Bull. Aust. Math. Soc. **79** (2009), 327–336
doi:10.1017/S0004972708001330

INJECTIVE TRANSFORMATIONS WITH EQUAL GAP AND DEFECT

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(Received 14 July 2008)

Abstract

Suppose that X is an infinite set and $I(X)$ is the symmetric inverse semigroup defined on X. If $\alpha \in I(X)$, we let $\operatorname{dom}\alpha$ and $\operatorname{ran}\alpha$ denote the domain and range of α, respectively, and we say that $g(\alpha) = |X \setminus \operatorname{dom}\alpha|$ and $d(\alpha) = |X \setminus \operatorname{ran}\alpha|$ is the 'gap' and the 'defect' of α, respectively. In this paper, we study algebraic properties of the semigroup $A(X) = \{\alpha \in I(X) \mid g(\alpha) = d(\alpha)\}$. For example, we describe Green's relations and ideals in $A(X)$, and determine all maximal subsemigroups of $A(X)$ when X is uncountable.

2000 *Mathematics subject classification*: primary 20M20.

Keywords and phrases: factorizable inverse semigroup, gap, defect, Green's relations, ideals, maximal subsemigroup.

1. Introduction

Let $I(X)$ denote the *symmetric inverse semigroup* on X: that is, the semigroup (under composition) consisting of all *one-to-one partial transformations* whose *domain*, $\operatorname{dom}\alpha$, and *range*, $\operatorname{ran}\alpha$, are subsets of X (see [3, Volume 1, p. 29]). For each $\alpha \in I(X)$, we write

$$g(\alpha) = |X \setminus \operatorname{dom}\alpha|, \quad d(\alpha) = |X \setminus \operatorname{ran}\alpha|$$

and refer to these cardinal numbers as the *gap* and *defect* of α, respectively. In this paper, we study various properties of a subsemigroup of $I(X)$ defined by

$$A(X) = \{\alpha \in I(X) \mid g(\alpha) = d(\alpha)\}.$$

Note that if X is finite, then $A(X) = I(X)$. Consequently *throughout this paper, X denotes a set with cardinal $n \geq \aleph_0$.*

Chen and Hsieh showed in [2, Section 3] that any inverse semigroup can be embedded in some $A(X)$ and that $A(X)$ is the largest *factorizable* subsemigroup of $I(X)$ (a semigroup S is *factorizable* if $S = GE$ where G is a subgroup of S and E

is a set of idempotents in S). Later, Howie [6] used certain subsemigroups of $A(X)$, namely

$$A(X, q) = \{\alpha \in A(X) \mid d(\alpha) = q\}$$

where $\aleph_0 \leq q < |X|$, to construct a class of bisimple congruence-free inverse semigroups; and Sullivan [10, Corollary 4] showed that if X is infinite, then

$$NI(X) = \{\alpha \in A(X) \mid d(\alpha) = |X|\}$$

is the semigroup generated by all of the nilpotents in $I(X)$.

In Section 2, we use the Vagner–Preston theorem to show that any factorizable inverse semigroup S can be embedded directly into $A(S)$. We characterize Green's relations on $A(X)$ in Section 3, and show that its ideals form a chain similar to that formed by the ideals in $E(X)$, the semigroup generated by all nonidentity idempotents in $T(X)$, the *total* transformation semigroup on X (see [9, Lemma 2]). In a subsequent paper, we shall use this latter result to describe the congruences on $A(X)$. Finally, in Section 4 we describe all maximal subsemigroups of $A(X)$ when X is uncountable; and we show that, in one case, this involves a maximal subsemigroup of $G(X)$, the symmetric group on X.

2. Basic ideas and results

In what follows, $Y = A \,\dot\cup\, B$ means Y is a *disjoint* union of A and B. For each nonempty $Y \subseteq X$, we write id_Y for the identity transformation with domain Y. In particular, id_X denotes the identity of $I(X)$ and the empty set $\emptyset$ acts as a zero for $I(X)$.

We extend the convention introduced in [3, Volume 2, p. 241]: namely, if $\alpha \in I(X)$ is nonzero then we write

$$\alpha = \begin{pmatrix} a_i \\ x_i \end{pmatrix}$$

and take as understood that the subscript i belongs to some (unmentioned) index set I, that the abbreviation $\{x_i\}$ denotes $\{x_i \mid i \in I\}$, and $\operatorname{ran}\alpha = \{x_i\}$, $\operatorname{dom}\alpha = \{a_i \mid i \in I\}$ and $x_i\alpha^{-1} = a_i$ for each i.

