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ORTHOGONALITY OF CERTAIN FUNCTIONS WITH RESPECT TO COMPLEX VALUED WEIGHTS

GEORGE GASPER

1. Introduction. In his work on the Dirichlet problem for the Heisenberg group Greiner [**5**] showed that each L_α-spherical harmonic is a unique linear combination of functions of the form

$$e^{in\psi} \sin^{|n|/2}\theta H_k^{(\alpha,n)}(e^{i\theta})$$

with $k = 0, 1, 2, \ldots$ and $n = 0, \pm 1, \pm 2, \ldots$, where $H_k^{(\alpha,n)}(\theta^{i\theta})$ is defined by the generating function

$$(1 - te^{-i\theta})^{(n-|n|+\alpha-1)/2}(1 - te^{i\theta})^{-(n+|n|+\alpha+1)/2} = \sum_{k=0}^{\infty} H_k^{(\alpha,n)}(e^{i\theta})t^k.$$

Since $H_k^{(0,0)}(e^{i\theta}) = P_k(\cos\theta)$, where $P_k(x)$ is the Legendre polynomial of degree k, and these functions satisfy the orthogonality relation

$$\int_0^\pi P_j(\cos\theta)P_k(\cos\theta)\sin\theta\, d\theta = 0, \quad j \neq k,$$

Greiner raised the question of whether the functions $H_k^{(\alpha,n)}(e^{i\theta})$ are orthogonal or biorthogonal with respect to some complex valued weight function.

In this paper we shall show that this is indeed the case and even show that the more general class of functions $C_k^{(\alpha,\beta)}(e^{i\theta})$ defined by

$$(1.1) \qquad (1 - te^{-i\theta})^{-\alpha}(1 - te^{i\theta})^{-\beta} = \sum_{k=0}^{\infty} C_k^{(\alpha,\beta)}(e^{i\theta})t^k$$

satisfy the orthogonality relation

$$(1.2) \qquad \int_0^{2\pi} C_j^{(\alpha,\beta)}(e^{i\theta})C_k^{(\alpha,\beta)}(e^{i\theta})w^{(\alpha,\beta)}(\theta)d\theta = 0, \quad j \neq k,$$

when $\alpha, \beta, \alpha + \beta > -1$, where

$$w^{(\alpha,\beta)}(\theta) = (1 - e^{-2i\theta})^\alpha(1 - e^{2i\theta})^\beta.$$

Hence the functions $H_k^{(\alpha,n)}(e^{i\theta})$ are orthogonal on $(0, 2\pi)$ with respect to the weight function

$$(1 - e^{-2i\theta})^{(|n|-n-\alpha+1)/2}(1 - e^{2i\theta})^{(n+|n|+\alpha+1)/2}.$$

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Unfortunately, the theory of polynomials orthogonal on the unit circle [**11**, Chapter XI], [**2**, Chapter V], [**4**] cannot be applied to $C_k^{(\alpha,\beta)}(e^{i\theta})$ since these functions are not polynomials in powers of $z = e^{i\theta}$ and the weight function $w^{(\alpha,\beta)}(\theta)$ is not real valued when $\alpha \neq \beta$. It is easy to see that when $\alpha \neq \beta$ the orthogonality relation (1.2) cannot hold with the upper limit of integration 2π replaced by π (even if $w^{(\alpha,\beta)}(\theta)d\theta$ is replaced by $d\mu(\theta)$ where μ is a positive measure) or if $C_k^{(\alpha,\beta)}(e^{i\theta})$ is replaced by its complex conjugate (also see the remarks at the end of Section 2). We shall also consider q-(basic) analogs of $C_k^{(\alpha,\beta)}(e^{i\theta})$ which contain the continuous q-ultraspherical polynomials in [**1**] as a special case and give conditions under which they are orthogonal.

