*Compositio Mathematica* **112:** 33–40, 1998.



# Stable complete minimal surfaces in hyperkähler manifolds

CLAUDIO AREZZO*
*University of Warwick, United Kingdom, and Università di Genova, Italy;*
*e-mail: claudio@maths.warwick.ac.uk*

Received: 19 September 1996; accepted in final form 28 February 1997

**Abstract.** In this paper we prove that an isometric stable minimal immersion of a complete oriented surface into a hyperkähler 4-manifold is holomorphic with respect to an orthogonal complex structure, if it satisfies a Bernstein-type assumption on the Gauss-lift. This result generalizes a theorem of Micallef for minimal surfaces in the euclidean 4-space. An example found by Atiyah and Hitchin shows that the assumption on the Gauss-lift is necessary.

**Mathematics Subject Classifications (1997):** 58E12, 53C42, 53A10

**Key words:** minimal surfaces, Kähler angle, hyperkähler manifolds

## 1. Introduction

It is well known that any holomorphic map of a Riemann surface into a Kähler manifold minimizes area in its homology class. The main question we study in this paper is the following:

PROBLEM 1. *Given an isometric stable minimal immersion $F: M \to N$ of a* <u>*complete*</u> *oriented surface $M$ into a hyperkähler* 4*-manifold $N$, is $F$ holomorphic with respect to some orthogonal complex structure on $N$?*

In general the answer to the above problem is negative: Atiyah and Hitchin ([1]) have found an example of a minimal two-sphere in the hyperkähler 4-manifold $\tilde{\mathcal{M}}_2^0$, the universal cover of the centered 2-monopoles in $\mathbb{R}^3$ with finite action, which is not holomorphic w.r.t. any compatible complex structure on $\tilde{\mathcal{M}}_2^0$, and which has been proved to be stable by Micallef and Wolfson ([7]).

In this paper we find a sufficient condition on the immersion for the problem to have positive answer.

We recall that for locally embedded submanifolds $M$ in $N$ the property to be a complex (or anti-complex) submanifold of $(N, J)$ can be expressed by saying that the tangent space $T_pM$ is $J$-invariant for each $p \in M$. When $N$ has real dimension 4 a way to measure the $J$-invariance of $TM$ is given by the Kähler angle: it follows

* The author was partially supported by Consiglio Nazionale delle Ricerche, Italy.

from Wirtinger's inequality that if $\omega$ is the restriction of the Kähler form of $(N, J)$ to $TM$, we can write $\omega = \cos\alpha_J\, \mathrm{d}Vol\, M$ and that $M$ is a complex submanifold w.r.t. $J$) if and only if $\alpha_J = 0$ on $M$.

It is possible to express the stability condition in terms of the Kähler angle. Micallef and Wolfson ([7]) proved that if $M$ is stable and $\sigma$ is a section with compact support of the normal bundle then

$$\int_M \{|\bar{\partial}\sigma|^2 - 2[|\mathrm{d}\alpha_J|^2 + \tfrac{1}{4} S \sin^2\alpha_J]|\sigma|^2\}\mathrm{d}Vol \geqslant 0,$$

where $S$ is the scalar curvature of $N$. Using this formula, they proved (Corollary 5.3 page 260) that if $N$ is hyperkähler (see section 2 for the definition), $M$ is compact and the normal bundle admits a holomorphic section, then the immersion $F$ is holomorphic with respect to one of the complex structures of $N$.

We'll apply the previous formula in the case $N$ is hyperkähler and $M$ not necessarily compact. The crucial problem is then to produce a holomorphic section of the normal bundle with appropriate growth and to do this we'll need some further hypothesis.

To overcome this problem we assume that the composition of the Gauss lift (see section 2 for the definition) with the projection over the sphere $S^2$ omits an open set. Eells and Salamon ([5]) proved that, under our assumptions, this map is anti-holomorphic, extending the analogy with the Gauss map of minimal surfaces in the euclidean space. This will allow us to prove the main result of this paper:

THEOREM 1.1. *Let $F\colon M \to N$ be an isometric stable minimal immersion of a complete oriented surface $M$ into a* 4*-dimensional hyperkähler manifold $N$. If the Gauss lift $\tilde{F}_+\colon M \to S_+ = N \times S^2$ omits an open set of $S^2$, then $F$ is holomorphic with respect to some orthogonal complex structure of $N$.*

About the assumption on the Gauss lift in the above theorem, we recall that the image of the Gauss lift of the stable two sphere found by Atiyah and Hitchin mentioned before, is the whole $S^2$.

