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NORMED LINEAR SPACES THAT ARE UNIFORMLY CONVEX IN EVERY DIRECTION

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The concept of uniform convexity in a normed linear space is based on the geometric condition that if two members of the unit ball are far apart, then their midpoint is well inside the unit ball. We consider here a generalization of this concept whose geometric significance is that the collection of all chords of the unit ball that are parallel to a fixed direction and whose lengths are bounded below by a positive number has the property that the midpoints of the chords lie uniformly deep inside the unit ball. This notion, called uniform convexity in every direction (*UCED*), was first used by A. L. Garkavi [**5**; **6**] to characterize normed linear spaces for which every bounded subset has at most one Čebyšev center. We discuss questions of renorming spaces so as to be *UCED* and forming products of spaces that are uniformly convex in every direction. We examine the relationship of this concept with normal structure and deduce some results of Belluce, Kirk, and Steiner and of V. Zizler as corollaries of our theorems.

Definition 1. A normed linear space X is *uniformly convex in every direction* (*UCED*) if and only if, for every nonzero member z of X and $\epsilon > 0$, there exists a $\delta > 0$ such that $|\lambda| < \epsilon$ if $||x|| = ||y|| = 1$, $x - y = \lambda z$, and $||\frac{1}{2}(x + y)|| > 1 - \delta$.

The following theorem gives several properties that are equivalent to *UCED*. Property (*I*) is a rather obvious restatement of Definition 1; (*II*) is interesting in that it is not assumed that each $x_n - y_n$ is a multiple of z; and (*III*), for $p = 2$, was introduced by V. Zizler in a slightly different context (see [**10**, Proposition 1, (10)]. It might also be noted that, in (*I*) and in Definition 1, "$\leqq 1$" can be substituted for "$= 1$" in the restrictions on x_n, y_n, x, and y.

THEOREM 1. *Each of the following is a necessary and sufficient condition for a normed linear space X to be UCED.*

(*I*) *If there are sequences $\{x_n\}$ and $\{y_n\}$ and a nonzero member z of X for which*

(a) $||x_n|| = ||y_n|| = 1$, *for every n*,

(b) $x_n - y_n = \alpha_n z$, *for every n*,

(c) $||x_n + y_n|| \to 2$,

then $\alpha_n \to 0$.

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(*II*) *If there are sequences* $\{x_n\}$ *and* $\{y_n\}$ *in* X *such that*
(α) $||x_n|| \leqq 1$ *and* $||y_n|| \leqq 1$, *for every* n,
(β) $x_n - y_n \to z$,
(γ) $||x_n + y_n|| \to 2$,
then $z = 0$.
(*III-p*) *For no nonzero* z *is there a bounded sequence* $\{x_n\}$ *in* X *such that*

$$2^{p-1}(||x_n + z||^p + ||x_n||^p) - ||2x_n + z||^p \to 0.$$

(*p can be any number for which* $2 \leqq p < \infty$.)
(IV) *For each nonzero* z *in* X, *there is a positive number* Δ *such that* $||x + \frac{1}{2}z|| < 1 - \Delta$, *whenever* $||x|| \leqq 1$ *and* $||x + z|| \leqq 1$.

Proof. Suppose first that X is *UCED* and that $\{x_n\}$, $\{y_n\}$ and z satisfy (a)–(c) of (I). For $\epsilon > 0$, choose δ as described in Definition 1. Since $||\frac{1}{2}(x_n + y_n)|| > 1 - \delta$ and, therefore, $|\alpha_n| < \epsilon$ if n is large enough, it follows that $\alpha_n \to 0$. Therefore, (I) is implied by *UCED*.