2. The function $C_k^{(\alpha,\beta)}(e^{i\theta})$. Since

$$(2.1) \qquad (1 - z)^{-a} = \sum_{k=0}^{\infty} \frac{(a)_k}{k!} z^k, \quad |z| < 1,$$

where $(a)_0 = 1$ and $(a)_k = a(a + 1) \dots (a + k - 1)$ for $k \geqq 1$, it follows from (1.1) that

$$(2.2) \qquad C_k^{(\alpha,\beta)}(e^{i\theta}) = \sum_{j=0}^{k} \frac{(\alpha)_{k-j}(\beta)_j}{(k - j)!j!} e^{i(2j-k)\theta}$$

$$= \frac{(\alpha)_k}{k!} e^{-ik\theta} {}_2F_1(-k, \beta; 1 - \alpha - k; e^{2i\theta}).$$

Thus

$$C_k^{(\alpha,\beta)}(e^{i\theta}) = C_k^{(\beta,\alpha)}(e^{i\theta}) \quad \text{and} \quad C_k^{(\alpha,\alpha)}(e^{i\theta}) = C_k^{\alpha}(\cos\theta),$$

where $C_k^{\alpha}(x)$ is the ultraspherical polynomial of degree k and order α.

To prove the orthogonality relation (1.2) we first need to show that for any integer n that if $\alpha, \beta, \alpha + \beta > -1$ then

$$(2.3) \qquad \int_0^{2\pi} e^{in\theta} w^{(\alpha,\beta)}(\theta)d\theta = \begin{cases} 0, & n \text{ odd}, \\ h_{n/2}^{(\alpha,\beta)}, & n \text{ even}, \end{cases}$$

where

$$h_m^{(\alpha,\beta)} = \frac{2\pi\Gamma(\alpha + \beta + 1)}{\Gamma(\alpha + 1)\Gamma(\beta + 1)} \frac{(-\alpha)_m}{(\beta + 1)_m}.$$

Note that, since $(a)_{-m} = (-1)^m/(1 - a)_m$, $(-\alpha)_m/(\beta + 1)_m$ can be replaced by $(-\beta)_{-m}/(\alpha + 1)_{-m}$ when m is a negative integer. By (2.1), if $|t| < 1$ then

$$\int_0^{2\pi} e^{in\theta}(1 - te^{-2i\theta})^{\alpha}(1 - te^{2i\theta})^{\beta}d\theta$$

$$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-\alpha)_j(-\beta)_k}{j!k!} t^{j+k} \int_0^{2\pi} e^{i(n-2j+2k)\theta}d\theta,$$

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which equals zero when n is odd and equals

$$2\pi \sum_{k=0}^{\infty} \frac{(-\alpha)_{k+m}(-\beta)_k}{(k+m)!k!} t^{2k+m} = \frac{2\pi(-\alpha)_m t^m}{m!} {}_2F_1(m-\alpha, -\beta; m+1; t^2)$$

when $n = 2m$, $m = 0, 1, 2, \ldots$. Letting $t \to 1-$ and using Gauss' formula

$$ {}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad c > a+b,$$

gives (2.3) for $n \geqq 0$; while the case $n < 0$ follows analogously (or by using the fact that $w^{(\alpha,\beta)}(-\theta) = w^{(\beta,\alpha)}(\theta)$).

In view of (2.2) to prove (1.2) it suffices to show that

$$\text{(2.4)} \qquad \int_0^{2\pi} e^{in\theta} C_k^{(\alpha,\beta)}(e^{i\theta}) w^{(\alpha,\beta)}(\theta) d\theta = 0, \quad k > |n|.$$

By (2.2) and (2.3) this holds if $k - n$ is odd; while if $k - n = 2m$ is even and $k - 2 \geqq |n|$ then the above integral equals

$$\text{(2.5)} \qquad \frac{(\alpha)_k}{k!} \sum_{j=0}^{k} \frac{(-k)_j(\beta)_j}{j!(1-\alpha-k)j} \int_0^{2\pi} e^{i(2j-2m)\theta} w^{(\alpha,\beta)}(\theta) d\theta$$
$$= \frac{2\pi\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \frac{(\alpha)_k(-\beta)_m}{k!(\alpha+1)_m} \times {}_3F_2\left(\begin{matrix} -k, \beta, -\alpha-m \\ 1+\beta-m, 1-\alpha-k \end{matrix}; 1\right) = 0$$

from the case $p = 2$ of the formula

$$\text{(2.6)} \qquad {}_{p+1}F_p\left(\begin{matrix} a, b_1+m_1, \ldots, b_p+m_p \\ b_1, \ldots, b_p \end{matrix}; 1\right) = 0,$$
$$\operatorname{Re}(-a) > m_1 + \ldots + m_p,$$

where, as elsewhere, it is assumed that $m_1, \ldots, m_p$ are non-negative integers and that no denominator parameter is a negative integer or zero; see [**7**] and [**8**]. This completes the proof of (2.4) and hence of (1.2).

