

A UNIVERSAL FACTORIZATION THEOREM IN TOPOLOGY

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1. Introduction. The purpose of this paper is to prove and generalize the following theorem: Given any topological space X , of all the T_2 spaces Z which are continuous images of X , there is a maximal one Y ; that is, one over which all others factor, as in Figure 1.

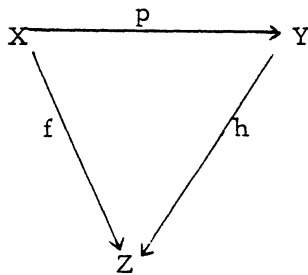


Figure 1.

In pursuit of this result, the authors define a certain species of functors and natural transformations on the category of all topological spaces and maps. A subspecies is singled out which yields the main result. As well it leads to a uniform definition of many separation axioms, and universal proofs for some of the simple properties of these axioms.

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2. Topological Equivalence Relations and Quotients.

In this section we introduce the basic machinery in two equivalent forms. The idea is similar in spirit to rewriting the Stone-Čech compactification in terms of an induced natural transformation.

Definition 1. A quotient on Top (the category of topological spaces and maps, the latter meaning continuous mappings) is a pair (F, n) where $F: \text{Top} \rightarrow \text{Top}$ is a covariant functor, $n: I \rightarrow F$ is a natural transformation of the identity functor on Top into F and n_X is onto for all X in Top.

If $Q=(F, n)$ is a quotient (on Top) we shall say that a space X (in Top) is Q -invariant when n_X is a homeomorphism.

In the following, if R is an equivalence relation defined on a space X (in Top) we denote the set of equivalence classes endowed with the quotient topology by X/R . Also, X will be called R -discrete when xRx' implies $x=x'$ for all x, x' in X .

Definition 2. An equivalence relation R which is defined on every topological space is called topological when, for any map $f: X \rightarrow Y$, xRx' implies $f(x)Rf(x')$.

PROPOSITION 1. There is a 1-1 correspondence between topological equivalence relations and quotients for which the topology on $F(X)$ is the topology induced by n_X .

Proof. Given a topological equivalence relation R , define $F: \text{Top} \rightarrow \text{Top}$ by: $F(X) = X/R$ and $F(f)$ is the map determined uniquely by the commutative diagram in Fig. 2. Here, n_X is the canonical map of X onto X/R .

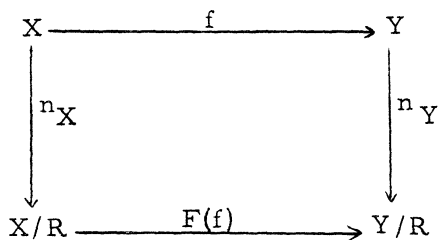


Figure 2.

Uniqueness, together with Figure 3, shows that F is a covariant functor.

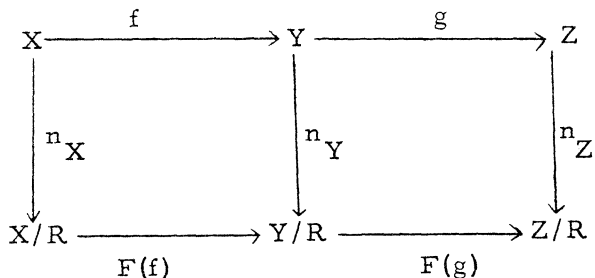


Figure 3.

Conversely, given a quotient $Q = (F, n)$, we define R on X by xRx' if and only if $n_X(x) = n_X(x')$. R is a topological equivalence relation, for, given $f: X \rightarrow Y$ and xRx' in X , then $n_Y(f(x)) = F(f) \circ n_X(x) = F(f) \circ n_X(x') = n_Y(f(x'))$. Hence $f(x)Rf(x')$.

Finally, it is easy to see that the above correspondences are the inverse of one another. This completes the proof.

COROLLARY 1. Suppose Q and R induce each other as in Proposition 1. Then X (in Top) is Q -invariant if and only if it is R -discrete.