As we will see in the proof of the main theorem the condition on the Gauss lift is equivalent to the requirement for the Kähler angle to omit an open set of $[0, \pi]$.

When $M$ is compact, Wolfson ([10]) has proved that the conclusion of Theorem 1.1 holds even without the stability assumption and with a milder one on the Gauss lift. His result is the following:

THEOREM 1.2. *Let $M$ be a compact oriented minimal surface in a hyperkähler* 4*-manifold $N$. If the Gauss lift omits two antipodal points of the* 2*-sphere* (*i.e. the surface is* totally real)*, then $F$ is holomorphic with respect to some orthogonal complex structure of $N$.*

Wolfson's result leads naturally to conjecture that the stability assumption might be unnecessary also in Theorem 1.1; we leave this intriguing problem for further research.

https://doi.org/10.1023/A:1000358906964 Published online by Cambridge University Press

Theorem 1.1 is a generalization of a theorem of Micallef ([6]) in the case when $N$ is the euclidean 4-space which is the easiest example of hyperkähler manifold. In particular he proved the following:

THEOREM 1.3. *Let $F\colon M \to \mathbb{R}^4$ be an isometric stable minimal immersion of a complete oriented surface $M$ into the euclidean* 4*-space. If one of the projections of the Gauss map onto $S^2 \times S^2$ omits an open set, then $F$ is holomorphic with respect to some orthogonal complex structure of $\mathbb{R}^4$.*

In the second section we show how this theorem follows from our result. For sake of completedness we recall that by a famous theorem due to Chern ([3]) and Osserman ([8]) we know that, in the case of the euclidean 4-space, if both projections on the spheres of the Gauss map omit an open set then $M$ is a plane.

## 2. Notations and Definitions

Let $N$ be a riemannian manifold with metric $g$, $M$ a Riemann surface and $F: M \to N$ a map. Let $\nabla$ denote the Levi–Civita connections on $TM$ and $F^{-1}TN$.

Let now assume that $\dim N = 4$ and $N$ is oriented. In this case the Hodge-star operator $*\colon \Lambda^2(TN) \to \Lambda^2(TN)$, gives rise to a decomposition

$$\Lambda^2(TN) = \Lambda^2_+(TN) \oplus \Lambda^2_-(TN),$$

where $\Lambda^2_\pm(TN)$ are the eigenspaces corresponding to the eigenvalues $\pm 1$. The elements of $\Lambda^2_\pm$ are called *self-dual* and *antiself-dual* forms respectively. Let $S_\pm = S(\Lambda^2_\pm)$ be the two-sphere bundle of unit vectors. The *Grassmann bundle* $\tilde{G}_2$ is the bundle whose fibre at $x \in N$ is $\tilde{G}_2(T_xN)$, the space of oriented two dimensional subspaces of $T_xN$.

We can associate to an immersion $F\colon M \to N$ another map, called the *Gauss lift* of $F$, $\tilde{F}\colon M \to \tilde{G}_2$ defined by

$$\tilde{F}(p) = F_*(T_pM),$$

which is an element of $\tilde{G}_2(T_xN)$ where $F(p) = x$. In the case of immersions in the euclidean space it is possible to avoid the difficulty of working with bundles in the following way: given $F\colon M \to \mathbb{R}^n$ define $\gamma_F\colon M \to \tilde{G}_2(\mathbb{R}^n)$ where $\gamma_F(p)$ is the two plane $F_*(T_pM)$ translated to the origin. $\gamma_F$ is called the *Gauss map*. We recall that $\tilde{G}_2(\mathbb{R}^n)$ may be identified with a quadric $Q_{n-2}$ in $\mathbb{C}\mathbf{P}^{n-1}$, and that a conformal immersion is harmonic if and only if $\gamma_F\colon M \to Q_{n-2}$ is anti-holomorphic (see Chern [3]). It is well known that $Q_2$ is diffeomorphic to $S^2 \times S^2$ using Plücker coordinates (e.g. see Chern–Spanier [4]). The same happens also in the general case: indeed $\tilde{G}_2(T_xN)$ is isomorphic to $(S_+)_x \times (S_-)_x$ and so we have two projections $p_\pm\colon \tilde{G}_2(TN) \to S_\pm$ and two new maps $\tilde{F}_\pm\colon M \to S_\pm$, $\tilde{F}_\pm = p_\pm \circ \tilde{F}$. Hence if

$N = \mathbb{R}^n$, $\tilde{F}_+$ is the projection of $\gamma_F$ onto the first $S^2$ and this gives the relation between our theorem and Micallef's one.

