On a Linear Partial Differential Equation of Hyperbolic Type.

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§ 1. Riemann's method of solution of a linear second order partial differential equation of hyperbolic type was introduced in his memoir on sound waves.* It has been used by Darboux † in discussing the equation

$$\frac{\partial^2 z}{dx\, dy} + \alpha \frac{\partial z}{\partial x} + \beta \frac{\partial x}{\partial y} + \gamma z = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots 1.1$$

where α, β, γ are functions of x and y.

The method involves finding a particular solution of the partial equation adjoint to 1.1, viz.,

$$\frac{\partial^2 u}{\partial x\, \partial y} - \frac{\partial (du)}{\partial x} - \frac{\partial (\beta u)}{\partial y} + \gamma u = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots 1.2$$

This particular solution has to satisfy certain first order differential equations on the characteristics $x = \xi$, $y = \eta$ through the point at which z is to be determined.

§ 2. Suppose that we are given the value of a function $z(x, y)$ and its first derivatives on the straight lines $x = \alpha$ $(\alpha > 0)$ and $y = 0$, and that z satisfies a partial differential equation which is a particular case of 1.1, viz.,

$$F(z) \equiv \frac{\partial^2 z}{\partial x\, \partial y} + \frac{a}{x} \frac{\partial z}{dx} + \frac{b}{x} \frac{\partial z}{\partial y} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots 2.1$$

where a, b are constants $(a > 0, b > 1)$. It is required to find z at the point (ξ, η). The adjoint equation is

$$G(u) \equiv \frac{\partial^2 u}{\partial x\, \partial y} - \frac{a}{x} \frac{\partial u}{\partial x} - \frac{b}{x} \frac{\partial u}{\partial y} + \frac{au}{x^2} = 0 \quad \ldots\ldots\ldots\ldots 2.2$$

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* *Ueber die Fortpflanzung ebener Luftwellen* (*Werke*, p. 145).

† *Théorie Générale des Surfaces*, t. II., ch. IV.

We easily obtain that

$$\int_c (Mdy - Ndx) = \iint_S \{uF(z) - zG(u)\}\, dx\, dy$$

where C is the closesd boundary of a region S in the (x, y) plane, and

$$M = \frac{auz}{x} + \tfrac{1}{2}\left(u\frac{\partial z}{\partial y} - z\frac{\partial u}{\partial y}\right)$$

$$N = \frac{buz}{x} + \tfrac{1}{2}\left(u\frac{\partial z}{\partial x} - z\frac{\partial u}{\partial x}\right)$$

If z and u are solutions of equations 2.1 and 2.2 respectively throughout the region S, then

$$\int_c (M\,dy - N\,dx) = 0.$$

Applying this result to the rectangle whose sides are $x = \alpha$, $x = \xi$, $y = 0$, $y = \eta$, we obtain the solution

$$z(\xi, \eta) = u(\alpha, 0\,;\, \xi, \eta)\, z(\alpha, 0)$$

$$+ \int_a^\xi u(x, 0\,;\, \xi, \eta)\left\{\frac{\partial z(x, 0)}{\partial x} + \frac{b}{x} z(x, 0)\right\} dx$$

$$+ \int_0^\eta u(\alpha, y\,;\, \xi, \eta)\left\{\frac{\partial z(\alpha, y)}{\partial y} + \frac{a}{\alpha} z(\alpha, y)\right\} dy \qquad \text{.........2.3}$$

where $u(x, y\,;\, \xi, \eta)$ is a solution of equation 2.2 which satisfies the relations

(i) $\dfrac{\partial u}{\partial y} - \dfrac{au}{x} = 0$ when $x = \xi$,

(ii) $\dfrac{\partial u}{\partial x} - \dfrac{bu}{x} = 0$ when $y = \eta$,

(iii) $u(\xi, \eta\,;\, \xi, \eta) = 1$.

In fact, $u(x, y\,;\, \xi, \eta) = \left(\dfrac{x}{\xi}\right)^b e^{a(y-\eta)/\xi}\, F\left\{-\dfrac{(x-\xi)(y-\eta)}{x\xi}\right\}$

where $F(t) = 1 + \sum_{n=1}^{\infty} \dfrac{(b-1)(b-2)\dots(b-n)}{n!\,n!}(-at)^n$ 2.4

§ 3. Now let α tend to zero. The solution 2.3 becomes

$$z(\xi, \eta) = \int_0^\xi u(x, 0; \xi, \eta) \left\{ \frac{\partial z(x, 0)}{\partial x} + \frac{b}{x} z(x, 0) \right\} dx$$

$$+ \underset{\alpha \to 0}{\mathrm{Lt}} \int_0^\eta u(\alpha, y; \xi, \eta) \left\{ \frac{\partial z(\alpha, y)}{\partial y} + \frac{a}{\alpha} z(\alpha, y) \right\} dy$$

