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A MAXIMAL GROSS-STADJE NUMBER IN THE EUCLIDEAN PLANE

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Let X be a compact, connected Hausdorff space and f a real valued, symmetric, continuous function on $X \times X$. Then the Gross-Stadje number $r(X, f)$ is the unique real number with the property that for each positive integer n and for all (not necessarily distinct) $x_1, \dots, x_n$ in X, there exists some x in X such that $\sum_{i=1}^{n} f(x_i, x) = nr(X, f)$. This paper solves the following open question in distance geometry: What is the least upper bound $g_2(\mathbf{R}^2)$ of $r(X, d^2)$, where X ranges over all compact, connected subsets of the Euclidean plane with diameter one and where d^2 denotes the squared, Euclidean distance. We show: $g_2(\mathbf{R}^2) = 3 - \sqrt{6}$.

1. INTRODUCTION

Let X be a compact, connected Hausdorff space and f a real valued, symmetric, continuous function on $X \times X$. Then there is a unique real number $r(X, f)$ with the property that for each positive integer n and for all (not necessarily distinct) $x_1, \dots, x_n$ in X, there exists some x in X such that

$$\frac{1}{n} \sum_{i=1}^{n} f(x_i, x) = r(X, f).$$

For the case when f is a metric on $X \times X$ this result was proved by O. Gross [**2**] in 1964. The more general result stated above was proved by W. Stadje [**3**] (independendly from Gross) in 1981. The number $r(X, f)$ is called Gross-Stadje number and is associated with X and the function f. If f is a metric d, then $r(X, d)$ is also often called the rendezvous number of the metric space (X, d). An excellent survey on this topic is given in [**1**].

In this paper we consider the case that X is a subset of the Euclidean plane and f is the squared, Euclidean distance d^2 (by $\|.\|$ we denote the Euclidean norm). In general the explicit calculation of the number $r(X, f)$ for a given compact, connected Hausdorff space X and a real valued, symmetric, continuous function f on $X \times X$ is rather difficult. It turns out that the calculation of $r(X, d^2)$ is much easier.

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THEOREM 1. (Wilson) *Let X be a compact, connected subset of $\mathbb{R}^n$. Let B_1 be a closed ball and B_2 an open ball such that X is contained in $B_1 \backslash B_2$ and the centre of each ball lies in the closed convex hull of the intersection of X with the boundary of the other. Further, let B_1 have centre u and radius R and let B_2 have centre v and radius r. Then*

$$r(X, d^2) = R^2 + r^2 - \|u - v\|^2.$$

For a proof see [**4**]. The existence of the balls in Theorem 1 is also shown in Wilson's paper.

For example let X be the Reuleaux triangle with diameter 1. Choose B_1 as the convex hull of the circumscribed circle und B_2 as the interior of the convex hull of the inscribed circle. Then we get with the help of Wilson's Theorem $r(X, d^2) = \left(5 - 2\sqrt{3}\right)/3$. (Remember that $r(X, d)$ of the Reuleaux triangle is still unknown.) For more examples see [**1, 4**].

Define the number $m(X, d^2)$ as $r(X, d^2)/D(X, d^2)$, where $D(X, d^2) = \sup\{\|x - y\|^2 \mid x, y \in X\}$ and $g_2(\mathbb{R}^2)$ as the supremum of the numbers $m(X, d^2)$ as X ranges over all compact, connected subsets of $\mathbb{R}^2$. In [**1**] the authors ask for the value of $g_2(\mathbb{R}^n)$, which is defined analogously. All values $g_2(\mathbb{R}^n)$, $n \geqslant 2$, are still unknown. The first information about the magnitude of $g_2(\mathbb{R}^2)$ is given in the following inequality: For all compact, connected metric spaces (X, d) we have

$$\frac{1}{4} \leqslant m(X, d^2) < 1.$$

For a proof of this inequality see for example [**1**]. Wilson conjectured in [**4**] that $g_2(\mathbb{R}^2) = \left(3 - \sqrt{11/3}\right)/2$, which is the number $m(X, d^2)$ for two sides of a Reuleaux triangle. But we shall show that this value is a little bit too small.