We begin with a simple result that is used often in the following.

LEMMA 1. *If $\alpha \in I(X)$ and $Y, Z \subseteq X$, then $(Y \setminus Z)\alpha = Y\alpha \setminus Z\alpha$, where, as usual, we interpret $Y\alpha$ as $(Y \cap \operatorname{dom}\alpha)\alpha$.*

In [7, pp. 237–238], the authors remarked that, if X is infinite, then $A(X)$ is a subsemigroup of $I(X)$, and we repeat the proof here for convenience. In [2, Theorem 3.1], the authors say this is 'routine'.

LEMMA 2. *If X is an infinite set, then $A(X)$ is an inverse subsemigroup of $I(X)$ with identity* id_X. *Moreover, the group of units in $A(X)$ is $G(X)$, the symmetric group on X.*

https://doi.org/10.1017/S0004972708001330 Published online by Cambridge University Press

PROOF. Using Lemma 1, we obtain the following equalities for any $\alpha, \beta \in A(X)$.

$$\begin{aligned}
d(\alpha\beta) &= |X \setminus X\beta| + |X\beta \setminus X\alpha\beta| \\
&= d(\beta) + |[(X \setminus X\alpha) \cap \operatorname{dom}\beta]\beta| \\
&= g(\beta) + |(X \setminus X\alpha) \cap \operatorname{dom}\beta| \\
&= |(X \setminus \operatorname{dom}\beta) \cap X\alpha| + |(X \setminus \operatorname{dom}\beta) \cap (X \setminus X\alpha)| + |(X \setminus X\alpha) \cap \operatorname{dom}\beta| \\
&= |(X \setminus \operatorname{dom}\beta)\alpha^{-1}| + d(\alpha) \\
&= |(X \setminus \operatorname{dom}\beta)\alpha^{-1}| + g(\alpha) \\
&= |X\alpha^{-1} \setminus (X\alpha \cap \operatorname{dom}\beta)\alpha^{-1}| + |X \setminus X\alpha^{-1}| \\
&= |X \setminus (X\alpha \cap \operatorname{dom}\beta)\alpha^{-1}| \\
&= g(\alpha\beta).
\end{aligned}$$

Therefore, $A(X)$ is a semigroup. In fact, since $d(\alpha^{-1}) = g(\alpha)$ and $g(\alpha^{-1}) = d(\alpha)$, $A(X)$ is an inverse subsemigroup of $I(X)$. Clearly, $G(X) \subseteq A(X)$, and if $\alpha \in A(X)$ and $\alpha\beta = \mathrm{id}_X$ for some $\beta \in A(X)$, then $g(\alpha) = 0$, so $\alpha \in G(X)$. Thus, $G(X)$ is the group of units in $A(X)$. □

In passing, we assert that $A(X) = A$ (say) is not a maximal inverse subsemigroup of $I(X)$. To see this, choose $\alpha \in I(X)$ with $g(\alpha) = 1$ and $d(\alpha) = 2$, and let B denote the subsemigroup of $I(X)$ generated by $A \cup \{\alpha, \alpha^{-1}\}$. Clearly $A \subsetneq B$ and B is inverse. Suppose that $B = I(X)$ and let β be an element of $I(X)$ with $g(\beta) = 0$ and $d(\beta) = 1$. Then $\beta \notin A$. If $\beta = \alpha^k\gamma$ for some $k \in \mathbb{Z}^+$ and $\gamma \in B$, then $g(\alpha) = 0$, which is a contradiction. On the other hand, if $\beta = \lambda\alpha^k\gamma$ for some $\lambda \in A$, $k \in \mathbb{Z}^+$ and $\gamma \in B$, then $g(\lambda) = 0$ and so $\lambda \in G(X)$. Therefore, $\lambda^{-1}\beta = \alpha^k\gamma$ and, since $\lambda^{-1}\beta \in T(X)$, it follows that $g(\alpha) = 0$, a contradiction. If $\beta = \alpha^{-k}\gamma$ for some $k \in \mathbb{Z}^+$ and $\gamma \in B$, then $d(\alpha) = g(\alpha^{-1}) = 0$, a contradiction. In addition, if $\beta = \lambda\alpha^{-k}\gamma$ for some $\lambda \in A$, $k \in \mathbb{Z}^+$ and $\gamma \in B$, then $\lambda \in G(X)$ and $\lambda^{-1}\beta = \alpha^{-k}\gamma$, so $d(\alpha) = g(\alpha^{-1}) = 0$, which is a contradiction. Hence, $A \subsetneq B \subsetneq I(X)$, and the assertion follows.

