Can. J. Math., Vol. XXVI, No. 1, 1974, pp. 1–6

ON THE EXISTENCE OF NORMAL METACOMPACT MOORE SPACES WHICH ARE NOT METRIZABLE

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It is known that the following classes of spaces (all spaces in this article are assumed T_1) are identical:

1. Images of metric spaces under continuous open maps with compact point inverses.
2. Spaces with uniform bases (in the sense of Alexandrov [**1**]).
3. Metacompact developable spaces.
4. Spaces with σ-point-finite bases in which closed sets are G_δ's.

For proofs, see Arhangel'skiĭ [**2**, Theorem 1], Hanai [**13**, Theorem 5], Čoban [**9**, Theorem 11], Aull [**5**, Theorem 5], Heath [**14**, Theorem 4]. (Three more equivalents of lesser interest are in Shiraki [**18**].) Thus the problem considered by Heath [**14**, p. 770], Traylor [**24**], and Borges [**8**, p. 795] of whether every metacompact normal Moore space is metrizable – a modest version of the normal Moore space conjecture – is the same as the problem of Alexandrov (problems 1.2 and 1.3 of Arhangel'skiĭ's survey [**3**]) concerning the metrizability of normal spaces with uniform bases. We shall construct a normal non-metrizable member of this class, assuming the existence of a simpler space.

THEOREM 1. *If there is a normal first countable space which is not collectionwise Hausdorff* (i.e. it contains a discrete collection of points which is *unseparated* in the sense that there do not exist simultaneously disjoint open sets about the members of the collection), *then there is a normal non-metrizable metacompact Moore space.*

This improves Heath [**14**, Theorem 3] where the stronger assumption that there exists a separable normal non-metrizable Moore space is needed to get the same conclusion. The space constructed has a number of interesting properties – it is a complete Moore space, it is locally metrizable, and it has an open dense metrizable subspace. From this last fact we shall get

COROLLARY 2. *If there is a normal non-metrizable Moore space, there is one with a dense metrizable subspace.*

Another result of interest is

Received July 28, 1971 and in revised form, November 16, 1973. Some of this material appeared in the author's doctoral dissertation [**21**], which was supported by NSF grants GP-5913 and GP-8501. The author also acknowledges support from grant A-7354 of the National Research Council of Canada.

COROLLARY 3. *If there is a locally compact, subparacompact, perfectly normal space which is not paracompact, then there is a normal non-metrizable metacompact Moore space.*

The idea of our construction is due to Bing [**7**, p. 618] who used it for a different purpose. Let X_0 be a normal first countable space which is not collectionwise Hausdorff. Then there exist disjoint subsets D, Y of X_0, such that D is dense in X_0, and Y is the union of a discrete unseparated collection of points. For each $y \in Y$, let $\{M(y, n)\}_{n<\omega}$ be a base at y, such that for each n, $M(y, n) \cap (Y - \{y\}) = \emptyset$ and $M(y, n) \supseteq M(y, n + 1)$.

The points of the desired space X will consist of the points of Y and the members of

$$D^{\#} = \{\langle d, \{y, y'\}\rangle : d \in D, \quad y, y' \in Y, y \neq y'\}.$$

($\{y, y'\}$ is the *un*ordered pair.) Abbreviate "$\langle d, \{y, y'\}\rangle$" by "$d_{y,y'}$". Thus each $d \in D$ splits into as many points as there are non-trivial pairs in Y. The topology for $X = Y \cup D^{\#}$ is defined as follows: each $\{p\}$, $p \in D^{\#}$, is open; a base at $y \in Y$ is $\{N(y, n)\}_{n<\omega}$, where $N(y, n) = \{y\} \cup \{d_{y,y'} : d \in M(y, n)\}$. It is immediately evident that X is first countable, $D^{\#}$ is dense in X, and $\{\{y\} : y \in Y\}$ is a discrete collection in X.

Note that for arbitrary $y, z \in Y, n, k \in \omega$,

$$N(y, n) \cap N(z, k) = \emptyset \quad \text{if and only if} \quad M(y, n) \cap M(z, k) = \emptyset.$$

It follows that $\{\{y\} : y \in Y\}$ has the same separation properties in X as it has in X_0. Therefore, after making the trivial observation that $d_{y,y'}$ can be separated from $z \in Y$ by simply taking n sufficiently large so that $d \notin M(z, n)$, it can be seen that since X_0 is normal (and T_1), so is X; and since $\{\{y\} : y \in Y\}$ is not separated in X_0, it is not separated in X.

