Can. J. Math., Vol. XXIII, No. 6, 1971, pp. 969–976

EXTREME POINTS IN THE HARDY CLASS H^1 OF A RIEMANN SURFACE

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Introduction. The theorems presented here extend known results on the set of extreme points of the unit ball of the Hardy class H^1 of a disk to the situation of an arbitrary Riemann surface. Several new results are obtained. The initial motivation for this work was provided by the theorem of de Leeuw and Rudin [**2**, p. 471] characterizing the extreme points in the case of a disk. Careful scrutiny of the proof of that theorem yields one necessary and one sufficient condition for being an extreme point in H^1 of an arbitrary surface (Theorems 1 and 4 below). The material presented here on compact bordered surfaces is closely related to the beautiful results of Gamelin and Voichick [**4**] and the results of Forelli [**3**].

For a subharmonic function u, which has a harmonic majorant on the Riemann surface R, Mu will denote the least harmonic majorant of u. For a superharmonic function v, the corresponding greatest harmonic minorant will be denoted by mv. The notation is that of Heins [**6**, p. 6]. $H^p(R)$ is the class of functions f, holomorphic on R, for which $|f|^p$ has a harmonic majorant. Here, $0 < p < \infty$. $H^\infty(R)$ is the Banach space of bounded holomorphic functions on R. For $1 \leqq p$, H^p is a Banach space with norm $||f||_p = (M|f|^p(\zeta))^{1/p}$, where $\zeta \in R$ (*cf.* [**6**, p. 11]). Any Riemann surface for which some H^p space has non-constant members has the disk $U = \{z| \, |z| < 1\}$ as its canonical universal covering surface. This means that many results may be proved by lifting to the disk. It also allows one to prove that $H^p(R)$ is isometrically isomorphic to a closed subspace of $H^p(U)$, when $1 \leqq p$. Unfortunately, this technique of lifting to U is not available in other situations, e.g., when R is a domain in C^n, $n > 1$. The techniques used in this paper are intrinsic to the surface and, since the Hardy classes may be defined by the use of majorants in other situations (*cf.* [**5**; **10**, p. 52]), it is possible that these methods will extend also. The techniques in [**3**] and [**4**] do depend on the uniformization theorem; those of this paper do not. Some of the results of those papers may be proved using the method below.

The first two theorems are proved using elementary means; the hypotheses of Theorem 2 are then weakened. The remaining theorems rely heavily on results of M. Heins [**6**, p. 17].

In what follows, extreme point always means extreme point of the unit ball of H^1.

Received February 18, 1971, in revised form June 4, 1971, and again in revised form, August 20, 1971.

Extreme points in H^1. The results of this section apply to an arbitrary Riemann surface R.

Because H^1 is a Banach space, the function f of norm 1 is an extreme point of the unit ball if and only if $g \in H^1$ and $\|f \pm g\| = 1$ imply that g is the zero function. If $g \in H^1$, $2M|f| \leqq M|f + g| + M|f - g|$; equality holds in this last inequality if $\|f \pm g\| = 1$, since both harmonic functions which appear are positive and have the same value at ζ. So

$$M|f| = (M|f + g| + M|f - g|)/2. \tag{1}$$

This equation may be complemented with the following one which holds without restriction.

$$(M|f + g| + M|f - g|)/2 = M((|f + g| + |f - g|)/2). \tag{2}$$

More generally, suppose that u and v are non-negative subharmonic functions which have harmonic majorants. From the fact that $M(u + v) - v$ is superharmonic and the fact that $u \leqq M(u + v) - v$, we have $Mu \leqq M(u + v) - v$. Then $v \leqq M(u + v) - Mu$, so $Mv \leqq M(u + v) - Mu$. Thus, $Mu + Mv \leqq M(u + v)$. The reverse inequality is clear. Hence, equality holds; i.e., $Mu + Mv = M(u + v)$.

For H^1 of the circle, a function f of norm 1 is an extreme point if and only if each real-valued function $h \in L^\infty$, which satisfies $hf \in H^1$, is constant [**7**, p. 139]. This condition has the following analogue.