The Vagner–Preston theorem [3, Volume 1, Theorem 1.20] states that any inverse semigroup S can be embedded in $I(S)$ via the representation $\rho : S \to I(S)$, $a \to \rho_a$, where $\rho_a : Sa^{-1} \to Sa$, $x \to xa$, for each $a \in S$. In [2, Theorem 3.4], the authors prove that any infinite inverse semigroup S can be embedded in some $A(X)$. They do this by first using the representation ρ to embed S in $I(S)$, and then showing that $I(S) \subseteq A(X)$ where $X = S \dot{\cup} Y$ for some set Y whose cardinal equals that of S (compare with [5, Proposition 5.9.6]).

Clearly, if $\alpha \in A(X)$, then there is a bijection $\theta : X \setminus \operatorname{dom}\alpha \to X \setminus \operatorname{ran}\alpha$. In addition, if we regard α, θ as subsets of $X \times X$, then $\pi = \alpha \cup \theta$ is a permutation of X such that $\alpha = \pi \cdot \alpha^{-1}\alpha$. It follows that $A(X) = G(X)E$ where E is the set of all idempotents in $I(X)$, and so $A(X)$ is factorizable. We now show that any factorizable inverse semigroup S can be embedded directly into $A(S)$ using the representation ρ. In other words, $A(X)$ is a 'model' for all factorizable inverse semigroups. This is similar to the Vagner–Preston theorem, as well as Teissier's theorem which states that any

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right simple, right cancellative semigroup without idempotents can be embedded into a transformation semigroup with the same properties (see [3, Volume 2, Theorem 8.5]).

THEOREM 3. *If S is a factorizable inverse semigroup, then the representation ρ embeds S into $A(S)$.*

PROOF. We need to show that $\rho_a \in A(S)$ for each $a \in S$: that is, $|S \setminus Sa^{-1}| = |S \setminus Sa|$ for each $a \in S$. It is easy to see that if $S = GE$ and S is inverse, then S contains an identity, G is the group of units of S and $S = EG$ (see [2, Lemma 2.1]). Therefore, if $a \in S$, then $a = eu$ for some $e \in E$ and $u \in G$. So, $ea = a$ and $au^{-1} = e$, hence $eaa^{-1} = aa^{-1}$ and $aa^{-1}e = e$, and thus $e = aa^{-1}$ (since idempotents commute in an inverse semigroup). Clearly, for each $u \in G$, $\operatorname{dom} \rho_u = Su^{-1} = S = Su = \operatorname{ran} \rho_u$, and so ρ_u is a permutation of S. In particular, if $a = aa^{-1} \cdot u$, then $Sa = Saa^{-1} \cdot u = Sa^{-1} \cdot u$: that is, ρ_u maps $\operatorname{dom} \rho_a$ into $\operatorname{ran} \rho_a$. Similarly, $a^{-1} = a^{-1}a \cdot u$ implies that $\rho_{u^{-1}}$ maps $\operatorname{ran} \rho_a$ into $\operatorname{dom} \rho_a$. Since ρ_u permutes S, it follows that ρ_u maps $S \setminus \operatorname{dom} \rho_a$ bijectively onto $S \setminus \operatorname{ran} \rho_a$, and thus $\rho_a \in A(S)$. □

3. Green's relations and ideals

Since $A(X)$ is an inverse subsemigroup of $I(X)$, Hall's theorem [5, Proposition 2.4.2] implies that the $\mathcal{L}$ and $\mathcal{R}$ relations on $A(X)$ are the restrictions of the corresponding relations on $I(X)$ to $A(X)$, and the latter are well known: namely, $\alpha \; \mathcal{L} \; \beta$ in $I(X)$ if and only if $\operatorname{ran} \alpha = \operatorname{ran} \beta$; and $\alpha \; \mathcal{R} \; \beta$ in $I(X)$ if and only if $\operatorname{dom} \alpha = \operatorname{dom} \beta$ (see [5, Exercise 5.11.2]). In what follows, we let $r(\alpha) = |\operatorname{ran} \alpha|$ denote the *rank* of $\alpha \in I(X)$.