(I) $\Rightarrow$ (II). Suppose that (I) is satisfied and that z and sequences $\{x_n\}$ and $\{y_n\}$ satisfy (α), (β) and (γ), but that $z \neq 0$. For each n, let θ_n be the smaller of 1 and $||x_n - z||^{-1}$. Then let

$$\xi_n = \theta_n x_n,\ \eta_n = \theta_n(x_n - z).$$

Then $0 < \theta_n \leqq 1$, and it follows from (α) and (β) that $\theta_n \to 1$. Thus,

$$||\xi_n|| \leqq 1 \text{ and } ||\eta_n|| \leqq 1, \text{ for every } n,$$

$$\xi_n - \eta_n = \theta_n z,$$

$$\lim_{n\to\infty}||\xi_n + \eta_n|| = 2.$$

For each n, let

$$u_n = \xi_n + \alpha_n z,\ v_n = \eta_n - \beta_n z,$$

where α_n and β_n are nonnegative numbers for which

$$(1) \qquad ||u_n|| = ||v_n|| = 1.$$

Then $0 \leqq \alpha_n \leqq 2/||z||$, $0 \leqq \beta_n \leqq 2/||z||$, and

$$(2) \qquad u_n - v_n = (\theta_n + \alpha_n + \beta_n)z,$$

where $\liminf_{n\to\infty}(\theta_n + \alpha_n + \beta_n) \geqq 1$. Also,

$$(3) \qquad \begin{aligned} u_n + v_n &= \xi_n + \eta_n + (\alpha_n - \beta_n)z \\ &= \theta_n x_n + \theta_n(x_n - z) + (\alpha_n - \beta_n)z \\ &= x_n + y_n + (\alpha_n - \beta_n)(x_n - y_n) \\ &\quad + [(\theta_n - 1)x_n + (\theta_n x_n - \theta_n z - y_n) + (\alpha_n - \beta_n)(z - x_n + y_n)]. \end{aligned}$$

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If $\alpha_n \geqq \beta_n$ and $||x_n + y_n|| > 2 - \Delta$, then

$$\begin{aligned} ||x_n + y_n + (\alpha_n - \beta_n)(x_n - y_n)|| &= ||(1 + \alpha_n - \beta_n)(x_n + y_n) \\ &\qquad - 2(\alpha_n - \beta_n)y_n|| \\ &> (1 + \alpha_n - \beta_n)(2 - \Delta) - 2(\alpha_n - \beta_n) \\ &= 2 - \Delta(1 + \alpha_n - \beta_n). \end{aligned} \tag{4}$$

Since the expression in brackets in (3) approaches zero, it follows from (4) and a similar inequality for the case $\alpha_n \leqq \beta_n$, that

$$\lim_{n\to\infty} ||u_n + v_n|| = 2. \tag{5}$$

We now have a contradiction, since (1), (2), and (5) imply that the space does not satisfy (I).

$(II) \Rightarrow (III\text{-}p)$. If $(III\text{-}p)$ is not satisfied, then there is a nonzero z and a bounded sequence $\{x_n\}$ such that

$$2^{p-1}(||x_n + z||^p + ||x_n||^p) - ||2x_n + z||^p \to 0. \tag{6}$$

Since $z \neq 0$, $||x_n||$ is bounded away from zero and there is no loss of generality to assuming that $||x_n|| \to 1$. It follows from the inequality

$$2^{p-1}(a^p + b^p) \geqq (a - b)^p + (a + b)^p,$$

for $p \geqq 2$ and $a \geqq b \geqq 0$, that

$$\begin{aligned} 2^{p-1}(||x_n + z||^p + ||x_n||^p) - ||2x_n + z||^p &\geqq 2^{p-1}(||x_n + z||^p + ||x_n||^p) \\ &\qquad - (||x_n + z|| + ||x_n||)^p \\ &\geqq (||x_n + z|| - ||x_n||)^p, \end{aligned}$$

so $||x_n + z|| - ||x_n|| \to 0$ and $||x_n + z|| \to 1$. It then follows from (6) that $||2x_n + z|| \to 2$. If $\xi_n = x_n/||x_n||$ and $\eta_n = (x_n + z)/||x_n + z||$, then z and the sequences $\{\xi_n\}$ and $\{\eta_n\}$ satisfy $(\alpha) - (\gamma)$, so $z = 0$.