It remains to verify that X is developable and metacompact. Let $G_n = \{N(y, n) : y \in Y\} \cup \{\{p\} : p \in D^{\#}\}$. We claim that $\{G_n\}_{n<\omega}$ is a development for X. If $y \in Y$, then $\bigcup \{g \in G_n : y \in g\} = N(y, n)$. So if U is an open set about y, then for some n, $N(y, n) \subseteq U$, so $\bigcup \{g \in G_n : y \in g\} \subseteq U$. Suppose $p \in D^{\#}$, $p = d_{y,y'}$. Then

$$\bigcup \{g \in G_n : p \in g\} = \{p\} \cup \{N(z, n) : d \in M(z, n)\}.$$

Let

$$n = 1 \text{ plus the least } k \text{ such that } d \notin M(y, k) \cup M(y', k).$$

Note that $d_{y,y'}$ can only be in basic neighbourhoods of y or y'. Thus for any U about p, $\bigcup \{g \in G_n : p \in g\} = \{p\} \subseteq U$. We have shown that X is developable.

Finally, let $\mathscr{U}$ be an open cover of X. Then there is a function $f : Y \to \omega$, such that for every $y \in Y$, there is a $U \in \mathscr{U}$ such that $N(y, f(y)) \subseteq U$. Let

$$\mathscr{V} = \{\{p\} : p \in D^{\#}\} \cup \{N(y, f(y)) : y \in Y\}.$$

Then $\mathscr{V}$ is a cover refining U, and each point is in at most three members of $\mathscr{V}$. Thus X is certainly metacompact.

$D^{\#}$ is clearly open, dense, and metrizable. As pointed out by the referee, X is locally metrizable, since e.g. it is developable and locally collectionwise normal.

Recall that a Moore space is *complete* if it has a development $\{G_n\}_{n<\omega}$ such that if $\{F_i\}_{i<\omega}$ is a decreasing sequence of nonempty closed sets such that for each n, there is a $g_n \in G_n$ with $F_n \subseteq g_n$, then $\bigcap_{i<\omega} F_i \neq \emptyset$. The development $\{G_n\}_{n<\omega}$ of X in fact satisfies these conditions. Let $F_n \subseteq g_n$ as above. If some $g_n = \{p\}$, then $\bigcap_{n<\omega} F_n = \{p\}$. Observe that for arbitrary $i, j, k \in \omega$, $v, w, z \in Y$, if v, w, z are distinct, then $N(v, i) \cap N(w, j) \cap N(z, k) = \emptyset$. Thus if each $g_n = N(y_n, n)$, there exist $y, y' \in Y$ such that for every n, y_n is either y or y'. But since X is Hausdorff and the basic neighbourhoods about a member of Y decrease in size as n gets larger, for n sufficiently large and all $m > n$, the y_m's are either all y or all y'. Thus, without loss of generality, we may in fact limit ourselves to the case in which there is a y such that for each n, $F_n \subseteq N(y, n)$. If y is not in some F_n, then since F_n is closed, there is an $m > n$ such that $N(y, m) \cap F_n = \emptyset$. This is impossible since $F_m \subseteq N(y, m) \cap F_n$. Thus $y \in \bigcap_{n<\omega} F_n$, and so X is complete.

Several remarks are in order concerning this construction. First, note that the normality of X_0 is used only to obtain the normality of X. If X_0 is only assumed to be Hausdorff, X is still completely regular, since each basic open set is closed. There *are* Hausdorff first countable spaces that are not collectionwise Hausdorff, e.g. the familiar tangent disk space [**20**, p. 100]. Second, it is not difficult to verify that X is locally completely metrizable. Take the elements of a point-three cover of X by basic open sets and disjointify them. The natural map from this disjoint sum onto X expresses X as a continuous open three-to-one image of a completely metrizable space. Thus there is a "genuine" example of such a non-metrizable image.

Third, instead of using pairs of elements of Y, one could use for example unordered n-tuples for a fixed n, or all finite subsets, or all countable subsets of Y. The first variation has all the properties of X. The second, all except possibly completeness. The third, all except possibly completeness, developability, and metacompactness, yet it does have a point-countable base. By putting more restrictions on X_0, completeness can be attained in at least the second variation. See [**22**] where this maneuver yields an *absolute* G_δ space which is not *cocompact.*

It is not known whether there exist normal first countable spaces which are not collectionwise Hausdorff, but the set-theoretic consistency of their existence is known to follow from Martin's Axiom [**15**; **23**] plus $2^{\aleph_0} > \aleph_1$, since these hypotheses imply the existence of an uncountable set of reals S, such that in the subspace topology on S, every subset of S is an F_σ [**15**, p. 162]. Example E of Bing [**6**], a separable normal non-metrizable Moore space, then suffices.

https://doi.org/10.4153/CJM-1974-001-8 Published online by Cambridge University Press

Thus the consistency of the existence of normal non-metrizable metacompact Moore spaces is established. Since Martin's Axiom plus $2^{\aleph_0} > \aleph_1$ also implies Souslin's Hypothesis [**19**], the failure of the normal Moore space conjecture does not imply the failure of Souslin's Hypothesis, answering Question 6 of [**27**].

