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GENERALIZED SVERDRUP'S LEMMA AND THE TREATMENT OF LESS THAN FULL RANK REGRESSION MODEL

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SUMMARY. Generalized Sverdrup's lemma, Kabe [5], is used here to give a more direct treatment of less than full rank regression model.

1. **Introduction.** Consider the usual normal multivariate linear regression model $Y=BX+E$, $Y\,p\times N$, $B\,p\times q$, $X\,q\times N$, and E is $N(0,\Sigma\otimes I)$. When X is of rank $k(<q<N)$, the model may be reparametrized to one of full rank, see Graybill [3]. However, the reparametrization procedure gives the estimates of the pk linear estimable functions of the elements of B, and does not give an estimate of B itself. To remedy this situation efforts have been made to obtain estimates, not necessarily unique, of the elements of B, see e.g., Chakrabarti ([2], p. 4), Plackett [8], John [4], Odell and Lewis [7]. These authors obtained particular solutions of the normal equations for estimating B by using certain pseudo inverses. The theory of pseudoinverses did not succeed in showing the exact parallelism that existed between the model of full rank and the model of less than full rank. Moreover, in some cases the computation of particular solution was also involved and complicated. Rao [9] obtained a generalized inverse, simple to compute that exhibited the above mentioned parallelism, but Rao ([9], pp. 179–200) derived the distributions by using the theory of minimum values of quadratic forms subjected to linear restrictions. These distributions are derived here more directly by using generalized Sverdrup's lemma, stated in the next section.

All distributions are derived without their normalizing constants and K (as a general symbol) denotes these constants. We assume that all integrals occurring in this paper are evaluated over appropriate ranges of the variables of integration.

2. **Some useful results.** Let Y be a $p\times N$ matrix, D a given $q\times N$ matrix of rank $q<N$, A an $N\times N$ positive definite symmetric matrix, μ a $p\times N$ matrix of constant terms, $N\geq p+q$. Then Kabe [5] proves that

$$\int_{(Y-\mu)A(Y-\mu)'=G,\,DY'=V'} f((Y-\mu)A(Y-\mu)')\,dY \tag{1}$$
$$= Kf(G)\,|G-(V-\mu D')(DA^{-1}D')^{-1}(V-\mu D')'|^{1/2(N-p-q-1)}$$

where G is $p\times p$ and V is $p\times q$.

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When D is of rank $k(<q<N)$, then (1) is written as

$$\int_{(Y-\mu)A(Y-\mu)'=G,\, PDY'=PV'=W'} f((Y-\mu)A(Y-\mu)')\,dY \tag{2}$$
$$= Kf(G)\,|G-(W-\mu D'P')(PDA^{-1}D'P')^{-1}(W-\mu D'P')'|^{1/2(N-p-k-1)}$$

where P denotes the $k\times q$ matrix of the first k latent vectors of $DA^{-1}D'$, corresponding to the first k nonzero roots of $DA^{-1}D'$. Note that $P'(PDA^{-1}D'P')^{-1}P$ is the unique Moore-Penrose inverse of $DA^{-1}D'$, see Rao ([9], p. 25). We denote this inverse by $(DA^{-1}D')^{-}$.

We note from (2) that

$$\begin{aligned}&\text{Min (over } Y) \operatorname{tr}(Y-\mu)A(Y-\mu)', \text{ subject to } DY'=V',\\ &\text{i.e., } PDY'=W' \text{ is}\end{aligned} \tag{3}$$

$$\begin{aligned}&\operatorname{tr}(W-\mu D'P')(PDA^{-1}D'P')^{-1}(W-\mu D'P')'\\ &\quad= \operatorname{tr}(V-\mu D')P'(PDA^{-1}D'P')^{-1}P(V-\mu D')'\\ &\quad= \operatorname{tr}(V-\mu D')(DA^{-1}D')^{-}(V-\mu D')'.\end{aligned} \tag{4}$$

In case D is of full rank, then in (4) we replace $(DA^{-1}D')^{-}$ by $(DA^{-1}D')^{-1}$.

3. **Some estimation and testing theory.** The normal equations for the estimation of B are $XY'=XX'\hat{B}$, which we write as $RXY'=RXX'R'R\hat{B}'$, where R is the matrix of the first k latent vectors of XX'.

The normal equations are now solvable, and $R\hat{B}'=(RXX'R')^{-1}RXY'$ or $\hat{B}'=(XX')^{-}XY'$ is a solution of the normal equations. Note that RB' is uniquely estimable, and thus any linear estimable function of B must be a linear function of RB' or of RXB' or of $RXX'B'$. Now we set $YY'=G$, $(RXX'R')^{-1}RXY'=R\hat{B}'=W'$ and integrate the density of Y over these restrictions by using (2), and we find that $YY'-W(RXX'R')W'=YY'-\hat{B}XX'\hat{B}'=\hat{\Sigma}$, and $(W-BR')RXX'R'(W-BR')'=(\hat{B}-B)XX'(\hat{B}-B)'$ are independently distributed. $\hat{\Sigma}$ has a Wishart density with $N-k$ degrees of freedom and $(\hat{B}-B)$ has a pk variate normal density with mean vector zero and singular covariance matrix $\Sigma\otimes(XX')^{-}$. The transformation of the singular distribution of $\hat{B}$ to the nonsingular distribution of W allows us to carry on all test procedures in terms of W first and retransform them in terms of $\hat{B}$ afterwards. These procedures are carried on in statistical literature by using pseudo and generalized inverses and we carry them on by using (2).

Let F be a $g\times q$ matrix of rank $t<g<q$, and let FB' be an estimable function, then $F\hat{B}'$ is an estimate of FB'. The sum of products matrix Λ, due to the hypothesis that FB' has a specified value, is found by using

$$\text{Min (over } Y) \operatorname{tr}(Y-BX)(Y-BX)', \text{ subject to } F(XX')^{-}XY' = F\hat{B}', \tag{5}$$

which is given by (4) as

$$\operatorname{tr}(\hat{B}F'-BF')(F(XX')^{-}F')^{-}(\hat{B}F'-BF')' = \operatorname{tr}\Lambda. \tag{6}$$

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Since $\hat{\Sigma}$ is independent of $\hat{B}$, Λ is independent of $\hat{\Sigma}$ and Λ has a Wishart distribution with $t>p$ degrees of freedom. Thus our criterion is $|\Lambda|/|\Lambda+\hat{\Sigma}|$, which is distributed as $U_{P,N-k,t}$ see Anderson (1, pp. 191–193).

The results given by Kabe [6] for the full rank model may now be generalized to that of less than full rank model by using (2).

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