$(III\text{-}p) \Rightarrow (IV)$. It follows from $(III\text{-}p)$ that, for each nonzero z, there is a number Δ such that $0 < \Delta < \frac{1}{2}$ and

$$2^{p-1}(||x + z||^p + ||x||^p) - ||2x + z||^p > 2^p p\Delta, \quad \text{if } ||x|| \leqq 1.$$

If $||x|| \leqq 1$ and $||x + z|| \leqq 1$, this implies that

$$||2x + z||^p < 2^p - 2^p p\Delta,$$

so

$$||x + \tfrac{1}{2}z|| < (1 - p\Delta)^{1/p} < 1 - \Delta.$$

To complete the proof of Theorem 1, we need to show that (IV) implies $UCED$. Suppose that $z \neq 0$ and that $\epsilon > 0$. Use (IV) to obtain a positive number Δ such that

$$||\xi + \tfrac{1}{2}\epsilon z|| < 1 - \Delta, \text{ if } ||\xi|| \leqq 1 \text{ and } ||\xi + \epsilon z|| \leqq 1.$$

Let x and y satisfy $||x|| = 1$, $||y|| = 1$, and $x - y = \lambda z$. If $|\lambda| \geqq \epsilon$, let $\xi = -(\text{sign } \lambda)x$. Then $||\xi|| = 1$ and $||\xi + |\lambda|z|| = ||y|| = 1$, so

$$||\xi + \epsilon z|| = \left\| \left(1 - \left|\frac{\epsilon}{\lambda}\right|\right)\xi + \left|\frac{\epsilon}{\lambda}\right| (\xi + |\lambda|z) \right\| \leqq 1,$$

and $||\xi + \frac{1}{2}\epsilon z|| < 1 - \Delta$. Therefore,

$$\begin{aligned} ||\tfrac{1}{2}(x + y)|| &= ||x - \tfrac{1}{2}\lambda z|| \\ &= ||\xi + \tfrac{1}{2}|\lambda|z|| \\ &\leqq \frac{|\lambda|}{2|\lambda| - \epsilon} ||\xi + \tfrac{1}{2}\epsilon z|| + \frac{|\lambda| - \epsilon}{2|\lambda| - \epsilon} ||\xi + |\lambda|z|| \\ &< \frac{|\lambda|(1 - \Delta)}{2|\lambda| - \epsilon} + \frac{|\lambda| - \epsilon}{2|\lambda| - \epsilon} \\ &= \frac{(2 - \Delta)|\lambda| - \epsilon}{2|\lambda| - \epsilon}. \end{aligned}$$

This and $\Delta > 0$ imply that $||\frac{1}{2}(x + y)|| < 1 - \frac{1}{2}\Delta$, and we can let δ in Definition 1 be $\frac{1}{2}\Delta$.

If X is *UCED*, then X is strictly convex. The converse is not true. For example, the space $C[0, 1]$ of all real continuous functions on the unit interval with the norm

$$||f|| = \sup \{|f(t)|\} + \left(\int_0^1 |f(t)|^2 dt\right)^{\frac{1}{2}}$$

is strictly convex, but this space is not *UCED* [**6**, pp. 126–127]. For a set, Γ, $C_0(\Gamma)$ is the space of all functions on Γ such that, for each $\epsilon > 0$, $|f(t)| < \epsilon$, for all but finitely many values of t, with $||f|| = \max \{|f(t)|: t \in \Gamma\}$. For all Γ the space $C_0(\Gamma)$ can be renormed so as to be strictly convex [**4**, Theorem 10, p. 523], but if Γ is not countable, then $C_0(\Gamma)$ can not be renormed so as to be *UCED*.