LEMMA 4. *Suppose that $|X| = n \geq \aleph_0$ and let $\alpha, \beta \in A(X)$. Then:*

(a) *$\beta = \lambda\alpha$ for some $\lambda \in A(X)$ if and only if* $\operatorname{ran} \beta \subseteq \operatorname{ran} \alpha$*;*
(b) *$\beta = \alpha\mu$ for some $\mu \in A(X)$ if and only if* $\operatorname{dom} \beta \subseteq \operatorname{dom} \alpha$*;*
(c) *$\beta = \lambda\alpha\mu$ for some $\lambda, \mu \in A(X)$ if and only if $r(\beta) \leq r(\alpha)$ and $d(\beta) \geq d(\alpha)$;*
(d) *$\mathcal{D} = \mathcal{J}$.*

PROOF. It remains to show (c) and (d). Clearly, if $\beta = \lambda\alpha\mu$ for some $\lambda, \mu \in A(X)$, then $r(\beta) \leq r(\alpha)$. Moreover,

$$\begin{aligned} d(\beta) &= |X \setminus X\mu| + |X\mu \setminus X\lambda\alpha\mu|, \\ &= g(\mu) + |(X \setminus X\lambda\alpha)\mu|, \\ &\geq |(X \setminus X\alpha) \cap (X \setminus \operatorname{dom} \mu)| + |(X \setminus X\alpha)\mu| + |(X\alpha \setminus X\lambda\alpha)\mu|, \\ &\geq |(X \setminus X\alpha) \cap (X \setminus \operatorname{dom} \mu)| + |(X \setminus X\alpha) \cap \operatorname{dom} \mu|, \\ &= d(\alpha). \end{aligned} \tag{1}$$

Conversely, suppose that the condition holds and write

$$\alpha = \begin{pmatrix} a_i & a_j \\ x_i & x_j \end{pmatrix}, \quad \beta = \begin{pmatrix} b_i \\ y_i \end{pmatrix}.$$

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Define $\lambda, \mu \in I(X)$ by $\lambda : b_i \mapsto a_i$ and $\mu : x_i \mapsto y_i$. If $r(\alpha) < n$, then $d(\lambda) = |J| + g(\alpha) = n$ and $g(\lambda) = g(\beta) = n$, so $\lambda \in A(X)$, and similarly $\mu \in A(X)$. If $r(\alpha) = n$ and $r(\beta) < n$, then $|J| = n$ and so $\lambda, \mu \in A(X)$ as before. If $r(\alpha) = r(\beta) = n$ then, since $d(\beta) \geq d(\alpha)$, we can choose J so that $|J| + g(\alpha) = g(\beta)$. Then $d(\lambda) = d(\beta)$ and $g(\lambda) = g(\beta)$, so $\lambda \in A(X)$ and, with the same choice of J, we also see that $\mu \in A(X)$.

To show (d), suppose that $\alpha \mathcal{J} \beta$ in $A(X)$. Then $r(\alpha) = r(\beta)$ and $d(\alpha) = d(\beta)$, and we can write

$$\alpha = \begin{pmatrix} a_i \\ x_i \end{pmatrix}, \quad \beta = \begin{pmatrix} b_i \\ y_i \end{pmatrix}, \quad \gamma = \begin{pmatrix} b_i \\ x_i \end{pmatrix}.$$

Now $d(\gamma) = d(\alpha)$ and $g(\gamma) = g(\beta) = d(\beta) = d(\alpha)$, so $\gamma \in A(X)$ and, thus, $\alpha \mathcal{L} \gamma \mathcal{R} \beta$ by (a) and (b). That is, $\mathcal{J} \subseteq \mathcal{D}$, and this completes the proof. □

Let $|X| = n \geq \aleph_0$ and, for each cardinal p, let p' denote the successor of p. In addition, for each r, d such that $1 \leq r \leq n'$ and $0 \leq d \leq n$, let

$$A(r, d) = \{\alpha \in A(X) \mid r(\alpha) < r \text{ and } d(\alpha) \geq d\}.$$

