Planetary waves in a
rotating global ocean

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A theoretical study is made of the free periods of oscillation of
a global layer of inviscid, incompressible fluid under the action
of gravitational and Coriolis forces. The leading approximations
to the eigenvalues are found to be sensitive to variations in the
Froude number and also to the shape of the globe. It is shown
that for oceanic and atmospheric motions displaying essentially
the same features as the model, it is not sufficient to consider
the motion as horizontally non-divergent.

1.   Introduction

Stewartson and Rickard [5] analysed the free oscillations of an
incompressible, inviscid fluid contained between two rotating, rigid
concentric spherical shells. An extension to the case of rigid spheroidal
boundaries was given by Rickard [4].

It is the purpose of this paper to extend the analysis to the
corresponding 'free-surface' problem; that is, to consider the slow
periodic oscillations of a layer of fluid under the action of gravitational
and Coriolis forces, when the fluid is assumed to cover the whole surface
of the rotating globe. In particular, we wish to show that, for
atmospheric and oceanic motions based on this model, it is not sufficient
to consider the motion as non-divergent as have many other models display-
ing essentially the same features.

The problem is formulated for the case of an arbitrary shaped

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spheroidal globe, but only the special case when the globe is spherical is
considered in detail. Other cases will be treated in a later paper.

The manner in which the eigenvalues vary as functions of the Froude
number, F , suggests that surface topography is likely to play an
important role in any studies of the free periods of oscillation of a
rotating fluid layer. This point will also be covered in further detail in
a later paper.

2.   Formulation of the problem

Consider a thin layer of incompressible, inviscid fluid bounded
internally by the rigid spheroid defined by (2.7), and whose outer surface
is free, which is rotating as if rigid with angular velocity Ω about an
axis Oz where O is the centre of the spheroid. We shall assume that
the acceleration due to gravity, g , is constant and purely radial. A
small disturbance is given to the steady motion and we wish to determine
the periods of free oscillation of the subsequent motion of the fluid. Let
(R, θ, ϕ) denote spherical polar coordinates in which the line $\theta = 0$
coincides with the axis of rotation Oz , and let $\left(u_R, u_\theta, u_\phi\right)$ denote the
corresponding components of the velocity $\mathbf{u}$, measured relative to a set of
axes rotating about Oz with angular velocity Ω .

The equation of continuity is

(2.1) $\operatorname{div} \mathbf{u} = 0$,

and, neglecting squares and products of $\mathbf{u}$, the equation of momentum
reduces to

(2.2) $\dfrac{\partial \mathbf{u}}{\partial t} + 2\Omega \times \mathbf{u} = -\operatorname{grad} \bar{p}$,

where

(2.3) $\bar{p} = \dfrac{p}{\rho} - \tfrac{1}{2}\Omega^2 R^2 \sin^2\theta + gR$,

p is the pressure and ρ the density of the fluid. It may easily be
verified that these equations are separable in ϕ and t : indeed if Q
is one of the dependent variables u_R, u_θ, u_ϕ or $\bar{p}$, then Q may be
expressed in the form

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(2.4) $Q = R\ell\{q(R,\ \theta)e^{im\phi+i\omega^* t}\}$,

where m is an integer, which may be either positive or negative, ω^* is
a constant to be found, and $q(R,\ \theta)$ is a function of R and θ only.
From now on when the exponential factors are omitted it will be understood
that the real part is to be taken. Further let us write

$$\mu = \cos\theta \ , \quad \omega^* = \Omega\omega \ , \qquad u_\theta = U \ ,$$

(2.5)

$$u_\phi = iV \ , \qquad u_R = \frac{W^*}{\left(1-\mu^2\right)^{\frac{1}{2}}} \ , \quad \bar{p} = (i\Omega R\sin\theta)P \ .$$

The governing equations then reduce to

(2.6a) $$\frac{\omega W^*}{1-\mu^2} - 2V = -P - R\ \frac{\partial P}{\partial R} \ ,$$

(2.6b) $$\omega U - 2\mu V = \left(1-\mu^2\right)\frac{\partial P}{\partial\mu} - \mu P \ ,$$

(2.6c) $$\omega V - 2\mu U = -mP + 2W^* \ ,$$

and the equation of continuity to

(2.6d) $$R\ \frac{\partial W^*}{\partial R} + 2W^* - \left(1-\mu^2\right)\frac{\partial U}{\partial\mu} + \mu U - mV = 0 \ .$$