To show that

$$\text{(2.7)} \qquad \int_0^{2\pi} (C_k^{(\alpha,\beta)}(e^{i\theta}))^2 w^{(\alpha,\beta)}(\theta) d\theta = \frac{2\pi\Gamma(k+\alpha+\beta)(2k+\alpha+\beta)}{k!\Gamma(\alpha)\Gamma(\beta)(k+\alpha)(k+\beta)}$$

if suffices to apply the summation formula [**7**]

$$\text{(2.8)} \qquad {}_{p+2}F_{p+1}\left(\begin{matrix} a, b, b_1+m_1, \ldots, b_p+m_p \\ b+1, b_1, \ldots, b_p \end{matrix}; 1\right) = \frac{\Gamma(b+1)\Gamma(1-a)}{\Gamma(1+b-a)}$$
$$\times \frac{(b_1-b)_{m_1} \ldots (b_p-b)_{m_p}}{(b_1)_{m_1} \ldots (b_p)_{m_p}}, \quad \operatorname{Re}(-a) > m_1 + \ldots + m_p - 1,$$

to the cases $n = \pm k$ of the sum in (2.5) to obtain for $k > 0$ that

$$(2.9)\qquad \int_0^{2\pi} e^{ik\theta} C_k^{(\alpha,\beta)}(e^{i\theta}) w^{(\alpha,\beta)}(\theta) d\theta = \frac{2\pi\Gamma(k+\alpha+\beta)}{\Gamma(\alpha)\Gamma(k+\beta+1)},$$

$$(2.10)\qquad \int_0^{2\pi} e^{-ik\theta} C_k^{(\alpha,\beta)}(e^{i\theta}) w^{(\alpha,\beta)}(\theta) d\theta = \frac{2\pi\Gamma(k+\alpha+\beta)}{\Gamma(\beta)\Gamma(k+\alpha+1)},$$

and then to use (2.2) and (2.3). The ${}_3F_2$ cases of (2.6) and (2.8) can be proved by using the Thomae transformation formulas for ${}_3F_2$ series [**10**, Section 4.3].

Remarks. From (2.2) it is clear that neither $e^{i\theta}$ nor $e^{-i\theta}$ can be written as a linear combination of the functions $C_k^{(\alpha,\beta)}(e^{i\theta})$ when $\alpha \neq 0$ and $\beta \neq 0$. Since the functions $\{e^{in\theta}\}_{n=-\infty}^{\infty}$ are closed in the L^2 space of functions $f(\theta)$ with

$$\int_0^{2\pi} |f(\theta)|^2 |w^{(\alpha,\beta)}(\theta)| d\theta < \infty,$$

it is natural to consider how the system

$$(2.11)\qquad \{C_k^{(\alpha,\beta)}(e^{i\theta})\}_{k=0}^{\infty}$$

could be enlarged to form a closed orthogonal system. Observing that $C_k^{(\alpha,\beta)}(z)$ is what Jones and Thron [**6**] call a Laurent polynomial in the complex variable z, one is tempted in analogy with their theory of orthogonal Laurent polynomials on the interval $[0, \infty)$ to try to find constants a, b such that the function $ae^{-i\theta} + b$ is orthogonal (in the sense (1.2)) to each $C_k^{(\alpha,\beta)}(e^{i\theta})$. But, in general, the case $k = 1$ implies that $a = 0$ and so this fails.

However, when $\alpha = \beta$ we can use the fact that the ultraspherical polynomials

$$C_k^{\alpha+1}(\cos\theta) = C_k^{(\alpha+1,\alpha+1)}(e^{i\theta}),$$

which are even functions of θ, satisfy the orthogonality relation

$$\int_0^{\pi} C_j^{\alpha+1}(\cos\theta) C_k^{\alpha+1}(\cos\theta) \sin^2\theta\, w^{(\alpha,\alpha)}(\theta) d\theta = 0, \quad j \neq k,$$

to see that the functions

$$(2.12)\qquad \{C_k^{\alpha}(\cos\theta), \sin\theta\, C_k^{\alpha+1}(\cos\theta)\}_{k=0}^{\infty}$$