2. RESULTS

The following Proposition leads to the calculation of $g_2(\mathbb{R}^2)$.

PROPOSITION 1. *Let S_1 be a circle with centre u and radius R and let S_2 be a circle with centre v and radius r, $R \geqslant r \geqslant 0$, $R > 0$ and $0 \leqslant \|u - v\| \leqslant R$. Let X be a compact, connected subset of $\operatorname{conv} S_1 \backslash (\operatorname{conv} S_2)^\circ$ where v is in $\operatorname{conv}(S_1 \cap X)$ and u is in $\operatorname{conv}(S_2 \cap X)$. Then we have*

$$m(X, d^2) \leqslant 3 - \sqrt{6} \approx 0.5505102.$$

Now we get

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Theorem 2. *Define $g_2(\mathbb{R}^2)$ as in Section 1. Then we have*

$$g_2(\mathbb{R}^2) = 3 - \sqrt{6}.$$

Remark 1. The value $3-\sqrt{6}$ is attained, for example for the following set: Let S_1 be a circle with centre u and radius $R = 1$, S_2 be a circle with centre v and radius $r = \sqrt{3/\left(4\sqrt{6}-6\right)}$ and let $\|u-v\| = \sqrt{3/2}-1$. Let $\{x_1, x_2\}$ be the intersection of S_1 and S_2. Further let x_3 be the intersection point of $S_1 \setminus \operatorname{conv} S_2$ and the line which is determined by u and v and let x_4 be the intersection point of $S_2 \cap \operatorname{conv} S_1$ and the line which is determined by u and v. Then define the set A as follows: A consists of the arc joining x_1 and x_2 in $S_2 \cap \operatorname{conv} S_1$ and the line segment x_3x_4 (see Figure 1). Observe that $D(A, d^2) = \|x_1 - x_3\|^2 = (2r)^2$.

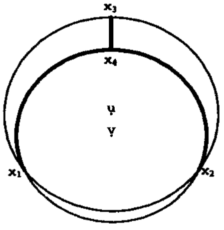

Figure 1: The set A.

3. Proofs

For the proof of Proposition 1 we need the following Lemmas:

Lemma 1. *Let S be a circle with centre u and radius R. Let v be a point in $\operatorname{conv} S$ and g be the line with v in g and g perpendicular to the line segment uv. Further let h be an arbitrary line with v in h. Then we have with $\{x_1, x_2\} = S \cap g$ and $\{y_1, y_2\} = S \cap h$*

$$\|x_1 - x_2\| \leqslant \|y_1 - y_2\|.$$

The proof is straight forward.

Lemma 2. *Let S be a circle with centre u and radius R. Let x_1, x_2, x_3 be points in S with u in $\operatorname{conv}\{x_1, x_2, x_3\}$. Then we have*

$$\max_{1 \leqslant i,j \leqslant 3} \|x_i - x_j\| \geqslant \sqrt{3}R.$$

The proof is straight forward.

Lemma 3. *Let S be a circle and let X be a subset of $\operatorname{conv} S$ with $S \cap X$ is not empty. Let v be a point in $\operatorname{conv}(S \cap X)$. Then there are points x_1, x_2, x_3 in $X \cap S$ with v in $\operatorname{conv}\{x_1, x_2, x_3\}$.*

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The Lemma follows from Caratheodory's Theorem.

Lemma 4. *Let S_1 be a circle with centre u and radius R and let S_2 be a circle with centre v and radius r, $R \geqslant r \geqslant 0$, $R > 0$ and $0 \leqslant \|u-v\| \leqslant R$. Assume v is in $\operatorname{conv}\bigl(\operatorname{conv} S_1 \backslash (\operatorname{conv} S_2)^\circ\bigr)$. Then we have*

$$\|u-v\|^2 + r^2 \leqslant R^2.$$

Proof: If $S_1 \cap S_2$ is empty, the assertion is trivial. Let $S_1 \cap S_2$ be not empty. Assume that $\|u-v\|^2 + r^2 > R^2$. Let $L := \operatorname{conv}(S_1 \cap S_2)$, $l := D(L, d)$ and $a := l/2$. Define $d := \min\bigl\{\|x-u\| : x \in L\bigr\}$. Then we have $a^2 + d^2 = R^2$ and $a^2 + \bigl(\|u-v\| - d\bigr)^2 = r^2$. From this we get

$$R^2 - d^2 = r^2 - \bigl(\|u-v\| - d\bigr)^2$$

and hence

$$d = \frac{R^2 - r^2 + \|u-v\|^2}{2\|u-v\|} < \frac{\|u-v\|^2 + \|u-v\|^2}{2\|u-v\|} = \|u-v\|.$$