It is possible to give an interpretation of the bundles $S_\pm$ in terms of almost complex structures over $N$. In fact if $w \in S_\pm$ on the $x$-fiber, then it is clearly possible to choose an oriented orthonormal basis $\{e_i\}$ of $T_xN$ such that $w = e_1 \wedge e_2 + e_3 \wedge e_4$. Defining $Je_1 = e_2, Je_3 = e_4$ and $J^2 = -1$, we get an almost complex structure over $N$ oriented consistently with $N$. If $\theta_i$ is the dual basis of $e_i$ then $\omega = \theta_1 \wedge \theta_2 + \theta_3 \wedge \theta_4$ is the fundamental 2-form associated to the almost complex structure $J$ given by $g(JX, Y)$. In the case of $S_-$ we get contrariwise oriented almost complex structures over $N$.

By definition a riemannian manifold is called *hyperkähler* if it admits a family of compatible complex structures parametrized by $S^2$, with respect to each of which the manifold is Kähler. In this case $S_+ = N \times S^2$ and every point of the sphere represents a complex structure on $N$.

Let $(N, g, J)$ be an hermitian manifold, i.e. $g$ is a riemannian metric, $J$ a complex structure such that $g(JX, JY) = g(X, Y)$ for every $X, Y \in TN$, $\omega$ the fundamental 2-form and $v$ the fundamental 2-vector of $N$. If $e_i$ is a unitary basis of $T^{(1,0)}N$, such that $g(e_i, \bar{e}_j) = \delta_{ij}$, we can write

$$v = -i \sum_{k=1}^{n} e_k \wedge \bar{e}_k.$$

Let us denote by $T_0^{(1,1)}N$ the space of $(1,1)$-vectors orthogonal to $v$. So we have

$$\Lambda_2^{\mathbb{C}}(TN) = T^{(2,0)}N \oplus T^{(0,2)}N \oplus \mathbb{C}v \oplus T_0^{(1,1)}N.$$

It is easy to prove that, if dim $N = 4$,

$$\Lambda_-^{\mathbb{C}}(TN) = T_0^{(1,1)}N \quad \Lambda_+^{\mathbb{C}}(TN) = T^{(2,0)}N \oplus T^{(0,2)}N \oplus \mathbb{C}v. \tag{1}$$

If $y$ is a point in the $x$-fibre of $S_+$, then $T_yS_+ = \mathcal{V}_y \oplus \mathcal{H}_y$ where $\mathcal{V}_y$ is the space of vertical vectors, i.e. those tangent to the fiber $(S_+)_x$. By (1) $\mathcal{V}_y^{\mathbb{C}}$ is isomorphic, via an isomorphism $\nu$, to $T_x^{(2,0)}N \oplus T_x^{(0,2)}N$.

By the above observation $y$ fixes an almost complex structure, $J_{\pi(y)}$, on $T_xN$. For any $y$ in the $x$-fibre of $S_+$, $\pi_{*|_{\mathcal{H}_y}}$ defines an isomorphism between $\mathcal{H}_y$ and $T_xN$. We will denote this isomorphism with $\mu$. So we can define an almost complex structure $J_1$ (*warning*: this is called $J_2$ in [5]) on $S_+$ by

$$J_1(v_y, h_y) = (I_y v_y, J_{\pi(y)} h_y),$$

where $I$ is minus the standard complex structure on $S^2$. This means that the vectors of type $(1,0)$ with respect to $J_1$ in $T_yS_+$ are given by $(T_x^{(1,0)}N)^\mu \oplus (T_x^{(0,2)}N)^\nu$. Let us recall the following (see Eells–Salamon [5]):

https://doi.org/10.1023/A:1000358906964 Published online by Cambridge University Press