form a closed (in L^2) orthogonal system on $(0, 2\pi)$ with respect to the weight function $w^{(\alpha,\alpha)}(\theta)$. Since

$$\sin\theta\, C_0^{\alpha+1}(\cos\theta) = \sin\theta = (e^{i\theta} - e^{-i\theta})/2i,$$

this suggests that when $|\alpha|, |\beta| < 1$, $\alpha \neq \beta$ we should first consider a function of the form

$$c_1^{(\alpha,\beta)}(e^{i\theta}) = ae^{-i\theta} + b + ce^{i\theta}.$$

In view of (2.3), (2.4), (2.9), and (2.10), in order for this function to be orthogonal to each function in (2.11) it is necessary and sufficient that $b = 0$ and $a\beta(\beta + 1) + c\alpha(\alpha + 1) = 0$. Thus we may take

$$c_1^{(\alpha,\beta)}(e^{i\theta}) = (\alpha)_2 e^{-i\theta} - (\beta)_2 e^{i\theta},$$

which will be linearly independent of the functions in (2.11) as long as $\alpha \neq 0$ and $\beta \neq 0$ (note that, since

$$C_k^{(0,\beta)}(e^{i\theta}) = \frac{(\beta)_k}{k!} e^{ik\theta},$$

$c_1^{(0,\beta)}(e^{i\theta})$ is a constant multiple of $C_1^{(0,\beta)}(e^{i\theta})$ and $c_1^{(\alpha,0)}(e^{i\theta})$ is a constant multiple of $C_1^{(\alpha,0)}(e^{i\theta})$). Similarly we find that

$$c_2^{(\alpha,\beta)}(e^{i\theta}) = (\alpha)_3 e^{-2i\theta} + \alpha\beta(\alpha - \beta) - (\beta)_3 e^{2i\theta}$$

and

$$c_3^{(\alpha,\beta)}(e^{i\theta}) = (\alpha)_4 e^{-3i\theta} + \beta(2\alpha + 3 - \beta)(\alpha)_2 e^{-i\theta} \\ - \alpha(2\beta + 3 - \alpha)(\beta)_2 e^{i\theta} - (\beta)_4 e^{3i\theta}$$

are orthogonal to each other, to $c_1^{(\alpha,\beta)}(e^{i\theta})$ and to the functions in (2.11). Unfortunately, when $\alpha \neq 0$ and $\beta \neq 0$ the succeeding functions of the form

$$c_k^{(\alpha\,\beta)}(e^{i\theta}) = \sum_{j=0}^{k} a_j(k; \alpha, \beta) e^{i(2j-k)\theta} \tag{2.13}$$

that need to be added to (2.11) to form a closed orthogonal system become progressively more complicated and so I have not been able to find any simple general formulas for them (except to say that if we normalize the right hand side of (2.13) so that $a_0(k; \alpha, \beta) = (\alpha)_{k+1}$ then $a_k(k; \alpha, \beta) = -(\beta)_{k+1}$).

3. The q-analog $C_k^{(\alpha,\beta)}(e^{i\theta}; q)$. For $|z| < 1$ the ${}_{p+1}\phi_p$ basic hypergeometric series is defined by

$${}_{p+1}\phi_p\left(\begin{matrix} a_1, \ldots, a_{p+1} \\ b_1, \ldots, b_p \end{matrix}; q, z\right) = \sum_{k=0}^{\infty} \frac{(a_1; q)_k \ldots (a_{p+1}; q)_k}{(q; q)_k (b_1; q)_k \ldots (b_p; q)_k} z^k,$$

where $(a; q)_k = (1 - a)(1 - aq) \ldots (1 - aq^{k-1})$, $(a; q)_0 = 1$ and, as elsewhere, it is assumed that $|q| < 1$ and no denominator parameter is 1 or a negative integer power of q.