So the line which is determinated by L separates $\operatorname{conv}(\operatorname{conv} S_1 \backslash (\operatorname{conv} S_2)^\circ)$ and v, which is a contradiction. ∎

Lemma 5. *Define the following functions:*

1. $f_1 : [0, 1/2] \longrightarrow \mathbb{R},\ x \longmapsto 5/3 - \bigl(2\sqrt{3}\sqrt{1 - x + x^2} + x\bigr)/3.$
2. *For* $0 \leqslant w \leqslant 1/2$: $f_2 : [0, 1] \longrightarrow \mathbb{R}$,
$$x \longmapsto \frac{(1 + x^2 - w^2)(1+w)^2}{\Bigl(x + \sqrt{(1+w)^3 - wx^2}\Bigr)^2}.$$
3. *For* $w > 0$: $f_3 : (0, 1] \longrightarrow \mathbb{R},\ x \longmapsto (1 + x^2 - w^2)/(4x^2).$
4. $f_4 : [0, 1/2] \longrightarrow \mathbb{R},\ x \longmapsto 1/4 + (1 + 3x - 4x^2)/\bigl(4(1+x)\bigr).$

Then we have:

1. $\max\limits_{0 \leqslant x \leqslant 1/2} f_1(x) = f_1\Bigl(\bigl(1 - \sqrt{3/11}\bigr)/2\Bigr) = \bigl(3 - \sqrt{11/3}\bigr)/2.$
2. $$f_2'(x) \begin{cases} < 0 & \text{for } x < (1 - w^2)/\bigl(\sqrt{1 + 6w + w^2}\bigr) \\ = 0 & \text{for } x = (1 - w^2)/\bigl(\sqrt{1 + 6w + w^2}\bigr) \\ > 0 & \text{for } x > (1 - w^2)/\bigl(\sqrt{1 + 6w + w^2}\bigr) \end{cases}.$$
3. *f_3 is monotonic decreasing.*
4. $\max\limits_{0 \leqslant x \leqslant 1/2} f_4(x) = f_4\Bigl(\sqrt{3/2} - 1\Bigr) = 3 - \sqrt{6}.$

The proof is straight forward.

PROOF OF PROPOSITION 1: Without loss of generality, let $R = 1$. From Lemma 4 we have

$$\|u - v\|^2 + r^2 \leqslant 1 \tag{1}$$

and from Theorem 1 we have

$$m(X, d^2) = \frac{1 + r^2 - \|u - v\|^2}{D(X, d^2)}. \tag{2}$$

If $r = 0$ we get $u = v$ and therefore u is in $\mathrm{conv}(S_1 \cap X)$. From this we get $D(X, d^2) \geqslant 3$ and hence

$$m(X, d^2) \leqslant \frac{1}{3}.$$

So assume $r > 0$. Then it is easy to see that $|X \cap S_1| > 1$.

CASE 1. $|X \cap S_1| = 2$. So $X \cap S_1 = \{y_1, y_2\}$ and $D(X, d^2) \geqslant \|y_1 - y_2\|^2$. Let g be the line with v is in g, with g perpendicular to the line segment uv and let $\{x_1, x_2\} = S_1 \cap g$. Then we have $\|x_1 - x_2\|^2 = 4(1 - \|u - v\|^2)$. Since v is in $\mathrm{conv}\{y_1, y_2\}$ we get from Lemma 1 $\|y_1 - y_2\| \geqslant \|x_1 - x_2\|$ and therefore

$$D(X, d^2) \geqslant 4(1 - \|u - v\|^2).$$

Now we get with (1) and (2):

$$m(X, d^2) \leqslant \frac{1 + r^2 - \|u - v\|^2}{4(1 - \|u - v\|^2)} \leqslant \frac{2(1 - \|u - v\|^2)}{4(1 - \|u - v\|^2)} = \frac{1}{2}.$$

CASE 2. $|X \cap S_1| > 2$. From Lemma 3 we get points y_1, y_2, y_3 in $S_1 \cap X$ with v in $\mathrm{conv}\{y_1, y_2, y_3\}$.