We shall now suppose that the inner rigid spheroidal boundary is
defined by

(2.7) $$R = b + \tfrac{1}{4}(a+b)k_1\left(1-\mu^2\right) \ ,$$

where a, b, and k_1 are constants. The boundary condition to be
satisfied on this rigid boundary is that the normal component of velocity
should vanish, or equivalently

(2.8) $$\frac{D}{Dt}\left[R-b-\tfrac{1}{4}(a+b)k_1\left(1-\mu^2\right)\right] = 0 \ ,$$

where $R = b + \tfrac{1}{4}(a+b)k_1\left(1-\mu^2\right)$, where D/Dt denotes differentiation
following the motion of the fluid. On using (2.5), (2.8) reduces to

(2.9) $$W^* = \frac{\tfrac{1}{2}k_1(a+b)\mu\left(1-\mu^2\right)U}{b+\tfrac{1}{4}k_1(a+b)\left(1-\mu^2\right)} \ ,$$

where W^*, U are to be evaluated at $R = b + \frac{1}{4}(a+b)(1-\mu^2)$.

Let us now define a non-dimensional Froude number

$$(2.10) \qquad\qquad F_R = \frac{2a^2\Omega^2}{(a+b)g}$$

and the free surface by

$$(2.11) \qquad\qquad R = a + \frac{1}{4}(a+b)F_R(1-\mu^2) + \eta(\theta, \phi, t) \ ,$$

where $\eta(\theta, \phi, t)$ is some function of θ, ϕ, t to be determined, and squares and products of η will be neglected.

For the Earth F_R is approximately 4×10^{-3} and we shall assume here that $F_R \ll 1$.

The kinematic and dynamic boundary conditions at the free surface are

$$(2.12) \qquad\qquad \frac{D}{Dt}\left[R-a-\frac{1}{4}(a+b)F_R(1-\mu^2)-\eta(\theta, \phi, t)\right] = 0 \ ,$$

$$(2.13) \qquad\qquad\qquad \text{pressure} = p = \text{constant.}$$

On neglecting squares and products of η, U, V, W^*, P , (2.12) reduces to

$$(2.14) \qquad\qquad \frac{\partial\eta}{\partial t} = \frac{W^*}{(1-\mu^2)^{\frac{1}{2}}} - \frac{\frac{1}{2}\mu(1-\mu^2)^{\frac{1}{2}}(a+b)F_R U}{a+\frac{1}{4}(a+b)F_R(1-\mu^2)} \ ,$$

and on substituting (2.13) into (2.3) and using Taylor's Theorem to expand $P\left[a+\frac{1}{4}(a+b)F_R(1-\mu^2)+\eta, \ \mu\right]$ in ascending powers of η we obtain a second condition on the free surface in terms of η , which, on differentiation with respect to t gives

$$(2.15) \quad \frac{\partial\eta}{\partial t}\left[1 - \frac{a+b}{2a}(1-\mu^2)F_R-\frac{1}{2}\left(\frac{a+b}{2a}\right)^2 F_R^2(1-\mu^2)^2\right]$$
$$= -\omega F_R(1-\mu^2)^{\frac{1}{2}}\frac{a+b}{2a}\left[a+\frac{1}{4}(a+b)F_R(1-\mu^2)\right]P\left[a+\frac{1}{4}(a+b)F_R(1-\mu^2), \ \mu\right] \ .$$

If we now write

$$(2.16) \qquad \varepsilon = \frac{a-b}{a+b} \ , \quad R = \frac{1}{2}(a+b)(1+\varepsilon\zeta) \ , \quad W^* = \varepsilon W \ , \quad F_R = \varepsilon F \ ,$$

then (2.14), (2.15) may be written

(2.17)
$$\frac{\partial \eta}{\partial t} = \left[\frac{\varepsilon W}{\left(1-\mu^2\right)^{\frac{1}{2}}} - \frac{\mu\left(1-\mu^2\right)^{\frac{1}{2}}\varepsilon FU}{1+\varepsilon+\frac{1}{2}\varepsilon F\left(1-\mu^2\right)} \right] \ ,$$