Corson and Michael [**10**, p. 353] ask whether every normal absolute G_δ space with a point-countable base is metrizable. The concepts "complete" and "absolute G_δ" coincide in completely regular Moore spaces [**11**], so a negative answer is consistent.

There has been some interest in the question of when do Moore spaces have dense metrizable subspaces. For a survey and references, see Reed [**17**]. Given a normal Moore space which is not collectionwise Hausdorff, our construction has yielded a normal non-metrizable Moore space with a dense metrizable subspace. On the other hand, Fitzpatrick [**12**, Corollary 1] proves that if a normal Moore space is collectionwise Hausdorff, it already has a dense metrizable subspace. Thus Corollary 2 is established.

In Przymusiński and Tall [**16**], it is shown that Martin's Axiom in conjunction with $2^{\aleph_0} > \aleph_1$ also implies the existence of an "uncompletable", normal non-metrizable, metacompact Moore space in which every metrizable subspace is nowhere dense. Traylor [**25**, p. 381] proves that if there is a complete, normal, non-metrizable Moore space, then there is one which is also connected, locally connected, has a dense metrizable subspace, but is not locally metrizable at any point.

In order to prove Corollary 3, we employ a method of getting nice unseparated collections apparently first used by Traylor [**26**, Theorem 6].

Definition. A property P is *weakly hereditary* if it is inherited by closed subspaces. A space is *locally* P if for each point x and each open set U containing x, there is an open V and a W having property P, such that $x \in V \subseteq W \subseteq U$.

It is well-known that paracompactness is equivalent to collectionwise normality plus either metacompactness or subparacompactness (F_σ-screenability). The standard proofs can be modified in a straightforward fashion to prove

LEMMA 4. *Suppose X is normal, metacompact or subparacompact, but not paracompact. Further assume that X is locally P, where P is weakly hereditary. Then there is a discrete unseparated collection of closed subsets of X, each having property P.*

It is also well-known that in a compact perfectly normal space, each closed set F has countable *character*, i.e. there exists a countable collection of open sets including F such that every open set including F includes one of them.

Given then a locally compact, subparacompact, perfectly normal space which is not paracompact, the Lemma assures us that there exists in that space

https://doi.org/10.4153/CJM-1974-001-8 Published online by Cambridge University Press

a discrete unseparated collection of closed sets, each included in a compact perfectly normal space (perfect normality is hereditary) and hence having countable character. Identify the members of the discrete collection to points. It is easy to verify that the resulting quotient space satisfies the hypotheses of Theorem 1, yielding Corollary 3.

One could similarly prove that if there is a locally compact, metacompact, perfectly normal space which is not paracompact, then there is a normal, non-metrizable, metacompact Moore space. However, in response to an earlier version of this note, A. V. Arhangel'skiĭ informed me that he had some years previous proved the nonexistence of the former spaces. He has now published this result as [**4**].

It is unknown whether there exist locally compact, normal, metacompact or subparacompact spaces which are not paracompact. Assuming the existence of an uncountable set of reals S, such that in the subspace topology on S, every subset of S is an F_σ, the "rational sequence topology" [**20**, p. 87] can be modified to produce a locally compact, normal, non-metrizable Moore space, *a fortiori* a locally compact, normal, subparacompact space which is not paracompact. Worrell [**28**, p. 558] has a complicated example of a locally compact Hausdorff space which is first countable, metacompact, and subparacompact. It is probably not paracompact or normal.

It remains open whether it is consistent with the axioms of set theory that every (metacompact) normal Moore space be metrizable. Some progress was made in Tall [**21**] and W. Fleissner is currently achieving promising results in this direction at the University of Wisconsin.

Added in proof. It follows from Theorem 1 and a result just announced by Fleissner that the existence of a metacompact normal non-metrizable Moore space is consistent with the continuum hypothesis and hence with the metrizability of separable normal Moore spaces.

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