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Non-local rings whose ideals are all quasi-injective: Addendum

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In [1] I made a Remark which, when subsequently challenged by Anne Koehler, I was unable to justify. While writing up my PhD thesis, I tried to both prove and disprove the statement but failed on both counts. However, I did reduce it to a highly plausible statement which is presented here.

CONJECTURE. *Every Q-ring is a direct sum of a finite number of indecomposable non-local Q-rings and a Q-ring all of whose idempotents are central.*

The conjecture is obviously true if and only if every left ideal in a Q-ring is contained in an ideal which is generated by a central idempotent and is minimal with respect to these properties. This is equivalent to the condition that the intersection of all left ideals which have (in the ring) a common homomorphic image disjoint from them is non-zero. The equivalence of the latter statement with the conjecture is proved in the following paragraph.

It is clear that Q-rings which satisfy the conjecture have this property. To show the converse, let L be a left ideal in a Q-ring R, f a central idempotent such that $L \subseteq Rf$ and let Rf_0, f_0 an idempotent, be an injective hull of L. As Rf is injective, $f_0Rf \neq 0$, which implies that $f_0f \neq 0$. Since $Rf_0 = Rf_0f \oplus Rf_0(1-f)$ and $L \subseteq Rf$ is essential in Rf_0, the ideal $Rf_0(1-f)$ is trivial. Therefore $Rf_0 \subseteq Rf$. It follows that the intersection of all ideals which contain L and are generated by central idempotents is injective. Let Re, e an

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idempotent, be this two-sided ideal. If $x \in R(1-e)$ then $ex \in Re \cap R(1-e)$ and so $ex = 0$. If $(1-e)Re \neq 0$ then there is a minimal ideal $M \subseteq Re$ which is a homomorphic image of $R(1-e)$ (Lemma 2). Moreover, for any central idempotent f with the property that $Re \subseteq Rf$, the product $(1-e)f \neq 0$. This means that each Rf contains a left ideal $R(1-e)f$ which has M as an image. By assumption, the intersection K of these ideals is non-zero. But as each $R(1-e)f = Rf(1-e)$ is contained both in $R(1-e)$ and in Rf , the ideal K is contained in both Re and $R(1-e)$: a contradiction. Therefore, $(1-e)Re = 0$ and e is a central idempotent, since Re and $R(1-e)$ annihilate each other. Hence Re is the unique smallest ideal which contains L and is generated by a central idempotent.

Reference

[1] G. Ivanov, "Non-local rings whose ideals are all quasi-injective", *Bull. Austral. Math. Soc.* 6 (1972), 45-52.