(2.18) $\frac{\partial \eta}{\partial t} \left[\left(1+\varepsilon\right)^2 - \varepsilon(1+\varepsilon)\left(1-\mu^2\right)F - \frac{1}{2}\varepsilon^2\left(1-\mu^2\right)^2 F^2 \right]$

$$= -\omega\varepsilon\left(1-\mu^2\right)^{\frac{1}{2}} F\left(1+\varepsilon+\frac{1}{2}\varepsilon F\left(1-\mu^2\right)\right)P \ ,$$

respectively, where U, W, P are evaluated at

$$R = a + \frac{1}{4}(a+b)F_R\left(1-\mu^2\right) \ ,$$

or

(2.19)
$$\zeta = 1 + \frac{1}{2}F\left(1-\mu^2\right) \ .$$

As a consequence of (2.16), the boundary condition on the inner rigid spheroidal boundary, (2.9), reduces to

(2.20)
$$W\left(1-\varepsilon+\frac{1}{2}\varepsilon K_1\left(1-\mu^2\right)\right) = \mu U\left(1-\mu^2\right)K_1 \ ,$$

where

(2.21)
$$\varepsilon K_1 = k_1 \ ,$$

and U, W are evaluated at $\zeta = -1 + \frac{1}{2}K_1\left(1-\mu^2\right)$.

The free surface boundary conditions (2.17), (2.18) may be combined to give the single boundary condition

(2.22) $\left[\mu\left(1-\mu^2\right)FU - \left(1-\varepsilon+\frac{1}{2}\varepsilon F\left(1-\mu^2\right)\right)W\right]\left[\left(1+\varepsilon\right)^2 - \varepsilon(1+\varepsilon)\left(1-\mu^2\right)F - \frac{1}{2}\varepsilon^2\left(1-\mu^2\right)^2 F^2\right]$

$$= \omega F\left(1-\mu^2\right)\left[1+\varepsilon+\frac{1}{2}\varepsilon F\left(1-\mu^2\right)\right]^2 P \ ,$$

where U, W, P are evaluated at $\zeta = 1 + \frac{1}{2}F\left(1-\mu^2\right)$.

Following Stewartson and Rickard [5], Rickard [4], we now attempt to set up an analytic expansion procedure with the aim of finding some properties of free oscillations in fluid layers of finite depth. We shall assume expansions of the form

(2.23a)
$$U = U_1(\mu, \ \zeta) + \varepsilon U_2(\mu, \ \zeta) + \dots \ ,$$

(2.23b)
$$V = V_1(\mu, \ \zeta) + \varepsilon V_2(\mu, \ \zeta) + \dots \ ,$$

(2.23c) $W = W_2(\mu, \zeta) + \epsilon W_3(\mu, \zeta) + \dots ,$

(2.23d) $P = \chi_1(\mu, \zeta) + \epsilon \chi_2(\mu, \zeta) + \dots ,$

(2.23e) $\omega = \omega_1 + \epsilon \omega_2 + \dots .$

On substituting these expansions into (2.6), bearing in mind (2.16), and comparing coefficients of equal powers of ϵ , we find, from (2.6a),

$$(2.24) \qquad\qquad \frac{\partial \chi_1}{\partial \zeta} = 0 ,$$

so that χ_1 is independent of ζ . It follows immediately from (2.6b), (2.6c), that U_1, V_1 are independent of ζ ; in fact they are given in terms of χ_1 by

$$(2.25) \qquad \left(\omega_1^2 - 4\mu^2\right) U_1 = \omega_1\left(1 - \mu^2\right) \frac{d\chi_1}{d\mu} - \mu\left(\omega_1 + 2m\right)\chi_1 ,$$

$$(2.26) \qquad \left(\omega_1^2 - 4\mu^2\right) V_1 = 2\mu\left(1 - \mu^2\right) \frac{d\chi_1}{d\mu} - \left(m\omega_1 + 2\mu^2\right)\chi_1 .$$

Continuing the expansion formally we find on comparing coefficients of ϵ , in (2.6a), that

$$(2.27) \qquad\qquad \chi_2(\mu, \zeta) = \left(2V_1 - \chi_1\right)\zeta + \bar{\chi}_2(\mu) ,$$

where $\bar{\chi}_2(\mu)$ is at present an arbitrary function of μ . From (2.6d) it follows that $W_2(\mu, \zeta)$ is given by

$$(2.28) \qquad W_2(\mu, \zeta) = \left\{\left(1 - \mu^2\right) \frac{dU_1}{d\mu} - \mu U_1 + m V_1\right\}\zeta + \bar{W}_2(\mu) ,$$

where $\bar{W}_2(\mu)$ is at present also an arbitrary function of μ . The boundary conditions (2.20), (2.22) give