Proof. Suppose X is Q -invariant. Then xRx' means that $n_X(x) = n_X(x')$. But n_X is a homeomorphism. Conversely, if X is R -discrete, $X = X/R = F(X)$ and the canonical map is the identity.

Definition 3. Given a topological space X , a pair (Y, g) consisting of a space Y (in Top) and a map $g: X \rightarrow Y$, is said to have the R -factorization property for X when for any R -discrete space Z together with a map $f: X \rightarrow Z$ there is a map $h: Y \rightarrow Z$ so that $f = h \circ g$.

The observation that Fig. 2 collapses to a triangle when Y is R -discrete, gives

COROLLARY 2. Given a space X and a topological equivalence relation R , then $(X/R, n_X)$ has the R -factorization property for X .

Now that we have established the relation between quotients and topological equivalence relations, we will discuss only the latter.

3. The Limit Relation. Given a topological equivalence relation R , we define a new relation $\lim R$ as follows. For X a topological space containing x and x' , we have $x(\lim R)x'$ if and only if for all R -discrete spaces Z together with maps f from X to Z , then $f(x) = f(x')$. Note that such pairs (Z, f) always exist.

PROPOSITION 2. (i) $\lim R$ is a topological equivalence relation;
(ii) $X/(\lim R)$ is R -discrete;

(iii) X is R -discrete if and only if it is $\text{lim}R$ -discrete.

Proof. (i) If $f: X \rightarrow Y$ is a map and $f(x)$ and $f(x')$ are not $\text{lim}R$ related then there is a map $g: Y \rightarrow Z$ with Z R -discrete and $g(f(x)) \not\equiv g(f(x'))$. Since $g \circ f$ is a map, x and x' are not $\text{lim}R$ related.

(ii) Suppose that x and x' in X are not $\text{lim}R$ related. Then there is a map f from X into an R -discrete space Z with $f(x) \not\equiv f(x')$. From Corollary 2 of Proposition 1 there is an h such that $f = h \circ p$, where p is the canonical map of X onto $X/(\text{lim}R)$. Thus $h(p(x)) \not\equiv h(p(x'))$, and hence $p(x)$ and $p(x')$ are not R related.

(iii) If X is R -discrete and $x(\text{lim}R)x'$, the identity map on X gives $x = x'$. Conversely, if X is $\text{lim}R$ -discrete then $X = X/(\text{lim}R)$ is R -discrete by (ii).

COROLLARY. $X/(\text{lim}R)$ is $\text{lim}R$ -discrete.

The result we mentioned in the introduction can be stated as follows:

THEOREM. Given a topological space X and a topological equivalence relation R , there is a pair (Y, p) consisting of an R -discrete space Y and a map p of X onto Y , which has the R -factorization property for X . This pair is unique up to homeomorphism.

Proof. We take $X/(\text{lim}R)$ for Y , with p the natural map. By (ii) of Proposition 2, $X/\text{lim}R$ is R -discrete. The R -factorization property is an immediate consequence of (iii) of Proposition 2 and Corollary 2 of Proposition 1. For uniqueness, if (Y, p) and (Y', p') both satisfy the conditions of the theorem we have a diagram as in Fig. 4. Since the diagram commutes and p and p' are onto, we have that (Y, p) and (Y', p') are related by a homeomorphism.

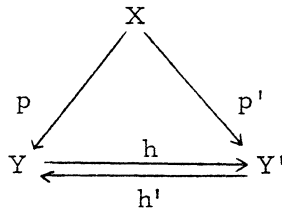


Figure 4.

An example will be given in Section 5 to show that X/R will not suffice for Y in this theorem.

4. Subspace and product theorems.

PROPOSITION 3. If $f: X \rightarrow Y$ is 1-1 and Y is R -discrete, then X is R -discrete.

Proof. If xRx' then $f(x)Rf(x')$. Hence $f(x) = f(x')$ and thus $x = x'$ since f is 1-1.

COROLLARY. A subspace of an R -discrete space is R -discrete.

PROPOSITION 4. Given a family of topological spaces Y_i , let Y denote the product space. Let R be a topological equivalence relation. Then yRy' in Y implies $y_i R y'_i$ in each Y_i . The converse holds if the family is finite.