THEOREM 2.1. *If $F: M \to N$ is a conformal and harmonic immersion, then $\tilde{F}_+$ is $J_1$-holomorphic.*

Then we are in the following situation: given an hyperkähler manifold $N$ and a minimal isometric immersion $F: M \longrightarrow N$ we have

$$M \xrightarrow{\tilde{F}_+} S_+ = N \times S^2 \xrightarrow{\pi} S^2 \xrightarrow{\rho} \mathbb{C} \cup \{\infty\},$$

where $\pi$ is the projection on the second factor and $\rho$ is the stereographic projection. On $S^2$ we are considering the usual complex structure so that $\pi \circ \tilde{F}_+$ is anti-holomorphic. If the Gauss lift $\tilde{F}_+$ omits an open set of $S^2$ we have, after composition with a stereographic projection with pole in this open set and conjugation, a bounded holomorphic map from $M$ to $\mathbb{C}$. This will be a crucial point in the proof of our theorem.

## 3. Proof of Theorem 1.1

Micallef and Wolfson ([7]) proved that the stability condition implies that, for every compactly supported section $\sigma$ of the complexified normal bundle $\nu_{\mathbb{C}}$:

$$\int_M \{|\bar{\partial}\sigma|^2 - 2[|\mathrm{d}\alpha|^2 + \tfrac{1}{4}S\sin^2\alpha]|\sigma|^2\}\,\mathrm{d}Vol \geqslant 0,$$

where $S$ is the scalar curvature of $N$. If $N$ is hyperkähler then $S = 0$. So we have

$$\int_M |\bar{\partial}\sigma|^2 \geqslant 2\int_M |\mathrm{d}\alpha|^2|\sigma|^2\ . \tag{2}$$

*Suppose there exists a global holomorphic section $\sigma$ of $\nu_{\mathbb{C}}$ in $L^2$*. Then, taking a cut-off function $f_R$ such that $f_R = 1$ on $B_R(p)$, $f_R = 0$ outside $B_{2R}(p)$ and $|\mathrm{d}(f_R)| < \frac{c}{R}$ everywhere, applying 2 to $f\sigma$ we get

$$2\int_M |\mathrm{d}\alpha|^2 f_R^2|\sigma|^2 \leqslant 2\int_M |\mathrm{d}(f_R)|^2|\sigma|^2.$$

Letting $R \to \infty$ we have that, since $\sigma \in L^2$, $\mathrm{d}\alpha = 0$ and so $\alpha$ is constant on $M$. Now, as in [10], choose a point $p \in M$ and a complex structure on $TN$ such that $T_pM$ is a complex subspace of $T_{F(p)}N$. The Kähler angle of this complex structure vanishes at $p$, but it is still constant on $M$. *Then the immersion is holomorphic with respect to this complex structure of $N$.*

So the following lemma concludes the proof of the theorem:

LEMMA 3.1. *If the hypothesis of the theorem hold then there exists a global holomorphic section $\sigma$ of the complexified normal bundle such that $\sigma$ is square integrable.*

https://doi.org/10.1023/A:1000358906964 Published online by Cambridge University Press

*Proof.* Let $W$ be an open set of $S^2$ s.t. $W \subset S^2 \setminus \tilde{F}_+(M)$, and $J$ a complex structure on $N$ corresponding, via the discussion in section 2, to a point in $W$. We then have $1 - \cos\alpha_J < 1 - \epsilon$, $\epsilon > 0$, everywhere on $M$. Since $F$ is conformal there exist $\{e_1, e_2\}$ local real vector fields in $F_*(TM)$ such that $F_*(\frac{\partial}{\partial z}) = \sqrt{\frac{\lambda}{2}}(e_1 - ie_2)$, where $\lambda$ is the conformal factor of the immersion; then we complete $\{e_1, e_2\}$ to a local orthonormal basis of $TN$, $\{e_1, \ldots, e_4\}$. Defining

$$\begin{aligned} f_1 &= e_1, \quad & f_2 &= Je_1 \\ f_3 &= e_4, \quad & f_4 &= -Je_4, \end{aligned} \tag{3}$$

we have directly that

$$\langle \tilde{F}_+(p), f_1 \wedge f_2 + f_3 \wedge f_4 \rangle = \cos\alpha_J(p). \tag{4}$$

This means, by the discussion in Section 2, that the angle between $J$ as a point of the sphere and $\tilde{F}_+(p)$ is precisely the Kähler angle at $p$ and therefore the stereographic projection of $\tilde{F}_+$, from the point corresponding to $J$ in $S^2$ has norm $\frac{\sin\alpha_J}{1-\cos\alpha_J}$.