Since the continuous q-ultraspherical polynomials $C_k(\cos\theta; \beta|q)$ are defined in [**1**] by

$$\frac{(\beta e^{-i\theta}t; q)_\infty (\beta e^{i\theta}t; q)_\infty}{(e^{-i\theta}t; q)_\infty (e^{i\theta}t; q)_\infty} = \sum_{k=0}^{\infty} C_k(\cos\theta; \beta|q) t^k, \tag{3.1}$$

where $(a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k)$, it is natural to consider for a q-analog of

$C_k^{(\alpha,\beta)}(e^{i\theta})$ the functions $C_k^{(\alpha,\beta)}(e^{i\theta}; q)$ defined by

$$\frac{(\alpha e^{-i\theta}t; q)_\infty(\beta e^{i\theta}t; q)_\infty}{(e^{-i\theta}t; q)_\infty(e^{i\theta}t; q)_\infty} = \sum_{k=0}^{\infty} C_k^{(\alpha,\beta)}(e^{i\theta}; q)t^k. \tag{3.2}$$

Then

$$C_k(\cos\theta; \beta|q) = C_k^{(\beta,\beta)}(e^{i\theta}; q) \quad \text{and} \quad \lim_{q\to 1-} C_k^{(\alpha,\beta)}(e^{i\theta}; q) = C_k^{(\alpha,\beta)}(e^{i\theta}).$$

From (3.2) and the q-binomial theorem [**10**, p. 92]

$$\frac{(az; q)_\infty}{(z; q)_\infty} = \sum_{k=0}^{\infty} \frac{(a; q)_k}{(q; q)_k} z^k, \quad |q| < 1, |z| < 1, \tag{3.3}$$

it follows that

$$\begin{aligned} C_k^{(\alpha,\beta)}(e^{i\theta}; q) &= \sum_{j=0}^{k} \frac{(\alpha; q)_{k-j}(\beta; q)_j}{(q; q)_{k-j}(q; q)_j} e^{i(2j-k)\theta} \\ &= \frac{(\alpha; q)_k}{(q; q)_k} e^{-ik\theta} {}_2\phi_1\left(\begin{matrix} q^{-k}, \beta \\ \alpha^{-1}q^{1-k} \end{matrix}; q, \alpha^{-1}qe^{2i\theta}\right). \end{aligned} \tag{3.4}$$

We shall show that if $|\alpha| < 1$ and $|\beta| < 1$ then these functions satisfy the orthogonality relation

$$\int_0^{2\pi} C_j^{(\alpha,\beta)}(e^{i\theta}; q)C_k^{(\alpha,\beta)}(e^{i\theta}; q)w^{(\alpha,\beta)}(\theta; q)d\theta = 0, \quad j \neq k, \tag{3.5}$$

where

$$w^{(\alpha,\beta)}(\theta; q) = \frac{(e^{-2i\theta}; q)_\infty(e^{2i\theta}; q)_\infty}{(\alpha e^{-2i\theta}; q)_\infty(\beta e^{2i\theta}; q)_\infty}.$$

Using (3.3) and the transformation formula [**1**, (4.7)]

$${}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, z\right) = \frac{(bz; q)_\infty(c/b; q)_\infty}{(z; q)_\infty(c; q)_\infty} {}_2\phi_1\left(\begin{matrix} abz/c, b \\ bz \end{matrix}; q, c/b\right)$$

we find, as in the proof of (2.3), that for any integer n

$$\int_0^{2\pi} e^{in\theta}w^{(\alpha,\beta)}(\theta; q)d\theta = \begin{cases} 0, & n \text{ odd}, \\ h_{n/2}^{(\alpha,\beta)}(q), & n \text{ even}, \end{cases} \tag{3.6}$$

where

$$h_m^{(\alpha,\beta)}(q) = \frac{2\pi(\alpha; q)_\infty(\beta q; q)_\infty}{(q; q)_\infty(\alpha\beta; q)_\infty} \frac{\alpha^m(\alpha^{-1}; q)_m}{(\beta q; q)_m} \left(1 + \frac{1-\beta}{1-\alpha} q^m\right)$$

and we used the fact that

$$(a; q)_{-n} = (-1)^n a^{-n} q^{n(n+1)/2}/(q/a; q)_n.$$

Then (3.4) and (3.6) yield that

$$\int_0^{2\pi} e^{in\theta}C_k^{(\alpha,\beta)}(e^{i\theta}; q)w^{(\alpha,\beta)}(\theta; q)d\theta \tag{3.7}$$