LEMMA 1. *The necessary and sufficient condition for a function $f \in H^1$ of norm 1 not to be an extreme point of the unit ball is that there be a non-constant meromorphic function ψ such that $\psi f \in H^1$ and*

$$M|f| = (M|f + \psi f| + M|f - \psi f|)/2. \tag{3}$$

Proof. The necessity of the condition has already been shown. For the sufficiency, suppose that a function ψ exists. f will not be an extreme point if $\|f \pm \psi f\| = 1$. Since (3) holds, we may suppose that $\|f + \psi f\| > 1$. We show how to modify ψ to produce the desired result. Those meromorphic functions ψ for which equation (3) holds and for which $\psi f \in H^1$ form a convex set. In fact,

$$\begin{aligned}|f - (t\psi_1 + (1 - t)\psi_2)f| + |f + (t\psi_1 + (1 - t)\psi_2)f| &\leqq t|f - \psi_1 f| \\ &+ (1 - t)|f - \psi_2 f| + t|f + \psi_1 f| + (1 - t)|f + \psi_2 f|.\end{aligned}$$

Therefore, the majorant of the function on the left side of the above inequality does not exceed $2M|f|$, since this function dominates that on the right side. The function on the left side dominates $2|f|$, so its majorant dominates $2M|f|$, whence (3) holds with $t\psi_1 + (1 - t)\psi_2$ replacing ψ. Since the functions ψ and -1 both belong to this set, the function $g = g_t = (t(-1) + (1 - t)\psi)f$ satisfies equation (1) for any t, $0 \leqq t \leqq 1$. The map $t \to \|f + g_t\|$ is continuous. Its value at 0 exceeds 1 while its value at 1 is 0; thus, there is a t for

https://doi.org/10.4153/CJM-1971-104-4 Published online by Cambridge University Press

which $||f + g_t|| = 1$. Equation (1) implies that $||f - g_t|| = 1$; g_t is not the zero function, for ψ is not constant. Hence, f is not an extreme point.

In the case of the disk an outer function has the property that its modulus is maximal among the moduli of those functions in H^1 which have the same least harmonic majorant. This property is characteristic of extreme points as shown by the following theorem.

THEOREM 1. *Suppose that f is an extreme point of H^1. If $F \in H^1$ satisfies $|f| \leqq |F|$ and $||F|| = 1$, then $|f| = |F|$.*

Proof. $|f| \leqq |F|$ and $||F|| = 1$, taken together, imply that $M|f| = M|F|$. The condition $|f| \leqq |F|$ means that $f = \phi F$, where $\phi \in H^\infty$ and $|\phi| \leqq 1$. Suppose that $\psi = (\phi + \phi^{-1})/2$; the function ψ is meromorphic and $\psi f \in H^1$. If $v = |f + \psi f| - M|f|$ and $s = M|f| - |f - \psi f|$, v is subharmonic and s is superharmonic. Also, $2|f \pm \psi f| = |1 \pm \phi|^2|F|$, $2(1 + |\phi|^2) = |1 + \phi|^2 + |1 - \phi|^2$, and $M|F| = M|f|$ together imply that $v \leqq s$. Therefore, $Mv \leqq ms$ which is the same as $M|f + \psi f| + M|f - \psi f| \leqq 2M|f|$, which implies equation (3). f being an extreme point, Lemma 1 implies that ψ is contant; so ϕ is constant. Then $M|f| = |\phi|M|F|$ and $M|f| = M|F|$; so $|\phi| = 1$ and $|f| = |F|$.

We follow the above necessary condition with one which is sufficient.

THEOREM 2. *If $f \in H^1$ and has norm* 1 *and satisfies the implication that for each subharmonic function q, $M(\exp q) \leqq M|f|$ implies $\exp(q) \leqq |f|$, then f is an extreme point.*

Proof. Suppose that $g \in H^1$ and that $||f \pm g|| = 1$; then (1) holds. Now, $(|f + g| + |f - g|)/2$ and its logarithm are subharmonic [**8**, p. 18]; so a suitable function q is $\log ((|f + g| + |f - g|)/2)$. The hypotheses on f imply that $(|f + g| + |f - g|)/2 \leqq |f|$, whence equality holds in this last inequality. From $|f| \geqq |g|$, we get $g = \phi f$ where $\phi \in H^\infty$ and $|\phi| \leqq 1$. Then $2 = |1 + \phi| + |1 - \phi|$. This equality implies that ϕ is real-valued; thus, it is constant. This is enough to imply that g is the zero function.

The above theorem implies that every constant function of modulus one is an extreme point. An inner function ϕ is a member of H^∞ such that $|\phi| \leqq 1$ and $M|\phi| = 1$. Theorem 1 implies an inner function which is an extreme point of H^1 must be constant. Inner functions are extreme points in H^p for every p, $1 < p \leqq \infty$, for an arbitrary surface. This is so for $1 < p < \infty$, since H^p is uniformly convex. When $p = \infty$, this is a consequence of the following sufficient condition for being an extreme point of H^∞: if $||f||_\infty = 1$, then f is an extreme point of the unit ball of H^∞ if the superharmonic function $\log(1 - |f|)$ does not have a greatest harmonic minorant. This condition is the analogue of the sufficient condition for the situation on the disk (cf. [**7**, p. 138]).