CASE 2.1. u is not in $\mathrm{conv}\{y_1, y_2, y_3\}$. Then there are two points in $\{y_1, y_2, y_3\}$, without loss of generality, y_1 and y_2, such that the line segment y_1y_2 does intersect the line segment uv. That is, $y_1y_2 \cap uv = \{\overline{v}\}$. It follows that $\|u - \overline{v}\| \leqslant \|u - v\|$. Define two lines g, h which are perpendicular to the line segment uv with $\overline{v}$ on g and v on h. Let $\{y_1', y_2'\} = S_1 \cap g$ and $\{x_1, x_2\} = S_1 \cap h$. From Lemma 1 we get $\|y_1' - y_2'\| \leqslant \|y_1 - y_2\|$. Further we get

$$\left(\frac{\|x_1 - x_2\|}{2}\right)^2 = 1 - \|u - v\|^2 \leqslant 1 - \|u - \overline{v}\|^2 = \left(\frac{\|y_1' - y_2'\|}{2}\right)^2$$

and hence

$$D(X, d^2) \geqslant \|y_1 - y_2\|^2 \geqslant \|y_1' - y_2'\|^2 \geqslant \|x_1 - x_2\|^2 = 4(1 - \|u - v\|^2).$$

Again we use (1) and (2) and get

$$m(X,d^2) \leqslant \frac{1}{2}.$$

CASE 2.2. u is in $\operatorname{conv}\{y_1,y_2,y_3\}$. From Lemma 2 we have

$$\max_{1\leqslant i,j\leqslant 3} \|y_i - y_j\| \geqslant \sqrt{3}$$

and so $D(X,d^2) \geqslant 3$. Assume $\|u-v\| > 1/2$. Then we get together with (1) and (2)

$$m(X,d^2) \leqslant \frac{2(1-\|u-v\|^2)}{3} < \frac{2(1-1/4)}{3} = \frac{1}{2}.$$

So in the following we only have to consider the case $\|u-v\| \leqslant 1/2$.
We have r in the interval $I=(0,1]$. Define the intervals

$$I_1 := \left(0, \sqrt{3} - \sqrt{1-\|u-v\|+\|u-v\|^2}\right],$$

$$I_2 := \left[\sqrt{3} - \sqrt{1-\|u-v\|+\|u-v\|^2}, \frac{1+\|u-v\|}{\sqrt{4\|u-v\|+1}}\right]$$

and

$$I_3 := \left[\frac{1+\|u-v\|}{\sqrt{4\|u-v\|+1}}, 1\right].$$

Therefore r is in $I_1 \cup I_2 \cup I_3$.

CASE 2.2.1. r is in I_1.
Since $D(X,d^2) \geqslant 3$ we get together with (2) and Lemma 5,

$$\begin{aligned} m(X,d^2) &\leqslant \frac{1+r^2-\|u-v\|^2}{3} \\ &\leqslant \frac{5}{3} - \frac{2\sqrt{3}\sqrt{1-\|u-v\|+\|u-v\|^2}+\|u-v\|}{3} \\ &= f_1(\|u-v\|) \\ &\leqslant \frac{1}{2}\left(3-\sqrt{\frac{11}{3}}\right) \approx 0.5425728. \end{aligned}$$

CASE 2.2.2. r is in I_2. For $1 \leqslant i \leqslant 3$ define the lines g_i, $v + t(v-y_i)$ for $t \geqslant 0$. Since y_1,y_2,y_3 are points in X and X is connected there are at least two indices $i_1,i_2 \in \{1,2,3\}$, $i_1 \neq i_2$ and two points a_1,a_2 in X with $a_1 \in g_{i_1}$ and $a_2 \in g_{i_2}$. Then define $x_1 := y_{i_1}$, $x_2 := y_{i_2}$ and $x_3 := y_k$, where $k \neq i_1, i_2$. From this it is clear that $\|x_1 - a_1\| \geqslant \|x_1 - v\| + r$ and $\|x_2 - a_2\| \geqslant \|x_2 - v\| + r$. So we have

$$D(X,d^2) \geqslant \max\{\|x_1-x_2\|, \|x_2-x_3\|, \|x_1-x_3\|, \|x_1-v\|+r, \|x_2-v\|+r\}^2.$$

