} and H_{ni} $(1 \leqslant i \leqslant n-1)$ exists. Again by induction we can assume the GD-product II of P and G_n amalgamating the subgroups H_{in} and H_{ni} $(1 \leqslant i \leqslant n-2)$ exists. This implies that the subgroup K of P generated by the subgroups H_{in} $(1 \leqslant i \leqslant n-2)$ is isomorphic to the subgroup K_n of G_n generated by H_{ni} $(1 \leqslant i \leqslant n-2)$ under the isomorphism ϕ such that $\phi|_{H_{in}} = \theta_{in}$. Since G_n is cyclic, K_n is cyclic. If $K_n = 1$, then the assertion follows trivially. Let $K_n \neq 1$. By Theorem 3.2, we can assume that G_n is generated by K_n and $H_{n,n-1}$. Therefore, without loss of generality, we can let $K_n = \{a_n^r\}$ and $H_{n,n-1} = \{a_n^s\}$ with $(r, s) = 1$. By Corollary 5.2 we need only show that ϕ and $\theta_{n-1,n}$ are compatible. Let $D = K \cap H_{n-1,n}$ and $D_n = K_n \cap H_{n,n-1}$. Let x and b generate K and $H_{n-1,n}$, respectively. Then

$D = \{x^d\} = \{b^t\}$ and $D_n = \{a_n^{rs}\}$. Let Q be the subgroup of D generated by the subgroups $Q_i = H_{in} \cap H_{n-1,n}$ $(1 \leqslant i \leqslant n-2)$ and Q_n the subgroup of D_n generated by the subgroups $H_{ni} \cap H_{n,n-1}$ $(1 \leqslant i \leqslant n-2)$. Since $\phi|_{Q_i} = \theta_{in}|_{Q_i} = \theta_{n-1,n}|_{Q_i}$ for all $1 \leqslant i \leqslant n-2$, ϕ and $\theta_{n-1,n}$ are compatible on Q, i.e., ϕ and $\theta_{n-1,n}$ map Q isomorphically onto Q_n and $\phi|_Q = \theta_{n-1,n}|_Q$. Let $\phi(x) = a_n^r$ and $\theta_{n-1,n}(b) = a_n^s$. Also let $Q = \{x^{dq}\} = \{b^{tq}\}$. Then,

$$\phi(x^{dq}) = \theta_{n-1,n}(b^{tq}),$$

that is,

$$a_n^{rdq} = a_n^{stq}.$$

Since G_n is cyclic, $a_n^{rd} = a_n^{st}$. This implies that

$$\phi(x^d) = \theta_{n-1,n}(b^t),$$

that is,

$$\phi|_D = \theta_{n-1,n}|_D.$$

This yields $\phi(D) = \theta_{n-1,n}(D) \subseteq D_n$. But, by Lemma 5.3, D_n is generated by the subgroups

$$H_{ni} \cap H_{n,n-1} = \phi(Q_i) = \theta_{n-1,n}(Q_i), \qquad 1 \leqslant i \leqslant n-2.$$

Since ϕ and $\theta_{n-1,n}$ are isomorphisms, D is generated by Q_i $(1 \leqslant i \leqslant n-2)$. Hence ϕ and $\theta_{n-1,n}$ map D isomorphically onto D_n and are compatible on D. This completes the proof.

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