In Theorem 2, the hypotheses could be amended thus: for each non-negative subharmonic h, $Mh \leqq M|f|$ implies that $h \leqq |f|$. Theorem 2 and the proof of Theorem 1 were given as above to avoid making an appeal to deeper results which follow. Theorem 2 can be improved by making an appeal. To proceed further, the analogue of the factorization of H^p functions for a disk and the analogue of strong subharmonicity for subharmonic functions are needed. We state here the results of Heins [**6**, p. 17] (in abbreviated form) which accomplish these things and which will be used repeatedly.

THEOREM 3. *If u is a subharmonic function, $u \not\equiv -\infty$, and $M(\exp u) < \infty$, the following hold*:

(a) *u has a unique representation of the form*

$$u = Q - s - p, \tag{4}$$

where Q is the difference of quasi-bounded harmonic functions, s is a singular harmonic function, and p is a potential.

(b) $M(\exp Q) < \infty$, *and* $M(\exp Q) = M(\exp Mu) = M(\exp u)$.

(c) *If* $\Phi_u = \{v|v$ *is subharmonic and* $M(\exp v) = M(\exp u)\}$, *the upper envelope h of Φ_u belongs to Φ_u.*

(d) *h is the difference of quasi-bounded harmonic functions and* $\Phi_u = \{v|v$ *is subharmonic and $v = h - s_1 - p_1$ as in* (a)$\}$.

Here, potential refers to a Green potential; i.e., a non-negative superharmonic function whose greatest harmonic minorant is zero.

THEOREM 4. *If f is a member of H^1 of norm* 1 *which satisfies the implication that for each $g \in H^1$, $M|g| \leqq M|f|$ implies that $|g| \leqq |f|$, then f is an extreme point.*

Proof. Suppose that $g \in H^1$ and that $||f \pm g|| = 1$. Equation (1) holds and implies that $|g| \leqq M|f|$. Then $|g| \leqq |f|$, whence $g = \psi f$, where $\psi \in H^\infty$ and $|\psi| \leqq 1$. Now,

$$M(\exp Q) = M|f| = M((|f + \psi f| + |f - \psi f|)/2),$$

where $\log |f| = Q - s - p$ as in (4); so Theorem 3(d) implies that

$$\log((|f + \psi f| + |f - \psi f|)/2) = Q - s_1 - p_1.$$

This means that

$$w = \log((|1 + \psi| + |1 + \psi|)/2) = s - s_1 + p - p_1.$$

But $\psi \in H^\infty$ and w is subharmonic; therefore, $M(\exp w) < \infty$ and w must have the form of (4). The relation $s - s_1 + p - p_1 \geqq 0$ implies $s - s_1 \geqq 0$ and, consequently, $s - s_1$ is a singular harmonic function (cf. [**6**, p. 8]). If the representation of w from (4) is $Q' - s_2 - p_2$, then

$$p_1 + Q' = s_2 + s - s_1 + p + p_2.$$

The uniqueness of this representation means that $s_2 + s - s_1 = 0$, $Q' = 0$ and that $p_1 = p + p_2$. But s_2 and $s - s_1$ both singular imply that $s_2 = 0 = s - s_1$. Therefore, $p - p_1 \leqq 0$ and $p - p_1 \geqq 0$. So $p = p_1$. Hence, $|1 + \psi| + |1 - \psi| = 2$ which implies that ψ is real, so it is constant. $||f \pm g|| = 1$ imply that ψ is identically zero; therefore, f is an extreme point.

A corollary to the above theorem is that every "outer" function of norm 1 is an extreme point.

COROLLARY 1. *If f is a member of H^1 of norm* 1 *such that* $\log|f|$ *is the difference of quasi-bounded harmonic functions, then f is an extreme point.*

Proof. This result relies on an easy corollary to Theorem 3, viz., if $\log|f|$ has the form $Q - s - p$ of (4), then $m(\log M|f|) = Q$. Suppose that $f \in H^1$, f has norm 1, and $\log|f| = Q$. Then $M|g| \leqq M|f|$ implies $\log M|g| \leqq \log M|f|$; therefore, $m(\log M|g|) \leqq m(\log M|f|) = \log|f|$. From $\log|g| \leqq \log M|g|$, we have $\log|g| \leqq m(\log M|g|)$, whence $\log|g| \leqq \log|f|$; then Theorem 4 implies that f is an extreme point.