Clearly, if $\alpha \in A(r, d)$ and $\lambda, \mu \in A(X)$, then $r(\lambda\alpha\mu) < r$ and Lemma 4(c) shows that $d(\lambda\alpha\mu) \geq d(\alpha)$. Therefore, $A(r, d)$ is an ideal of $A(X)$. In particular, $A(1, n) = \{\emptyset\}$ and $A(n', 0) = A(X)$ and the sets $A(r, d)$ form a chain as follows.

$$\begin{aligned} A(1, n) &\subset A(2, n) \subset \cdots \subset A(n, n) \\ &\subset A(n', n) \subset \cdots \subset A(n', \aleph_0) \subset \cdots \subset A(n', 1) \subset A(n', 0). \end{aligned}$$

Note that each of these containments is proper. For example, if $X = A \,\dot\cup\, B \,\dot\cup\, C$ where $|A| = r < s = |B| < n$, then $|C| = n$ and $\mathrm{id}_B \in A(s', n) \setminus A(r', n)$; if $X = A \,\dot\cup\, B$ where $|A| = |B| = n$, then $\mathrm{id}_A \in A(n', n) \setminus A(n, n)$; and if $X = A \,\dot\cup\, B \,\dot\cup\, C$ where $n > |B| = d > e = |C|$, then $|A| = n$ and $\mathrm{id}_{A \cup B} \in A(n', e) \setminus A(n', d)$.

THEOREM 5. *The ideals of $A(X)$ are precisely the sets $A(r, d)$. Moreover, $A(r, d)$ is principal if and only if $r = s'$ for some cardinal s.*

PROOF. Suppose that I is a nontrivial ideal of $A(X)$ and let

$$\begin{aligned} r &= \min\{s \mid s > r(\alpha) \text{ for all } \alpha \in I\}, \\ d &= \min\{t \mid t = d(\beta) \text{ for some } \beta \in I\}. \end{aligned}$$

Note that r and d always exist since the cardinals are well ordered. Now, either $r(\alpha) < r \leq n$ for all $\alpha \in I$, or $r(\alpha) = n$ for some $\alpha \in I$. If the former occurs, then $d(\alpha) = n$ for all $\alpha \in I$, and so $I \subseteq A(r, n)$. Conversely, if $\beta \in A(r, n)$, then there exists $\alpha \in I$ with $r(\beta) \leq r(\alpha)$: otherwise, if $r(\alpha) < r(\beta) < r$ for all $\alpha \in I$, we contradict the choice of r. Moreover, if $r(\beta) \leq r(\alpha) < r$, then $d(\beta) = n = d(\alpha)$. So, by Lemma 4(c), $\beta = \lambda\alpha\mu$ for some $\lambda, \mu \in A(X)$, hence $\beta \in I$ and thus $A(r, n) \subseteq I$.

Now, assume that $r(\alpha) = n$ for some $\alpha \in I$, so that $I \subseteq A(n', d)$. Now, by definition, there exists $\pi \in I$ such that $d(\pi) = d$ and clearly $r(\pi) \leq n$. If $r(\pi) < n$, then $r(\alpha) \geq d = n$. In this case, if $\beta \in A(n', n)$, then $r(\beta) \leq r(\alpha)$ and $d(\beta) = n = d(\alpha)$, so Lemma 4(c) implies that $\beta \in I$. On the other hand, if $r(\pi) = n$, then the same argument (using π instead of α) shows that $A(n', d) \subseteq I$. That is, in both cases, $I = A(n', d)$.

In effect, the first paragraph of this proof shows that, if $r \leq n$, then $A(r, d) = \{\alpha \in I(X) : r(\alpha) < r\} = I_r$ (say) and it is well known that this is an ideal of $I(X)$. Moreover, I_r is principal if and only if $r = s'$ for some cardinal s (compare with [3, Volume 1, Theorem 2.9(ii)] for the ideals of $T(X)$). Thus, we assume that $r = n'$ and show that $A(n', d)$ is principal for each $d \geq 0$. Clearly, if $X = B \dot{\cup} C$ where $|B| = n$ and $|C| = d$ then $r(\mathrm{id}_B) = n$ and $d(\mathrm{id}_B) = d$. Moreover, if $\alpha \in A(n', d)$ then, by Lemma 4(c), $\alpha = \lambda \cdot \mathrm{id}_B \cdot \mu$ for some $\lambda, \mu \in A(X)$. Hence, $A(n', d) \subseteq A(X) \cdot \mathrm{id}_B \cdot A(X)$, and the reverse containment holds since $\mathrm{id}_B \in A(n', d)$ which is an ideal of $A(X)$. $\square$

4. Maximal subsemigroups

Some work has been done on determining all maximal subsemigroups of certain transformation semigroups defined on a finite set X: see [1] for $T(X)$, [11] for the ideals of $T(X)$, and [12] for $E(X)$. There are partial results in the same direction for Baer–Levi semigroups defined on an infinite set: see [4, 8]. Here we describe all maximal subsemigroups of $A(X)$ when X is uncountable.