Holmgren* has considered this limit problem by finding an asymptotic expansion of $F(t)$ for large negative values of the argument. If $z(x, 0) = \phi(x)$ $z(0, y) = \psi(y)$, Holmgren's result is

$$z(\xi, \eta) = \int_0^\xi u(x, 0; \xi, \eta) \left\{ \phi'(x) + \frac{b}{x} \phi(x) \right\} dx$$

$$+ \frac{a^b}{\Gamma(b)\, \xi^b} \int_0^\eta \psi(y)\, e^{(y-\eta)/\xi} (\eta - y)^{b-1} dy \quad \text{.........3.1}$$

The object of this note is to show how Holmgren's result may be obtained by the more simple series solution method of T. W. Chaundy.†

§ 4. Using the notation

$$\delta = x \frac{\partial}{\partial x}, \quad \delta' = y \frac{\partial}{\partial y}, \quad t = \frac{ay}{x}$$

the equation 2.1 can be written in the form

$$(\delta\delta' + t\delta + b\delta') z = 0 \quad \text{.........................} \quad 4.1$$

Assuming a series-solution of the form

$$z = x^\alpha y^\beta \{1 + c_1 t + c_2 t^2 + \ldots + c_n t^n + \ldots\} \quad \text{............} \quad 4.2$$

we have the identity

$$\begin{aligned} &\alpha\beta + c_1(\alpha - 1)(\beta + 1)t + \ldots + c_n(\alpha - n)(\beta + n)t^n + \ldots \\ &+ b\beta + \quad c_1 b(\beta + 1)t \quad + \ldots + \quad c_n b(\beta + n)t^n \quad + \ldots \\ &+ \qquad \alpha t \qquad + \ldots + \; c_{n-1}(\alpha - \overline{n-1})t^n + \ldots \equiv 0 \end{aligned}$$

This gives us the indicial equation $\alpha\beta + b\beta = 0$ and the recurrence formula $c_n = \dfrac{(n - 1 - \alpha)c_{n-1}}{(\alpha - n + b)(\beta + n)}$.

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* *Cinquième congrès des mathématiciens scandinaves.* Helsingfors (1922), p. 260.

† *Proc. Lond. Math. Soc.* Series 2. Vol. 21, p. 214.

We have therefore either $\beta = 0$ or $\alpha = -b$, and also

$$c_n = \frac{(n-1-\alpha)(n-2-\alpha)\ldots(-\alpha)}{(\beta+n)(\beta+n-1)\ldots(\beta+1)(\alpha+b-n)(\alpha+b-\overline{n-1})\ldots(\alpha+b-1)}$$

§ 5. Take $\alpha = -b$.

Then $$c_n = \frac{(-)^n \Gamma(n+b)\Gamma(\beta+1)}{n!\,\Gamma(b)\Gamma(\beta+n+1)}$$

$$= \frac{(-)^n \Gamma(n+b)\Gamma(\beta-b+1)}{n!\,\Gamma(\beta+n+1)} \cdot \frac{\Gamma(\beta+1)}{\Gamma(b)\Gamma(\beta-b+1)}$$

If we substitute in 4.2 and omit a constant factor throughout we obtain the solution

$$x^{-b} y^{\beta} \sum_0^{\infty} \int_0^1 \theta^{n+b-1}(1-\theta)^{\beta-b} \,.\, d\theta \,.\, \frac{(-t)^n}{n!}$$

$$= \int_0^1 x^{-b}\{y(1-\theta)\}^{\beta} e^{-t\theta}\left(\frac{\theta}{1-\theta}\right)^b \frac{d\theta}{\theta}.$$

Here β is arbitrary. Multiplying by a arbitrary constant eoefficient and summing for all possible values of β, we have the solution

$$\int_0^1 x^b f\{y(1-\theta)\}\, e^{-t\theta}\left(\frac{\theta}{1-\theta}\right)^b \frac{d\theta}{\theta}$$

f being an arbitrary function

$$= \int_0^y x^{-b} f(p)\, e^{-\frac{a(y-p)}{x}} \frac{(y-p)^{b-1}}{p^b}\, dp$$

$$= \int_0^y x^{-b} E(p)\, e^{-\frac{a(y-p)}{x}} (y-p)^{b-1}\, dp \quad \ldots\ldots\ldots\ldots\ldots\ 5.1$$

where E is an arbitrary function.

To consider what value this has when $x = 0$, put $\dfrac{a(y-p)}{x} = q$.