THEOREM 2. *If $X = C_0(\Gamma)$ and Γ is uncountable, then X is not isomorphic to a space that is UCED.*

Proof. Suppose that X has been given an equivalent norm $|||\ |||$ with respect to which X is *UCED*. Let

$$\mu = \sup \{|||x|||: ||x|| \leqq 1 \text{ and } x \in X\}.$$

Let $\{u_n\}$ be a sequence for which $||u_n|| = 1$ and $|||u_n||| \to \mu$. Since Γ is uncountable, there is a nonzero z such that

$$z(t)u_n(t) = 0,$$

for all n and for all t in Γ. For $0 < \epsilon < 1$ and $\delta > 0$, choose u_n for which

$|||\, u_n \,||| > \mu(1 - \delta)$. Then, if $x = u_n + \frac{1}{2}z$ and $y = u_n - \frac{1}{2}z$, we have

$$|||\, x \,||| \leqq \mu, \; |||\, y \,||| \leqq \mu, \; |||\, \tfrac{1}{2}(x + y) \,||| > \mu(1 - \delta),$$

but $x - y = z$.

As follows from the next two theorems, there are many spaces—even reflexive Banach spaces—that are *UCED*, but are not isomorphic to a uniformly convex space. However, it is not known whether every reflexive space can be renormed so as to be *UCED*.

It has been shown by V. Zizler [**10**, Proposition 14] that X can be renormed so as to be *UCED* if there is a continuous one-to-one linear map T of X into a space Y that is *UCED*. The argument is easy, the new norm being given by

$$|||\, x \,||| = (||x||^2 + ||Tx||^2)^{\frac{1}{2}}.$$

Use is then made of (*III*-2) of Theorem 1, assuming that there is a $z \neq 0$ and a bounded sequence $\{x_n\}$ in X for which

$$2(|||\, x_n + z \,|||^2 + |||\, x_n \,|||^2) - |||\, 2x_n + z \,|||^2 \to 0.$$

It then follows that $\{Tx_n\}$ is a bounded sequence in Y for which

$$2(||Tx_n + Tz||^2 + ||Tx_n||^2) - ||2Tx_n + Tz||^2 \to 0,$$

so that Y is not *UCED*. Zizler then used this argument in [**10**] to establish (a) of the next theorem.

THEOREM 3. *A normed linear space X is isomorphic to a space that is UCED if any one of the following conditions is satisfied.*

(a) *B^* contains a countable set total over B (e.g., if B is separable, or if B is the conjugate of a separable space).*

(b) *B is $l_1(\Gamma)$, for any set Γ.*

(c) *B is $L_\infty(\mu)$, for a σ-finite measure μ.*

Proof. In view of the preceding paragraph, it is sufficient to show, in each case, that there is a continuous one-to-one linear map of B into a Hilbert space. As noted by V. Zizler [**10**, Proposition 14, Corollary], the map for (a) can be given by $Tx = \{f_i(x)/2^i\}$, where T maps B into l_2 and $\{f_i\}$ is total over B with $||f_i|| = 1$. For (b), we can use the identity map of $l_1(\Gamma)$ into $l_2(\Gamma)$. For (c), the identity map of $L_\infty(\mu)$ into $L_2(\mu)$ can be used, if μ is a finite measure, while the identity need only be weighted on a countable number of subsets of finite measure in the general case.

The next theorem uses the concept of uniform non-squareness, originally introduced in [**7**]. A normed linear space X is uniformly non-square if there is a positive number δ such that there do not exist x and y in X for which $||x|| \leqq 1$, $||y|| \leqq 1$, $||\frac{1}{2}(x + y)|| > 1 - \delta$, and $||\frac{1}{2}(x - y)|| > 1 - \delta$. A uniformly convex space is uniformly non-square, but it is not known whether uniform non-squareness and uniform convexity are isomorphically equivalent.