If $\|x_1-x_2\| > 2\sqrt{1-\|u-v\|^2}$ we have $D(X,d^2) > 4(1-\|u-v\|^2)$ and therefore together with (1) and (2), we have $m(X,d^2) < 1/2$. So we only have to consider $\|x_1-x_2\| \leqslant 2\sqrt{1-\|u-v\|^2}$.

Consider the arc joining x_1 and x_2 on S_1 which contains x_3. Let x_3' be on this arc with $\|x_1-x_3'\| = \|x_2-x_3'\|$. Then we have

$$\max\{\|x_1-x_3\|, \|x_2-x_3\|\} \geqslant \|x_1-x_3'\|$$

and so we get

$$D(X,d^2) \geqslant \max\{\|x_1-x_2\|, \|x_1-x_3'\|, \|x_1-v\|+r, \|x_2-v\|+r\}^2.$$

Now let $T: S_1 \longrightarrow S_1$ be a rotation with centre u and $Tx_3' = u+t(u-v)$ for a $t>0$. Then we have

$$\|Tx_1-v\|+r = \|Tx_2-v\|+r.$$

Of course $\|x_1-x_2\| = \|Tx_1-Tx_2\|$, $\|x_1-x_3'\| = \|Tx_1-Tx_3'\|$ and

$$\max\{\|x_1-v\|+r, \|x_2-v\|+r\} \geqslant \|Tx_1-v\|+r.$$

So we get

$$D(X,d^2) \geqslant \max\{\|Tx_1-Tx_2\|, \|Tx_1-Tx_3'\|, \|Tx_1-v\|+r\}^2. \tag{3}$$

For short write again $x_1 := Tx_1, x_2 := Tx_2$ and $x_3 := Tx_3'$. Let $\overline{x}_i$ be the intersection point of the circle S_2 and the line segment ux_i, for $1 \leqslant i \leqslant 3$. Now define the following set X': X' is the arc joining $\overline{x}_1$ and $\overline{x}_2$ in S_2 with u in the convex hull of this arc, together with the line segments $x_i\overline{x}_i$, for $1 \leqslant i \leqslant 3$. Then we have

1. X' is a compact, connected subset of $\mathrm{conv}S_1 \backslash (\mathrm{conv}S_2)^\circ$.
2. u is in $\mathrm{conv}(S_2 \cap X)$, and since $\|x_1-x_2\| \leqslant 2\sqrt{1-\|u-v\|^2}$ also v is in $\mathrm{conv}\{x_1,x_2,x_3\}$.
3. $D(X',d^2) = \max\{\|x_1-x_2\|, \|x_1-x_3\|, \|x_1-v\|+r\}^2 \leqslant D(X,d^2)$ and therefore $m(X',d^2) \geqslant m(X,d^2)$.

So in the following we only consider sets of the kind of X'.

Let z be the intersection point of S_1 and the line $u+t(u-v)$ for $t \geqslant 0$. If $S_1 \cap S_2$ is not empty, let y be in $S_1 \cap S_2$. Otherwise let y be the intersection point of S_1 and the line $u+t(u-v)$ for $t \leqslant 0$. Let B be the shortest arc joining y and z on S_1. Let g be the line which is perpendicular to uv and which contains v and define p as the intersection point of B and g. Each point x on B corresponds to an angle ϕ between the line segments uv and ux. Therefore we write $x = x(\phi)$. Now define the angles ϕ_1 and ϕ_2 with $y = x(\phi_1)$ and $p = x(\phi_2)$. Clearly $\phi_1 \geqslant 0$. Assume $\phi_1 > \pi/3$. Then we have

$r > \sqrt{1 - \|u-v\| + \|u-v\|^2} \geqslant (1 + \|u-v\|)/\left(\sqrt{4\|u-v\|+1}\right)$ and therefore r is not in I_2. Hence we have $\phi_1 \leqslant \pi/3$. On the other hand we have $\cos\phi_2 = \|u-v\|$. Since $0 \leqslant \|u-v\| \leqslant 1/2$ we get $\pi/3 \leqslant \phi_2 \leqslant \pi/2$.