equals zero when $k - n$ is odd and equals

$$(3.8) \quad \frac{2\pi(\alpha; q)_\infty(\beta q; q)_\infty}{(q; q)_\infty(\alpha\beta; q)_\infty} \frac{(\alpha; q)_k(\beta q)^m(\beta^{-1}; q)_m}{(q; q)_k(\alpha q; q)_m} \times \left\{ {}_3\phi_2\left(\begin{matrix} q^{-k}, \beta, \alpha^{-1}q^{-m} \\ \beta q^{1-m}, \alpha^{-1}q^{1-k} \end{matrix}; q; q\right) + \frac{1-\beta}{1-\alpha} q^{-m} {}_3\phi_2\left(\begin{matrix} q^{-k}, \beta, \alpha^{-1}q^{-m} \\ \beta q^{1-m}, \alpha^{-1}q^{1-k} \end{matrix}; q, q^2\right) \right\}$$

when $k - n = 2m$ is even. From the q-analogs of (2.6) derived in [**3**, (8), (15)] it follows that both of the above ${}_3\phi_2$ series and hence the integral (3.7) are equal to zero when $k > |n|$; which, in view of (3.4), completes the proof of (3.5). This can also be derived by writing the sum in braces in (3.8) as a ${}_4\phi_3$ series and applying the transformation formula [**1**, (4, 10)].

Application of the two q-analogs of (2.8) in [**3**, (7), (14)] to (3.8) in the cases $n = \pm k \neq 0$ gives

$$(3.9) \quad \int_0^{2\pi} e^{in\theta} C_k^{(\alpha,\beta)}(e^{i\theta}; q) w^{(\alpha,\beta)}(\theta; q) d\theta = \begin{cases} \dfrac{2\pi(\alpha; q)_\infty(\beta q; q)_\infty(\alpha\beta; q)_k}{(q; q)_\infty(\alpha\beta; q)_\infty(\beta q; q)_k}, & n = k > 0, \\[2ex] \dfrac{2\pi(\alpha q; q)_\infty(\beta; q)_\infty(\alpha\beta; q)_k}{(q; q)_\infty(\alpha\beta; q)_\infty(\alpha q; q)_k}, & n = -k < 0, \end{cases}$$

from which it follows that

$$(3.10) \quad \int_0^{2\pi} (C_k^{(\alpha,\beta)}(e^{i\theta}; q))^2 w^{(\alpha,\beta)}(\theta; q) d\theta = \frac{2\pi(\alpha; q)_\infty(\beta; q)_\infty(\alpha\beta; q)_k}{(q; q)_\infty(\alpha\beta; q)_\infty(q; q)_k} \times \frac{2 - (\alpha + \beta)q^k}{(1 - \beta q^k)(1 - \alpha q^k)}.$$

Remarks. As in Section 2 one can show that the function

$$c_1^{(\alpha,\beta)}(e^{i\theta}; q) = (\alpha; q)_2 e^{-i\theta} - (\beta; q)_2 e^{i\theta}$$

is orthogonal to each $C_k^{(\alpha,\beta)}(e^{i\theta}; q)$ and compute the succeeding functions of the form

$$c_k^{(\alpha,\beta)}(e^{i\theta}; q) = \sum_{j=0}^{k} a_j^{(\alpha,\beta)}(k; q) e^{i(2j-k)\theta}$$

so that, for $|\alpha| < 1$, $|\beta| < 1$,

$$\{C_k^{(\alpha,\beta)}(e^{i\theta}; q), c_{k+1}^{(\alpha,\beta)}(e^{i\theta}; q)\}_{k=0}^{\infty}$$

is an orthogonal system on $(0, 2\pi)$ with respect to the weight function $w^{(\alpha,\beta)}(\theta; q)$. In the continuous q-ultraspherical case $\alpha = \beta$, Section 2.5 in

[11] yields for the polynomials $c_k(x;\beta|q)$ defined by

$$(1-x^2)c_k(x;\beta|q)=C_{k+2}(1;\beta|q)C_k(x;\beta|q)-C_k(1;\beta|q)C_{k+2}(x;\beta|q)$$

that

$$\int_0^\pi c_j(x;\beta|q)c_k(x;\beta|q)\sin^2\theta\, w^{(\beta,\beta)}(\theta;q)d\theta=0,\quad j\neq k,$$

and so, analogous to the system (2.12), the functions

$$\{C_k(\cos\theta;\beta|q),\sin\theta c_k(\cos\theta;\beta|q)\}_{k=0}^\infty$$

form a closed (in L^2) orthogonal system on $(0,2\pi)$ with respect to the weight function $w^{(\beta,\beta)}(\theta;q)$, $|\beta|<1$. This suggests that an H^p theory can be developed for the continuous q-ultraspherical polynomials analogous to that given for the ultraspherical polynomials in [9].