In what follows, we often write $A = A(X)$ and let M denote a maximal subsemigroup of A. Also, if $B \subseteq A$, then $\langle B \rangle$ denotes the subsemigroup of A generated by B. We begin with a simple result.

LEMMA 6. *Every maximal subsemigroup M of A contains id_X and $\emptyset$.*

PROOF. If $\mathrm{id}_X \notin M$, then $M \cup \{\mathrm{id}_X\}$ is a subsemigroup of A that properly contains M, so $A = M \cup \{\mathrm{id}_X\}$ (by maximality) and, hence, each nonidentity $\gamma \in G(X)$ belongs to M. In addition, $\gamma^{-1} \in M$ and so $\gamma\gamma^{-1} = \mathrm{id}_X \in M$, a contradiction. If $\emptyset \notin M$, then $A = M \cup \{\emptyset\}$, hence each nonzero idempotent of A belongs to M. In particular, if $X = Y \dot{\cup} Z$ where Y, Z are nonempty, then $\mathrm{id}_Y, \mathrm{id}_Z \in M$ and so $\emptyset \in M$, which is a contradiction. $\square$

So far, all results aimed at determining the maximal subsemigroups of a transformation semigroup reduce the given problem to knowing the maximal subgroups of $G(X)$. However, the main result of this section involves a maximal subsemigroup of $G(X)$.

LEMMA 7. *Suppose that M is a maximal subsemigroup of A and write $G = G(X)$. If $G \setminus M \neq \emptyset$, then $M \cap G$ is a maximal subsemigroup of G.*

PROOF. By Lemma 6, $M \cap G \neq \emptyset$ and clearly it is closed under composition. If $\pi \in G \setminus M$ then $A = \langle M, \pi \rangle$. Hence, if $\gamma \in G$, then there exist $\lambda, \mu, \ldots \in M$ and positive integers $r, s, \ldots$ such that $\gamma = \lambda\pi^r\mu\pi^s \ldots$. Then $\lambda \in G$ and $\pi^{-r}\lambda^{-1}\gamma = \mu\pi^s \ldots$,

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and so $\mu \in G$. Clearly, we can repeat this argument to show that each mapping that appears in the expression for γ is a permutation of X. Moreover, we reach the same conclusion if the expression for γ begins with π^r for some $r > 0$. In other words, if $\pi \notin M \cap G$, then $G \subseteq \langle M \cap G, \pi \rangle$, and equality follows. □

By a remark on [4, p. 157], there are infinitely many maximal subsemigroups of $G(X)$ which are not groups. Here we show that there are four types of maximal subsemigroup of $A(X)$ when X is uncountable. To do this, we need a special case of Lemma 4(c).

LEMMA 8. *Suppose that $\alpha, \beta \in A$ have rank n. If $d(\alpha) = d(\beta)$, then $\beta = \lambda\alpha\mu$ for some $\lambda, \mu \in G(X)$.*

PROOF. If $d(\alpha) = d(\beta) = d$ (say) then $g(\alpha) = g(\beta) = d$ and we can write

$$\operatorname{dom}\alpha = \{a_i\}, \quad X \setminus \operatorname{dom}\alpha = \{a_j\},$$
$$\operatorname{dom}\beta = \{b_i\}, \quad X \setminus \operatorname{dom}\beta = \{b_j\},$$

where $|J| = d$. If we adopt similar notation for $\operatorname{ran}\alpha$, $\operatorname{ran}\beta$ and their complements, then

$$\alpha = \begin{pmatrix} a_i \\ x_i \end{pmatrix}, \quad \beta = \begin{pmatrix} b_i \\ y_i \end{pmatrix}, \quad \lambda = \begin{pmatrix} b_i & b_j \\ a_i & a_j \end{pmatrix}, \quad \mu = \begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix}.$$

Clearly $\lambda, \mu \in G(X)$ and $\beta = \lambda\alpha\mu$, as desired. □

If $0 \le d \le e \le n$, we write

$$S[d, e] = \{\alpha \in A : d \le d(\alpha) \le e\},$$
$$S[d, e) = \{\alpha \in A : d \le d(\alpha) < e\},$$

and we use similar notation to denote other subsets of A corresponding to defects lying between specified cardinals. In particular, note that $S[0, 0] = G(X)$.