In terms of the covering of R by U, the functions of Corollary 3 are those which are projections of the outer functions on the disk. Looking at this another way: since $H^1(R)$ is isometrically isomorphic to a closed subspace of $H^1(U)$, the functions which are in $H^1(R)$ and are extreme points of $H^1(U)$ are extreme points of $H^1(R)$.

There is additional information to be obtained from the situation of Lemma 1. Suppose that $f \in H^1$, $||f|| = 1$; suppose further that ψ is meromorphic and not zero, $\Psi f \in H^1$, and equation (3) holds. We have, from (4), $\log|f| = Q - s - p$ and $\log|\psi f| = Q' - s' - p'$, whence

$$\log|\psi| = Q' - Q + s - s' + p - p'.$$

Now, $|\psi f| \leqq M|f|$ follows from (3), so $Q' - Q \leqq 0$, and it follows that $Q' - Q = -q$, where q is quasi-bounded. Then

$$\log|\psi| = -q + s - s' + p - p'. \tag{5}$$

Moreover, (2) implies (Theorem 3) that

$$\log((|1 + \psi| + |1 - \psi|)/2) = s - s'' + p - p'', \tag{6}$$

where s'' is a singular harmonic function and p'' is a Green potential. The relation $0 \leqq s - s'' + p - p''$ implies the relation $0 \leqq s - s''$. The inequality $0 \leqq s - s''$ implies that $s - s''$ is a singular harmonic function in (6).

The next theorem shows that if f is not an extreme point and if the "inner part" of f dominates the "inner part" of F, then F is not an extreme point.

THEOREM 5. *Suppose that f and F are members of H^1 of norm* 1; *then* $\log|f| = Q - s - p$ *and* $\log|F| = Q' - t - l$ *are the representations of f and*

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F given by (4). *If $t \geqq s$ and $l \geqq p$, f is not extreme point implies that F is not an extreme point.*

Proof. Suppose that f is not an extreme point. Lemma 1 implies the existence of a meromorphic ψ, with $\psi f \in H^1$ and (3) holds. By (5)

$$\log|\psi F| = Q' - q + s - s' - t + p - p' - l.$$

But $-s + s' + t$ is a singular harmonic function since $t \geqq s$; because l and p are sums of Green functions and $l \geqq p$, $p' + l - p$ is a Green potential. Therefore, $\psi F \in H^1$ by Theorem 3(d), since $Q' - q \leqq Q'$. From (6)

$$\log((|F + \psi F| + |F - \psi F|)/2) = Q' - (s'' - s + t) - (p'' - p + l),$$

and, similarly, Theorem 3(d) and Theorem 3(c) imply that (3) holds with F in place of f. Thus, F is not an extreme point.

COROLLARY 2. *If f and F are members of H^1 of norm 1 with the representations $\log|f| = Q - s - p$ and $\log|F| = Q' - s - p$, then f is an extreme point if and only if F is an extreme point.*

Thus, an extreme point is determined by its "inner part" (cf. [**4**, p. 292]).

It is, of course, of interest to know that there are extreme points in H^1. The constant functions of modulus one are always extreme points. If $H^1(R)$ has non-constant members, does the unit ball have non-constant extreme points? The answer is yes and can be inferred from the Krein-Milman theorem.

$H^1(R)$ is isometrically isomorphic to a closed subspace of $H^1(U)$ which is in turn isometrically isomorphic to a closed subspace of the space M of bounded Borel measures on the circle. The weak-star topology of M induces a topology on $H^1(R)$ called the weak-star topology of $H^1(R)$. With this topology the unit ball is compact [**2**, pp. 469–470]. A sequence of members of the unit ball converges to a function in $H^1(R)$ in the weak-star topology if and only if it converges to this function pointwise. These facts imply that the convex hull of the constant functions of modulus one is a closed, hence compact, subset of the ball. If the constant functions of modulus one happen to be the only extreme points, then the Krein-Milman theorem implies that the convex hull of this set is the ball. Hence, when non-constant functions exist in $H^1(R)$, non-constant extreme points also exist.

H^1 of a compact bordered surface. Throughout this section, the Riemann surface R will be restricted to be the interior of a compact bordered surface. R can be realized as a domain in a compact Riemann surface S, the Schottky double of R. As a subsurface of S, R has a boundary which consists of a finite number of disjoint analytic Jordan curves.

The following theorem is proved in both [**3**] and [**4**]. We give here a proof using the ideas of the last section. The argument follows closely that of Gamelin and Voichick [**4**, p. 292]. If f is a meromorphic function on R, ∂_f is the divisor of f; the degree of an arbitrary divisor δ is denoted by $d[\delta]$.