4. Additional orthogonal functions. The generating function (3.2) suggests that we should also consider the more general functions $C_k^{(\alpha,\beta,\gamma)}(e^{i\theta};q)$ defined by

$$(4.1)\qquad \frac{(\alpha e^{-i\theta}t;q)_\infty(\beta e^{i\theta}t;q)_\infty}{(e^{-i\theta}t;q)_\infty(\gamma e^{i\theta}t;q)_\infty}=\sum_{k=0}^\infty C_k^{(\alpha,\beta,\gamma)}(e^{i\theta};q)t^k.$$

Then

$$C_k^{(\alpha,\beta,1)}(e^{i\theta};q)=C_k^{(\alpha,\beta)}(e^{i\theta};q).$$

Note that if $(e^{-i\theta}t;q)_\infty$ is replaced by $(\delta e^{i\theta}t;q)_\infty$, $\delta\neq 0$, in (4.1), then this case can be reduced to (4.1) by replacing t, α, β, γ by t/δ, $\alpha\delta$, $\beta\delta$, $\gamma\delta$, respectively.

As in Section 3, it follows from (4.1) that

$$(4.2)\qquad C_k^{(\alpha,\beta,\gamma)}(e^{i\theta};q)=\sum_{j=0}^k\frac{(\alpha;q)_{k-j}(\beta/\gamma;q)_j}{(q;q)_{k-j}(q;q)_j}\gamma^j e^{i(2j-k)\theta}$$
$$=\frac{(\alpha;q)_k}{(q;q)_k}e^{-ik\theta}{}_2\phi_1\left(\begin{matrix}q^{-k},\ \beta/\gamma\\ \alpha^{-1}q^{1-k}\end{matrix};q,\gamma\alpha^{-1}qe^{2i\theta}\right).$$

Analogous to (3.5) one is led to expect that for $|\alpha|$, $|\beta|$, $|\gamma|<1$ the orthogonality relation

$$(4.3)\qquad \int_0^{2\pi}C_j^{(\alpha,\beta,\gamma)}(e^{i\theta};q)C_k^{(\alpha,\beta,\gamma)}(e^{i\theta};q)w^{(\alpha,\beta,\gamma)}(\theta;q)d\theta=0,\quad j\neq k,$$

should hold with

$$w^{(\alpha,\beta,\gamma)}(\theta;q)=\frac{(e^{-2i\theta};q)_\infty(\gamma e^{2i\theta};q)_\infty}{(\alpha e^{-2i\theta};q)_\infty(\beta e^{2i\theta};q)_\infty}.$$

Unfortunately this is not the case, as we shall show, without additional restrictions on the parameters.

Using (3.3) we find that

$$(4.4)\qquad \int_0^{2\pi} e^{in\theta} w^{(\alpha,\beta,\gamma)}(\theta;q)d\theta$$

equals zero if n is odd and equals

$$(4.5)\qquad 2\pi \sum_{k=0}^{\infty} \frac{(\alpha^{-1};q)_{k+m}(\gamma/\beta;q)_k}{(q;q)_{k+m}(q;q)_k}\alpha^{k+m}\beta^k = \frac{2\pi\alpha^m(\alpha^{-1};q)_m}{(q;q)_m}\, {}_2\phi_1\left(\begin{matrix} q^m/\alpha,\ \gamma/\beta \\ q^{m+1} \end{matrix}; q, \alpha\beta\right)$$

if $n = 2m$, $m = 0, 1, \ldots$.