LEMMA 9. *Suppose that $|X| = n > \aleph_0$. Let T be a maximal subsemigroup of $G = G(X)$ and let d be a cardinal such that $\aleph_0 \le d < n$. Then the following sets are maximal subsemigroups of A:*

(a) $M = T \cup S[1, n]$;
(b) $M = G \cup S(1, n]$;
(c) $M = G \cup S[1, d) \cup S(d, n]$;
(d) $M = G \cup S[1, n) \cup A(n, n)$.

PROOF. (a) In this case, M contains every element of A except those in $G \setminus T$. By supposition, if $\pi \in G \setminus T$, then $G = \langle T, \pi \rangle \subseteq \langle M, \pi \rangle$, and it follows that $A = \langle M, \pi \rangle$.

(b) In this case, M contains all elements of A except those with defect 1. If $\alpha \in S[1, 1]$, then Lemma 8 implies that any element of $S[1, 1]$ belongs to $\langle M, \alpha \rangle$, and thus $A = \langle M, \alpha \rangle$.

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(c) Here, M contains all elements of A except those α with infinite defect $d < n$, in which case Lemma 8 implies that any $\beta \in A$ with defect d belongs to $\langle M, \alpha\rangle$, and thus $A = \langle M, \alpha\rangle$.

(d) Now M contains all elements of A except those α with rank and defect n. Now Lemma 8 implies that any $\beta \in A$ with rank and defect n belongs to $\langle M, \alpha\rangle$, and thus $A = \langle M, \alpha\rangle$. □

The next three lemmas will show that a maximal subsemigroup of $A(X)$ is one of the above types when $|X| = n > \aleph_0$.

Lemma 10. *If M is a maximal subsemigroup of A and $\alpha \notin M$, then $r(\alpha) = n$ and either $d(\alpha) \leq 1$ or $d(\alpha) \geq \aleph_0$.*

Proof. Clearly $M \subsetneq M \cup A\alpha A$, and $M \cup A\alpha A$ is a subsemigroup of A, so $A = M \cup A\alpha A$ (by maximality). Now, each element of $A\alpha A$ has rank at most $r(\alpha)$. Hence, if $r(\alpha) < n$, then $d(\alpha) = n$ and M contains all elements of A with rank n. Write

$$\operatorname{dom}\alpha = \{a_i\}, \quad X \setminus \operatorname{dom}\alpha = \{b_j\} \mathbin{\dot\cup} \{c_j\},$$
$$\operatorname{ran}\alpha = \{x_i\}, \quad X \setminus \operatorname{ran}\alpha = \{y_j\} \mathbin{\dot\cup} \{z_j\},$$

where $|J| = n$. Thus,

$$\alpha = \begin{pmatrix} a_i \\ x_i \end{pmatrix}, \quad \beta = \begin{pmatrix} a_i & b_j \\ x_i & y_j \end{pmatrix}, \quad \gamma = \begin{pmatrix} x_i & z_j \\ x_i & y_j \end{pmatrix},$$

where $\beta, \gamma \in A$ and have rank n. Hence, $\beta, \gamma \in M$ and clearly $\alpha = \beta\gamma \in M$, a contradiction. Therefore, each $\alpha \in A \setminus M$ has rank n.

Next we assume that $d(\alpha) \geq 2$ and d is finite. Since $r(\alpha) = n$, Lemma 4(c) implies that $A\alpha A$ consists of all of the elements of A with defect at least d. Hence, M must contain all of the elements of A with defect less than d. Write

$$\operatorname{dom}\alpha = \{a_i\}, \quad X \setminus \operatorname{dom}\alpha = \{a_1, a_2, \ldots, a_d\},$$
$$\operatorname{ran}\alpha = \{x_i\}, \quad X \setminus \operatorname{ran}\alpha = \{x_1, x_2, \ldots, x_d\},$$

and define

$$\beta = \begin{pmatrix} a_i & a_1 \\ x_i & x_1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} x_i & x_2 \\ x_i & x_2 \end{pmatrix}.$$