By definition of X' we have now $x_3 = z$, $x_1 = x(\phi)$ for $\phi_1 \leqslant \phi \leqslant \phi_2$ and x_2 is the point on S_1 with $x_1 \neq x_2$ and $\|x_3 - x_1\| = \|x_3 - x_2\|$.

Now we have

$$\left\|x(\phi) - x_3\right\| = \sqrt{2}\sqrt{1+\cos\phi}$$

and

$$\left\|x(\phi) - v\right\| + r = \sqrt{1 - 2\|u-v\|\cos\phi + \|u-v\|^2} + r.$$

It is easy to see that $\left\|x(\phi) - x_3\right\|$ is a monotonic decreasing function of ϕ and $\left\|x(\phi) - v\right\| + r$ is a monotonic increasing function of ϕ. If $S_1 \cap S_2$ is not empty we have $\left\|x(\phi_1) - v\right\| + r = 2r$. Since ϕ_1 is the angle between the line segments uy, and uv we have

$$r^2 = 1 + \|u-v\|^2 - 2\|u-v\|\cos\phi_1$$

and therefore

$$\cos\phi_1 = \frac{1 + \|u-v\|^2 - r^2}{2\|u-v\|}.$$

Hence we get

$$\begin{aligned}\left\|x(\phi_1) - x_3\right\| &= \sqrt{2}\sqrt{1+\cos\phi_1}\\ &= \sqrt{\frac{\left(1+\|u-v\|\right)^2 - r^2}{\|u-v\|}}.\end{aligned}$$

Since r is in I_2 we get

$$\begin{aligned}\left\|x(\phi_1) - x_3\right\|^2 &= \frac{\left(1+\|u-v\|\right)^2 - r^2}{\|u-v\|}\\ &\geqslant \frac{1}{\|u-v\|}\left[\left(1+\|u-v\|\right)^2 - \frac{\left(1+\|u-v\|\right)^2}{4\|u-v\|+1}\right]\\ &= 4\frac{\left(1+\|u-v\|\right)^2}{4\|u-v\|+1}\\ &\geqslant 4r^2\end{aligned}$$

and therefore

$$\left\|x(\phi_1) - x_3\right\| \geqslant \left\|x(\phi_1) - v\right\| + r.$$

On the other hand we have $\|x(\phi_2) - x_3\| \leqslant \|x(\pi/3) - x_3\| = \sqrt{3}$ and $\|x(\phi_2) - v\| + r \geqslant \|x(\pi/3) - v\| + r \geqslant \sqrt{3}$ since r is in I_2. So there is ϕ_0 in $[\phi_1, \phi_2]$ with

$$\|x(\phi_0) - x_3\| = \|x(\phi_0) - v\| + r.$$

If $S_1 \cap S_2$ is empty we have $\phi_1 = 0$ and therefore we get $\|x(\phi_2) - x_3\| \leqslant \sqrt{3}$, $\|x(0) - x_3\| = 2$, $\|x(\phi_2) - v\| + r \geqslant \sqrt{3}$ and $\|x(0) - v\| + r \leqslant 1 + r \leqslant 2$. As above there is ϕ_0 in $[\phi_1, \phi_2]$ with

$$\|x(\phi_0) - x_3\| = \|x(\phi_0) - v\| + r.$$

For short we define $w := \|u - v\|$. Therefore we have

$$1 - 2w \cos\phi_0 + w^2 = 2(1 + \cos\phi_0) + r^2 - 2\sqrt{2}r\sqrt{1 + \cos\phi_0}.$$

Since $\cos^2(\phi_0/2) = (1 + \cos\phi_0)/2$ and with $\psi := \phi_0/2$ we have

$$1 - 2w(2\cos^2\psi - 1) + w^2 = r^2 + 4\cos^2\psi - 2\sqrt{2}\sqrt{2}r\cos\psi.$$