When $\gamma = q$ this ${}_2\phi_1$ can be summed by the q-analog of Gauss' theorem [**10**, p. 247] to give

$$(4.6)\qquad \int_0^{2\pi} e^{2im\theta} w^{(\alpha,\beta,q)}(\theta;q)d\theta = \frac{2\pi(\alpha q;q)_\infty(\beta;q)_\infty}{(q;q)_\infty(\alpha\beta;q)_\infty}\frac{\alpha^m(\alpha^{-1};q)_m}{(\beta;q)_m},$$

which also holds for negative integer values of m. Then, from (4.2),

$$(4.7)\qquad \int_0^{2\pi} e^{in\theta} C_k^{(\alpha,\beta,q)}(e^{i\theta};q) w^{(\alpha,\beta,q)}(\theta;q)d\theta$$

equals zero when $k - n$ is odd and equals

$$(4.8)\qquad \frac{2\pi(\alpha q;q)_\infty(\beta;q)_\infty}{(q;q)_\infty(\alpha\beta;q)_\infty}\frac{(\alpha;q)_k\beta^m(q/\beta;q)_m}{(q;q)_k(\alpha q;q)_m}\, {}_3\phi_2\left(\begin{matrix} q^{-k},\ \beta/q,\ \alpha^{-1}q^{-m} \\ \beta q^{-m},\ \alpha^{-1}q^{1-k} \end{matrix}; q, q^2\right)$$

when $k - n = 2m$ is even. By formula (8) in [**3**] the above ${}_3\phi_2$ and hence the integral in (4.7) is equal to zero when $k > |n|$. Hence (4.3) holds for $|\alpha|, |\beta| < 1$ when $\gamma = q$ and, by (3.5), when $\gamma = 1$. The integral in (4.3) can also be computed for $\gamma = q$, $j = k$, by applying formula (7) in [**3**] to (4.8) and using (4.2).

Since, by (4.1),

$$C_k^{(\alpha,\beta,\beta)}(e^{i\theta};q) = C_k^{(\alpha,q,q)}(e^{i\theta};q)$$

it follows from the above that (4.3) also holds when $\gamma = \beta$. A simple computation shows that (4.3) also holds when $\alpha = 1$.

To see that (4.3) does not hold for $|\alpha|, |\beta|, |\gamma|, |q| < 1$ without additional restrictions it suffices to consider the case $q = 0$. From (4.5),

$$(4.9)\qquad \int_0^{2\pi} e^{2im\theta} w^{(\alpha,\beta,\gamma)}(\theta;0)d\theta = \frac{2\pi}{1-\alpha\beta}\begin{cases} 1 - \beta + \gamma - \alpha\gamma, & m = 0, \\ \alpha^{m-1}(\alpha - 1)(1 - \alpha\gamma), & m \geqq 1, \\ \beta^{-m-1}(1-\beta)(\beta-\gamma), & m \leqq -1. \end{cases}$$

Hence

$$\int_0^{2\pi} C_2^{(\alpha,\beta,\gamma)}(e^{i\theta};0)w^{(\alpha,\beta,\gamma)}(\theta;0)d\theta = \frac{2\pi\alpha\gamma(1-\alpha)(1-\gamma)(\beta-\gamma)}{1-\alpha\beta},$$

which shows that in order for (4.3) to hold for $q = 0$ it is necessary that at least one of the following conditions must be satisfied

(4.10) $\alpha = 0, \gamma = 0, \alpha = 1, \gamma = 1, \gamma = \beta.$

That $\alpha = 0$ is also a sufficient condition when $q = 0$ follows easily by using (4.9), while the sufficiency of the last four conditions in (4.10) follows from our previous observations. This completes our analysis of (4.3) for $q = 0$. Due to computational difficulties we so far have not been able to completely determine when (4.3) holds for $q \neq 0$.

Added in proof. It should be noted that since, by page 3 of N. Levinson [Gap and Density Theorems, Amer. Math. Soc., 1940], the functions $e^{ik\theta}$, $k = 1, 2, \ldots$, are closed over any interval of length less than 2π, it follows that the functions $C_k^{(1,\beta)}(e^{i\theta}; q)$, $k = 0, 1, 2, \ldots$, cannot be orthogonal with respect to any weight function over any interval of length less than 2π. In addition, this suggests the conjecture that the functions $C_k^{(\alpha,\beta)}(e^{i\theta})$ and the functions $C_k^{(\alpha,\beta)}(e^{i\theta}; q)$, $k = 0, 1, 2, \ldots$, are closed (in L^1) over any interval of length less than 2π for suitable values of α, β, and q.

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Northwestern University,
Evanston, Illinois