The solution 5.1 becomes

$$\int_0^{\frac{ay}{x}} E\left(y - \frac{xq}{a}\right) e^{-q} q^{b-1} \frac{dq}{a^b}$$

which has the value

$$\int_0^\infty E(y)\, e^{-q}\, q^{b-1}\frac{dq}{a^b} \text{ when } x=0$$

$$=\frac{E(y)\,\Gamma(b)}{a^b}$$

$$\therefore \quad E(y)=\frac{a^b\,\psi(y)}{\Gamma(b)}\,.$$

The value of the expression 5.1 at the point $(\xi,\ \eta)$ is therefore

$$\frac{a^b}{\Gamma(b)\,\xi^b}\int_0^\eta \psi(y)\, e^{a(y-\eta)/\xi}\,(\eta-y)^{b-1}\,dy \ldots\ldots\ldots\ldots \quad 5.2$$

§ 6. In the case when $\beta=0$, equation 4.3 gives

$$c_n=\frac{\Gamma(\alpha+b-n)\,\Gamma(n-\alpha)}{\Gamma(\alpha+b)\,\Gamma(-\alpha)\,n!}\,.$$

The solution in this case is, omitting a constant factor throughout,

$$\sum_{n=0}^{\infty}\int_0^1 x^\alpha \frac{t^n}{n!}(1-\theta)^{\alpha+b-n-1}\,\theta^{n-\alpha-1}\,d\theta$$

$$=x^\alpha\int_0^1 e^{t\theta/(1-\theta)}\,(1-\theta)^{\alpha+b-1}\quad\theta^{-\alpha-1}\ d\,\theta$$

$$=x^\alpha e^{-t}\sum_{n=0}^{\infty}\frac{t^n}{n!}\ \frac{\Gamma(\alpha+b-n)\,\Gamma(-\alpha)}{\Gamma(b-n)}.$$

Omitting a constant factor, this may be written

$$x^\alpha\, e^{-t}\sum_{n=0}^{\infty}\frac{\Gamma(b)}{\Gamma(b-n)}\ \frac{t^n}{n!\ n!}\int_0^1 \theta^{\alpha+b-n-1}\,(1-\theta)^n\,d\theta$$

$$=x^\alpha\ e^{-t}\int_0^1 \theta^{\alpha+b-1}\,F\left\{-\frac{t}{a}\,\frac{(1-\theta)}{\theta}\right\}d\,\theta$$

where F is the function of 2.4

Here α is arbitrary. Multiplying by an arbitrary constant coefficient and summing for all values of α, we obtain the solution

$$e^{-t}\int_0^1 g(x\,\theta)\,\theta^b\,F\left\{-\frac{y}{x}\,\frac{(1-\theta)}{\theta}\right\}xd\,\theta$$

where g is an arbitrary function.

$$=\int_0^x e^{-\frac{ay}{x}}\,g(p)\left(\frac{p}{x}\right)^b F\left\{-\frac{y(x-p)}{xp}\right\}dp$$

$$=\int_0^x g(p)\,u(p,\ 0;\ x,\ y)\,dp$$

where u is the function defined in 2.4.

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As this expression has the value $\phi(x)$ when $y=0$, we easily find that

$$g(p)=p^{-b}\frac{d}{dp}\{\phi(p)\,p^{b}\}$$

The solution just obtained is then

$$\int_0^x p^{-b}\,\frac{d}{dp}\,\{p^b\,\phi(p)\}\;u\,(p,\,0\,;\;x,\,y)\,dp$$

$$=\int_0^x\left\{\phi'(p)+\frac{b}{p}\,\phi(p)\right\}\,u\,(p,\,0\,;\;x,\,y)\,dp.$$

The value of this at $(\xi,\,\eta)$ is

$$\int_0^\xi\left\{\,\phi'(x)+\frac{b}{x}\,\phi(x)\right\}u\,(x,\,0\,;\;\xi,\,\eta)\,dx \qquad\qquad 6.1$$

§ 7. From 5.2 and 6.1, we see that if z is a solution of the equation 2.1 which has the value

$\phi(x)$ when $y=0$
$\psi(y)$ when $x=0$

then

$$z(\xi,\,\eta)=\int_0^\xi u\,(x,\,0\,;\;\xi,\,\eta)\,\left\{\phi'(x)+\frac{b}{x}\,\phi(x)\right\}\,dx$$

$$+\,\frac{a^b}{\Gamma(b)\,\xi^b}\int^\eta\psi(y)\,e^{\,a\,(y-\eta)/\xi}\,(\eta-y)^{b-1}\,dy$$

which is Holmgren's result.