Then $\beta, \gamma \in A$ and $d(\beta) = d(\gamma) = d - 1$, so $\beta, \gamma \in M$ and clearly $\alpha = \beta\gamma \in M$, which is a contradiction. Therefore, if $\alpha \notin M$, then $d(\alpha) \leq 1$ or $d(\alpha) \geq \aleph_0$. □

Lemma 11. *Suppose that M is a maximal subsemigroup of A and $\alpha \notin M$. If $d(\alpha) = 1$ or $\aleph_0 \leq d(\alpha) = d < n$, then M contains no element of A with defect 1 or d.*

Proof. By the last result, $r(\alpha) = n$ and so Lemma 4(c) implies that $A\alpha A$ consists of all of the elements of A with defect at least 1 or at least d. Since $A = M \cup A\alpha A$, M must contain all of the elements of A with defect zero or defect less than d. Hence,

https://doi.org/10.1017/S0004972708001330 Published online by Cambridge University Press

in each case, $G(X) \subseteq M$. Now suppose that there exists $\beta \in M$ with $d(\beta)$ equal to 1 or $d < n$. Then $r(\beta) = n$ and, by Lemma 8, there exist $\lambda, \mu \in G(X) \subseteq M$ so that $\alpha = \lambda\beta\mu \in M$, which is a contradiction. Therefore, M contains no element of A with defect 1 or d. □

LEMMA 12. *If there exists $\alpha \notin M$ with defect d, then M contains every element of A with defect less than d or greater than d.*

PROOF. Suppose that there exists $\alpha \notin M$ with defect d and $\beta \notin M$ with defect greater than d. Then $A = \langle M, \beta\rangle$, so $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in A$ (one, but not both, of λ, μ may equal id_X). Then Lemma 4(c) implies that $d(\alpha) \geq d(\beta)$, a contradiction. Similarly, if there exists $\gamma \notin M$ with defect less than d, then $A = \langle M, \alpha\rangle$, so $\gamma = \lambda\alpha\mu$ for some $\lambda, \mu \in A$ and hence $d(\gamma) \geq d(\alpha)$, a contradiction. □

Suppose that M is a maximal subsemigroup of A. In view of the last result, at most one infinite cardinal can be the defect of some $\alpha \in A \setminus M$. Moreover, if there exists $\alpha \notin M$ with $n > d(\alpha) = d \geq \aleph_0$, then Lemma 11 implies that every element of A with defect d lies outside M.

THEOREM 13. *Suppose that $|X| = n > \aleph_0$. If M is a maximal subsemigroup of $A(X)$, then M equals one of the following sets, where T is a maximal subsemigroup of $G = G(X)$ and d is a cardinal such that $\aleph_0 \leq d < n$.*

(a) $T \cup S[1, n]$;
(b) $G \cup S(1, n]$;
(c) $G \cup S[1, d) \cup S(d, n]$;
(d) $G \cup S[1, n) \cup A(n, n)$.

PROOF. Let $\alpha \notin M$. If $d(\alpha) = 0$, then $M \cap G \subsetneq G$ and, by Lemma 7, $M \cap G = T$ (say) is a maximal subsemigroup of G. Also, Lemma 12 implies that $S[1, n] \subseteq M$, and we conclude that $M = T \cup S[1, n]$.

If $d(\alpha) = 1$ then, by Lemma 12, $G \cup S(1, n] \subseteq M$ and equality follows (since, by Lemma 11, no element of M has defect 1).

If $\aleph_0 \leq d(\alpha) = d < n$, then, by Lemma 12, $S[0, d) \cup S(d, n] \subseteq M$. Also, by Lemma 11, no element of M has defect d. Hence, in this case, $M = G \cup S[1, d) \cup S(d, n]$.

Finally, if $d(\alpha) = n$, then Lemma 12 implies that $S[0, n) \subseteq M$. In addition, if $\beta \in A$ and $r(\beta) < n$, then $\beta \in M$ (since if $\beta \notin M$, then Lemma 10 implies that $r(\beta) = n$, which is a contradiction). Thus, it follows that $M = G \cup S[1, n) \cup A(n, n)$. □

Acknowledgement

The second author is grateful to the Faculty of Science, Chiangmai University, Thailand, for its financial assistance during his visit in June 2008.

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