THEOREM 6. *Let $f \in H^1$ have norm 1. A necessary condition that f be an extreme point is that $d[\partial_f] \leqq r/2$, where r is the first Betti number of R.*

In their proof, Gamelin and Voichick use a version of the Riemann-Roch theorem for compact bordered surfaces as given by Royden [**9**, p. 49]. Let δ be a divisor on R. $\mathscr{M}(-\delta)$ is the vector space over the real numbers of all functions meromorphic on the closure of R and real on its boundary whose divisors dominate $-\delta$. $\mathscr{D}(\delta)$ is the analogous space of differentials which are real along the boundary of R and whose divisors dominate δ. The essential fact is embodied in the following equality:

$$\dim \mathscr{M}(-\delta) = \dim \mathscr{D}(\delta) + 2d[\delta] - r + 1. \tag{7}$$

Here, r is the first Betti number of R. This fact is an easy consequence of the Riemann-Roch theorem for the doubled surface of R. It can be obtained using the decompositions in Ahlfors [**1**, p. 108].

Proof of Theorem 6. Suppose that $r \geqq 2$ and that f is an extreme point for which $d[\partial_f] > r/2$. By (7), $\dim \mathscr{M}(-\partial_f) > 1$, so there is a non-constant function ψ, meromorphic on the closure of R and real on its boundary, such that $\partial_\psi \geqq -\partial_f$. By multiplying ψ by a constant, we may assume that $-1 \leqq \psi \leqq 1$ on the boundary of R. Let p be the sum of Green functions which correspond to the poles of ψ, and let p' be the sum of Green functions which correspond to the zeros of ψ. On R, $\log|\psi| - p + p'$ is harmonic. At each point of the boundary, it is non-positive. So $\log|\psi| - p + p' \leqq 0$ on R; hence,

$$\log|\psi| = -q - s + p - p',$$

where q is a quasi-bounded harmonic function and s is a singular harmonic function (every positive harmonic function is the unique sum of a quasi-bounded and a singular harmonic function). Moreover, if w represents the left side of (6), then $w - p$ is subharmonic on R, continuous on its closure, and identically zero on its boundary. This results in

$$0 \leqq \log((|1 + \psi| + |1 - \psi|/2) \leqq p. \tag{8}$$

Now, $\log|f| = Q - s - l$ and the divisor condition means $p \leqq l$. So

$$\log|\psi f| = Q - q - (s + s') - (l - p + p'),$$

whence $\psi f \in H^1$, by Theorem 3. Adding $\log|f|$ to the inequality in (8), we have

$$Q - s - p = \log|f| \leqq \log((|f + \psi f| + |f - \psi f|)/2) \leqq Q - s.$$

Therefore,

$$M(\exp(Q - s - p)) \leqq M((|f + \psi f| + |f - \psi f|)/2) \leqq M(\exp(Q - s)).$$

Using Theorem 3(b),

$$M|f| = M(\exp Q) \leqq M((|f + \psi f| + |f - \psi f|)/2) \leqq M(\exp Q).$$

Therefore, (3) holds; so Lemma 1 implies that f is not an extreme point.

On a compact bordered surface R the periods of the conjugate of a function which is harmonic on R and continuous on its closure may be specified arbitrarily (see [**1**, p. 110; **4**, p. 920]). This fact used in conjunction with Theorem 5 yields the following corollary of that theorem.

COROLLARY 3. *Suppose that δ is the divisor of an extreme point. If α is a non-negative divisor and $\delta \geqq \alpha$, then α is the divisor of an extreme point.*

Proof. Suppose that $f \in H^1$ is an extreme point with divisor δ, and suppose that α is a non-negative divisor, $\delta \geqq \alpha$. Let f have the representation $\log|f| = Q - s - p$. If p' is the sum of Green functions which correspond to α (this is a finite sum), then there is a harmonic function Q' so that $Q' - p'$ is the logarithm of the modulus of a holomorphic function; i.e., $\log|F| = Q' - p'$, for some F holomorphic on R, and Q' is continuous on the closure of R (to get F, a homology basis may be chosen which avoids the poles of p'; Q' is then chosen so the periods of its conjugate are the same as the periods of the conjugate of p'). Because Q' is bounded on R, $F \in H^1$. We may assume that $M(\exp Q')$ is 1 at ζ, since this entails only the addition of a constant to Q' which will not affect the period relations. Since $p \geqq p'$, Theorem 5 implies that F is an extreme point.

Theorem 6 and Corollary 3 serve merely to illustrate the techniques developed herein. Some of the other known facts concerning the set of extreme points of the ball may also be proved in the same manner.

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