THEOREM 4. *Let $B = \Pi X_\alpha$, where $\alpha \in \Gamma$ and each X_α is UCED. Then B is* *isomorphic to a space that is UCED if the product norm has either of the following properties.*

(i) *There is a positive function ϕ on Γ and a p, with $1 \leqq p < \infty$, such that*

$$||\{x(\alpha)\}||^p \geqq \sum \phi(\alpha)||x(\alpha)||^p,$$

for all $x = \{x(\alpha)\}$ belonging to B.

(ii) *There is a uniformly non-square normed linear space Y of real-valued functions on Γ for which the set of unit vectors $\{e_\alpha\}$ is an unconditional basis and*

$$\{||x(\alpha)||\} \in Y, \text{ if } \{x(\alpha)\} \in B.$$

Proof. If condition (*ii*) is satisfied, then [**8**] there is a p, with $1 < p < \infty$, and a number k such that

$$||\{x(\alpha)\}|| = ||\sum ||x(\alpha)||e_\alpha|| \geqq k(\sum ||x(\alpha)||^p)^{1/p}.$$

Therefore, condition (*i*) is satisfied. If $p < r$, then

$$[\sum \phi(\alpha)||x(\alpha)||^p]^{1/p} \geqq [\sum \phi(\alpha)^{r/p}||x(\alpha)||^r]^{1/r},$$

so there is no loss of generality in assuming for (*i*) that $p \geqq 2$.

Let B' be the same linear space as B, but with norm defined by

$$||| x(\alpha) ||| = [\sum \phi(\alpha)||x(\alpha)||^p]^{1/p}.$$

If B' is not *UCED*, then it follows from (*III*) of Theorem 1 that there is a nonzero z and a bounded sequence $\{x_n\}$ in B' for which

$$(7) \qquad 2^{p-1}(|||\, x_n + z \,|||^p + |||\, x_n \,|||^p) - |||\, 2x_n + z \,|||^p \to 0.$$

Choose α for which $z(\alpha)$ is nonzero. Since each term is nonnegative when (7) is written as a sum over Γ, it follows that

$$2^{p-1}(||x_n(\alpha) + z(\alpha)||^p + ||x_n(\alpha)||^p) - ||2x_n(\alpha) + z(\alpha)||^p \to 0.$$

Since $z(\alpha) \neq 0$ and $\{x_n(\alpha)\}$ is bounded, this contradicts the fact that X_α is *UCED*. We know that the identity map of B onto B' is continuous and that B' is *UCED*. Therefore, B is isomorphic to a space that is *UCED*.

COROLLARY. *The space B is isomorphic to a space that is UCED if any of the following conditions is satisfied.*

(i) *B is a countable product of spaces that are UCED.*

(ii) *B is uniformly non-square and B has an unconditional* (*not necessarily countable*) *basis* (see [**8**, Theorem 5]).

(iii) $1 \leqq p < \infty$, *and B is an $l_p[\Gamma]$ product of spaces that are UCED.*

Although *UCED* is inherited by all subspaces, it need not be inherited by a factor space of a *UCED* space. For each Banach space B and each dense subset S of its unit sphere, there is a continuous linear map from $l_1(S)$ onto B, defined by

$$Tx = \sum_{s \in S} x(s)s.$$

Hence, B is isomorphic to $l_1(S)/T^{-1}(0)$. If I is uncountable, then $m(I)$ has no strictly convex norm [**4**, Corollary, p. 521]. But if S is dense in the unit sphere of $m(I)$, then $m(I)$ is isomorphic to a factor space of $l_1(S)$. There is an isomorphic norm that makes $l_1(S)$ *UCED*, but the factor space isomorphic to $m(I)$ cannot even be strictly convex.

The Čebyšev centers of a subset A of a normed linear space X are those members x_0 of X such that

$$\sup_{y\in A}\|x_0 - y\| = \inf_{x\in X} \sup_{y\in A}\|x - y\|.$$

In order that each bounded set A in a normed linear space X have at most one Čebyšev center, it is necessary and sufficient that X be *UCED* [**6**, pp. 124–125]. Thus, the preceding theorems describe a large class of spaces that can be renormed so as to have at most one Čebyšev center.