We will indicate the hermitian product of $X$ and $Y$ by $g(X,Y)$ and $X \cdot Y = g(X, \overline{Y})$, so that the $\cdot$ product is *complex bilinear*. Define $s = [JF_*(\frac{\partial}{\partial z})]^\perp$, where $J$ is a complex structure on $N$ and $\perp$ is the projection onto the normal bundle $\nu_{\mathbb{C}}$.

$s$ is a local holomorphic section of $\nu_{\mathbb{C}}$, in fact:

$$\begin{aligned} D_{\bar{z}} s &= D_{\bar{z}} \left( \left( JF_* \left( \frac{\partial}{\partial z} \right) - \left[ JF_* \left( \frac{\partial}{\partial z} \right) \right]^T \right) \right) \\ &= \left( \nabla_{F_* \frac{\partial}{\partial \bar{z}}} \left( JF_* \left( \frac{\partial}{\partial z} \right) \right) \right)^\perp - \left( \nabla_{F_* \frac{\partial}{\partial \bar{z}}} \left( \left[ JF_* \left( \frac{\partial}{\partial z} \right) \right]^T \right) \right)^\perp , \end{aligned} \tag{5}$$

where $D$ is the covariant derivative in the normal bundle and $\nabla$ is the covariant derivative on $N$. The first term vanishes because $J$ is parallel and $F$ is harmonic; the second term vanishes also. To prove this first observe that

$$\begin{aligned} \left[ JF_* \left( \frac{\partial}{\partial z} \right) \right]^T = \frac{1}{\lambda} \Bigg[ & \left( JF_* \left( \frac{\partial}{\partial z} \right) \cdot F_* \left( \frac{\partial}{\partial \bar{z}} \right) \right) F_* \left( \frac{\partial}{\partial \bar{z}} \right) \\ & + \left( JF_* \left( \frac{\partial}{\partial z} \right) \cdot F_* \left( \frac{\partial}{\partial z} \right) \right) F_* \left( \frac{\partial}{\partial z} \right) \Bigg] \end{aligned} \tag{6}$$

but

$$JF_* \left( \frac{\partial}{\partial z} \right) \cdot F_* \left( \frac{\partial}{\partial \bar{z}} \right) = 0 \tag{7}$$

https://doi.org/10.1023/A:1000358906964 Published online by Cambridge University Press

and then

$$\left(\nabla_{F_*\frac{\partial}{\partial\bar z}}\left[JF_*\left(\frac{\partial}{\partial z}\right)\right]^T\right)^\perp$$
$$=\frac{1}{\lambda}\left[\left(JF_*\left(\frac{\partial}{\partial z}\right)\cdot F_*\left(\frac{\partial}{\partial z}\right)\right)\left(\nabla_{F_*\frac{\partial}{\partial\bar z}}F_*\frac{\partial}{\partial z}\right)\right]^\perp \tag{8}$$

and then again harmonicity (i.e. $\nabla_{F_*\frac{\partial}{\partial\bar z}}F_*\frac{\partial}{\partial z}=0$ ) proves our claim.

Then $\frac{\bar s}{|s|^2}$ is a local meromorphic section of $\nu_{\mathbb{C}}$; in fact we have $D_z s = f s$, where $f$ is a complex valued function such that $\frac{\partial(s\cdot\bar s)}{\partial z}=f s\cdot\bar s$ and therefore $f=\frac{\partial\log|s|^2}{\partial z}$.