Proof. The first part comes immediately by considering the projection maps. For the converse, define $j_i: Y_i \rightarrow Y$ for $i = 1, 2, \dots, n$ by $j_i(z) = (y_1, y_2, \dots, y_{i-1}, z, y_{i+1}, \dots, y_n)$. Now $y_i R y'_i$ implies $j_i(y_i) R j_i(y'_i)$ for $i = 1, 2, \dots, n$. The result now follows by applying the transitivity of R a finite number of times.

PROPOSITION 5. With the notation of Proposition 4, Y is R -discrete if and only if each Y_i is R -discrete. (The family need not be finite).

Proof. Proposition 4 gives the "if" part at once. Conversely, suppose Y is R -discrete and $y_i R y'_i$ in each Y_i . Define $j_i: Y_i \rightarrow Y$ by $j_i(z)(k) = y_k$ if $i \neq k$ and equal to z if $i = k$. We have $j_i(y_i) R j_i(y'_i)$ and hence $j_i(y_i) = j_i(y'_i)$. Thus we have $y_i = y'_i$. This proves the result.

5. Separation axioms. To construct topological equivalence relations corresponding to separation axioms, we make

Definition 4. An elementary topological relation is a

symmetric, reflexive relation defined on every topological space which is preserved under maps.

An elementary topological relation E_0 induces an equivalence relation E as follows: xEx' when there are points z_1, z_2, \dots, z_n with $z_1 = x$, $z_n = x'$ and $z_k E_0 z_{k+1}$ for $k = 1, 2, \dots, n-1$. The following is clear:

PROPOSITION 6. (i) E is a topological equivalence relation;
(ii) X is E_0 -discrete if and only if it is E -discrete.

Examples. The following are examples of topological equivalence relations and how they are formed.

(1) T_0 defined by: xT_0x' when every open set containing one of x, x' contains them both. In this case $R = \lim R$.

(2) T_1 induced by the elementary topological relation E_0 defined by: xE_0x' when every open set containing x contains x' or every open set containing x' contains x .

(3) P_1 induced by: xE_0x' when there is a sequence which converges to both x and x' .

(4) T_2 induced by: xE_0x' when every pair of open sets containing x and x' respectively overlap.

(5) $T_{3/2}$ defined by: $xT_{3/2}x'$ when for every map $f: X \rightarrow [0, 1]$ we have $f(x) = f(x')$.

Then, for example, a space X is a T_2 space if and only if it is T_2 -discrete in the sense defined by (4). Similar statements hold for the other examples; they can, if desired, be taken as definitions.

Next we come to the question of what separation axioms are not definable by topological equivalence relations. We shall show that T_3 and T_4 fall into this class.

First of all T_4 (normality) is not product invariant and so would contradict Proposition 5. For T_3 we will get a contradiction with Proposition 3. Let I denote $[0, 1]$ with the

usual topology. Let I' denote $[0, 1]$ with topology generated by the following subbasis: (i) all open sets of I ; (ii) the complement of K , which is the union of $[1/(2n+1), 1/(2n)]$ $n=1, 2, \dots$. Now I' is not regular (T_3) since 0 cannot be separated from the closed set K . However, we have a 1-1 map $I' \rightarrow I$. This is not compatible with Proposition 3.

Finally, we give an example where R and $\text{lim}R$ are not the same. We do this by giving an example of a space X for which X/T_2 is not Hausdorff. Let X be the union of:

$$S = \{\dots, p_{-n}, \dots, p_{-2}, p_{-1}, p_1, p_2, \dots\} \text{ and}$$

$S' = \{\dots, q_{-2}, q_{-1}, q_1, q_2, \dots\}$. The topology is generated by the following basis: (i) all subsets of S' ; (ii) complements of sets of the form $\{p_{a_1}, \dots, p_{a_r}, q_{e_1 a_1}, \dots, q_{e_r a_r}\}$ where e_i is 1 or -1 . It is readily verified that points of S belong to the same class while the subspace S' is T_2 . Also, points in S can be separated from those of S' . Thus X/T_2 is not T_2 , for the only open set containing the class S is X/T_2 .

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