We shall now recall the definitions of normal structure and complete normal structure and relate them to *UCED*.

Definition 2. For a bounded subset S of a normed linear space, a *diametral point* is a member s of S such that

$$\sup \{\|s - x\|: x \in S\} = \text{diameter of } S.$$

A convex subset K of a normed linear space has *normal structure* if and only if for each bounded convex subset W of K which contains more than one point there is a member x of W that is not a diametral point of W (see [**3**]).

For bounded subsets H and S of a normed linear space, let

$$\begin{aligned} r_s(H) &= \sup\{\|s - x\|: x \in H\},\\ r(H, S) &= \inf\{r_s(H): s \in S\},\\ \mathscr{C}(H, S) &= \{s: s \in S \text{ and } r_s(H) = r(H, S)\}. \end{aligned}$$

The members of $\mathscr{C}(H, S)$ are the *Čebyšev centers of H in S.*

Definition 3. Let K be a closed convex subset of a Banach space X. Then K has *complete normal structure* if and only if each bounded closed convex subset W of K has the property that the closure of $\bigcup_{\alpha\in A} \mathscr{C}(W_\alpha, W)$ is a nonempty proper subset of W whenever $\{W_\alpha: \alpha \in A\}$ is a decreasing net of subsets of W such that $r(W_\alpha, W) = r(W, W)$, for each α [**1**, p. 475].

THEOREM 5. *Let X be a normed linear space which is UCED and let H be a nonempty bounded subset of a convex subset S of X. Then $\mathscr{C}(H, S)$ has at most one member.*

Proof. Let H be a nonempty bounded subset of a convex subset S of a normed linear space X. Suppose that $\mathscr{C}(H, S)$ contains s_1 and s_2 with $s_1 \neq s_2$. If $x \in H$, then

$$\begin{aligned} \|s_1 - x\| &\leqq r_{s_1}(H) = r(H, S),\\ \|s_2 - x\| &\leqq r_{s_2}(H) = r(H, S). \end{aligned}$$

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Let $\sigma = \frac{1}{2}(s_1 + s_2)$. Uniform convexity in the direction $s_1 - s_2$ implies the existence of a positive number δ such that

$$\|\sigma - x\| \leqq (1 - \delta)r(H, S), \tag{8}$$

where δ does not depend on x. Since (8) is true for all x in H, we have

$$r_\sigma(H) < r(H, S).$$

Since $r(H, S) = r_s(H)$, for all s in S, it follows that $s_1 = s_2$, and that $\mathscr{C}(H, S)$ has at most one member.

By letting W of Definition 2 be both H and S of Theorem 5, we can obtain the following corollaries. Corollary 3 also implies that a normed linear space has normal structure if it is uniformly convex.

COROLLARY 3. *A normed linear space has normal structure if it is UCED* (*see* [**10**, *Proposition* 23]).

It is known that, if H is bounded and if S is weakly compact and convex, then $\mathscr{C}(H, S)$ is nonempty [**1**, Lemma p. 475]. This and Theorem 4 imply:

COROLLARY 4. *A reflexive Banach space has complete normal structure if it is UCED.*

Corollary 4 implies the theorem of Belluce and Kirk which states that K has complete normal structure if K is a bounded closed convex subset of a uniformly convex Banach space [**1**, Theorem 4.1, p. 477].

The following corollary was proved by Belluce, Kirk, and Steiner [**2**, Theorem 3.1, p. 437]. It follows easily from Theorem 4, since any space that is *UCED* is strictly convex and has normal structure.

COROLLARY 5. (Belluce, Kirk, and Steiner). *There exists a Banach space which is reflexive, strictly convex, and which possesses normal structure, but which is not isomorphic to any uniformly convex Banach space.*

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