So we have

$$\begin{aligned}D_{\bar z}\left(\frac{\bar s}{|s|^2}\right) &= \frac{\partial}{\partial\bar z}\left(\frac{1}{|s|^2}\right)\bar s+\frac{1}{|s|^2}D_{\bar z}(\bar s)=\left(\frac{\partial}{\partial\bar z}\left(\frac{1}{|s|^2}\right)+\frac{1}{|s|^2}\bar f\right)\bar s\\ &=\left[\frac{\partial}{\partial\bar z}\left(\frac{1}{|s|^2}\right)+\frac{1}{|s|^4}\frac{\partial}{\partial\bar z}\left(|s|^2\right)\right]\bar s=0.\end{aligned}\tag{9}$$

Since the Gauss lift omits the open set $W$ of $S^2$ and it is holomorphic, taking $p\in W$, the function $h$ defined by the conjugate of $\rho\circ\pi\circ\tilde F_+$, where $\rho$ is the stereographic projection from the point $p$, is bounded and holomorphic. So $M$ admits (see Ahlfors–Sario [2] and Springer [9]) a square integrable holomorphic differential $\beta$. In a local chart $(U,z)$, $\beta=\zeta\,\mathrm{d}z$. Hence $\sigma=\frac{\bar s}{|s|^2}h\zeta$ is a global meromorphic section of $\nu_{\mathbb{C}}$: in fact, if $(V,w)$ is another local chart such that $U\cap V\neq\emptyset$ we have $\beta=\zeta'\mathrm{d}w$, where $\zeta'=\frac{\partial z}{\partial w}\zeta$ and $\frac{\partial}{\partial w}=\frac{\partial z}{\partial w}\frac{\partial}{\partial z}$ on $U\cap V$ and so $\sigma(w)=\sigma(z)$ on $U\cap V$.

To prove that $\sigma$ is square integrable we look at the zeros of $s$ which are the points where

$$\left(g\left(JF_*\left(\frac{\partial}{\partial z}\right),\sqrt{\tfrac12}(e_3-ie_4)\right),g\left(JF_*\left(\frac{\partial}{\partial z}\right),\sqrt{\tfrac12}(e_3+ie_4)\right)\right)$$
$$=(\sqrt{\lambda}\sin\alpha_J,0)=(0,0).$$

Hence they are the anti-complex points of $F$ with respect to $J$ (because there are no complex points w.r.t $J$ by assumption).

We have then

$$\left|\frac{\bar s}{|s|^2}h\zeta\right|=\left|\frac{1}{1-\cos\alpha}\right|\frac{|\zeta|}{\sqrt{\lambda}},$$

which is a locally bounded function. Hence

$$\left|\frac{\bar s}{|s|^2}h\zeta\right|^2\leqslant C|\beta|^2$$

and then is in $L^2$, since $\beta$ is in $L^2$. $\square$

## Acknowledgements

This paper is part of the author's thesis at Warwick University. The author wishes to thank his advisor M.J. Micallef for having suggested this problem to him and for many helpful conversations about this subject.

## References

1. Atiyah, M. and Hitchin, N.: *The Geometry and Dynamics of magnetic monopoles*, Princeton University Press, Princeton, 1988.
2. Ahlfors, L. V. and Sario, L.: *Riemann Surfaces*, Princeton University Press, Princeton, 1960.
3. Chern, S. S.: Minimal surfaces in Euclidean space of $N$ dimensions, in *Differential and Combinatorial Topology: Symposium in honor of Marston Morse*, Princeton University Press, 1965, pp. 187–198.
4. Chern, S. S. and Spanier, E.: A theorem on orientable surfaces in four-manifold space, *Comm. Math. Helv.* 25 (1951) 205–209.
5. Eells, J. and Salamon, S.: Twistorial construction of Harmonic Maps of Surfaces into Four-Manifolds, *Ann. Scuola Normale Sup. – Pisa* 4 (1985) 589–640.
6. Micallef, M. J.: Stable minimal surfaces in Euclidean space, *J. Differ. Geom.* 19 (1984) 57–84.
7. Micallef, M. J. and Wolfson, J. G.: The second variation of area of minimal surfaces in four-manifolds, *Math. Ann.* 295 (1993) 245–267.
8. Osserman, R.: Global properties of minimal surfaces in $\mathbb{R}^3$ and in $\mathbb{R}^n$, *Ann. of Math. (2)* 80 (1964) 340–364.
9. Springer, G.: *Introduction to Riemann Surfaces*, Addison-Wesley, Reading, MA, 1957.
10. Wolfson, J. G.: Minimal surfaces in Kähler surfaces and Ricci curvature, *J. Differ. Geom.* 29 (1989) 281–294.