So we get the following equation for $\cos\psi$:

$$(4 + 4w)\cos^2\psi - 4r\cos\psi + r^2 - (1 + w)^2 = 0.$$

Solving this equation we get

$$\cos\psi = \frac{r \pm \sqrt{(1 + w)^3 - w\,r^2}}{2(1 + w)}.$$

Since $\sqrt{(1 + w)^3 - w\,r^2} > r$ we get

$$\cos\psi = \frac{r + \sqrt{(1 + w)^3 - w\,r^2}}{2(1 + w)}.$$

(Otherwise we have $\cos\psi < 0$ and that is a contradiction to $0 \leqslant \psi \leqslant \pi/4$.) Now we get together with (3)

$$\begin{aligned} D(X', d^2) &\geqslant \left(\sqrt{2}\sqrt{1 + \cos\phi_0}\right)^2 \\ &= 4\cos^2\psi \\ &= \left[\frac{r + \sqrt{(1 + w)^3 - w\,r^2}}{(1 + w)}\right]^2 \end{aligned}$$

and hence

$$m(X', d^2) \leqslant \frac{(1 + r^2 - w^2)(1 + w)^2}{\left(r + \sqrt{(1 + w)^3 - w\,r^2}\right)^2} = f_2(r).$$

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From Lemma 5 we have

$$\max_{x\in I_2} f_2(x) = \max\left\{f_2\left(\sqrt{3}-\sqrt{1-w+w^2}\right), f_2\left(\frac{1+w}{\sqrt{4w+1}}\right)\right\}$$

and

$$f_2\left(\sqrt{3}-\sqrt{1-w+w^2}\right) = f_1(w) \leqslant \frac{1}{2}\left(3-\sqrt{\frac{11}{3}}\right).$$

The value $f_2\Big((1+w)/\sqrt{4w+1}\Big)$ will be calculated later.

CASE 2.2.3. r is in I_3. We have $D(X,d^2) \geqslant 4r^2$ and so

$$m(X,d^2) \leqslant \frac{1+r^2-\|u-v\|^2}{4r^2} = f_3(r)$$

where w is chosen as $\|u-v\|$. We have

$$f_3\left(\frac{1+\|u-v\|}{\sqrt{4\|u-v\|+1}}\right) = f_2\left(\frac{1+\|u-v\|}{\sqrt{4\|u-v\|+1}}\right).$$

Since f_3 is a monotonic decreasing function on I_3 we have

$$\begin{aligned} f_3(r) &\leqslant f_3\left(\frac{1+\|u-v\|}{\sqrt{4\|u-v\|+1}}\right) \\ &= f_4(\|u-v\|) \\ &\leqslant f_4\left(\sqrt{\frac{3}{2}}-1\right) \\ &= 3-\sqrt{6}. \end{aligned}$$

So we have

$$m(X,d^2) \leqslant f_3(r) \leqslant 3-\sqrt{6}$$

and we are done.

∎

PROOF OF THEOREM 2: Let X be a compact, connected subset of $\mathbb{R}^2$. Then there is a circle S_1 with centre u and radius R and a circle S_2 with centre v and radius r with X contained in $\operatorname{conv}S_1\backslash(\operatorname{conv}S_2)^\circ$ and u in $\operatorname{conv}(S_2\cap X)$ and v in $\operatorname{conv}(S_1\cap X)$ (see Theorem 1). Therefore we get from Proposition 1

$$m(X,d^2) \leqslant 3-\sqrt{6}$$

and hence

$$g_2(\mathbb{R}^2) \leqslant 3-\sqrt{6}.$$

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Now we consider the set A from Remark 1 in Section 2. We have

$$D(A, d^2) = \frac{6}{2\sqrt{6} - 3}.$$

So we get with Wilson's Theorem,

$$M(A, d^2) = \frac{1 + \frac{3}{4\sqrt{6} - 6} - \left(\sqrt{3/2} - 1\right)^2}{\frac{6}{2\sqrt{6} - 3}}$$
$$= 3 - \sqrt{6}.$$

□

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