$$(2.29) \qquad\qquad W_2 = \mu\left(1 - \mu^2\right) K_1 U_1 ,$$

when

$$\zeta = -1 + \tfrac{1}{2} K_1\left(1 - \mu^2\right) ,$$

and

(2.30) $$W_2 = (1-\mu^2)F[\mu U_1 - \omega_1 \chi_1] \ ,$$

when

$$\zeta = 1 + \tfrac{1}{2}F(1-\mu^2) \ .$$

From equations (2.28), (2.29), it follows that

(2.31) $\overline{W}_2(\mu) = \mu(1-\mu^2)K_1 U_1 + \left[(1-\mu^2)\dfrac{dU_1}{d\mu} - \mu U_1 + mV_1\right]\left[1-\tfrac{1}{2}K_1(1-\mu^2)\right] \ .$

On now substituting (2.31) into (2.28) and using (2.30) it follows that

(2.32) $\left[(1-\mu^2)\dfrac{dU_1}{d\mu} - \mu U_1 + mV_1\right](2+\tfrac{1}{2}F(1-\gamma)(1-\mu^2))$

$$= \mu(1-\gamma)(1-\mu^2)FU_1 - (1-\mu^2)\omega_1 F\chi_1 \ ,$$

where

(2.33) $$\gamma = \frac{K_1}{F} \ .$$

On substituting for U_1, V_1 from (2.25), (2.26), equation (2.32) reduces to

(2.34) $\left[2+\tfrac{1}{2}F(1-\gamma)(1-\mu^2)\right]\Bigg\{\omega_1(1-\mu^2)^2\left[\omega_1^2-4\mu^2\right]\dfrac{d^2\chi_1}{d\mu^2} - 4\mu\omega_1(1-\mu^2)\left[\omega_1^2-2-2\mu^2\right]\dfrac{d\chi_1}{d\mu}$

$$- \left[(\omega_1+2m)(1-\mu^2)\left[\omega_1^2+4\mu^2\right]+\omega_1(m^2-\mu^2)\left[\omega_1^2-4\mu^2\right]\right]\chi_1\Bigg\}$$

$$= (1-\mu^2)F\left\{\mu\left[\omega_1^2-4\mu^2\right](1-\gamma)\left[\omega_1(1-\mu^2)\dfrac{d\chi_1}{d\mu} - \mu(\omega_1+2m)\chi_1\right]-\omega_1\left[\omega_1^2-4\mu^2\right]^2\chi_1\right\} \ .$$

It is clear that there exist no solutions of (2.34) other than the trivial one $\chi_1 = 0$, for which $\omega_1 = 0$; that is

(2.35) $$\omega_1 \neq 0 \quad \text{for any value of } F \ .$$

Further, if we assume that the ω_1 are continuous functions of F , then since all ω_1 are positive when $F = 0$ (see Stewartson and Rickard [5]), it follows from (2.35) that

(2.36) $\omega_1 > 0$ for all F .

From equations (2.7), (2.11), it follows that

(2.37) $R_0 - R_{i} = (a-b)\left[1+\tfrac{1}{4}F(1-\gamma)\left(1-\mu^2\right)\right] + \eta(\theta,\,\phi,\,t)$,

where R_0, R_{i} denote the values of R on the outer and inner spheroids
respectively for the same value of μ . We note that the equilibrium polar
fluid depth is $(a-b)$. Further since $\eta(\theta,\,\phi,\,t)$ is periodic in ϕ it
follows that we have regions of 'dry ocean' when

(2.38) $F(1-\gamma) \le -4$.

     At this point it is worth noting that for an ocean of polar
equilibrium depth h , that is, $a - b = h$, F is approximately $27.5h^{-1}$.
For the Pacific Ocean an average depth of 12,000 feet is usually
considered, and this gives $F \simeq 12$. Longuet—Higgins [2] considers the
case $F \ll 1$ for a hemispherical annulus of fluid.

     We now seek the eigenvalues of (2.34), that is, the values of ω_1 for
which χ_1 is bounded everywhere $(|\mu| \le 1)$, since p is bounded. As in
Stewartson and Rickard [5] and Rickard [4] we shall restrict our discussion
to the case $m = 1$, the generalisation to arbitrary m being straight-
forward.

     We shall now consider the special case $\gamma = 0$, that is, when the
rigid spheroidal boundary reduces to the sphere $R = b$. Additional
special cases will be considered in a subsequent paper.

3.   The special case $\gamma = 0$

     When $\gamma = 0$, $K_1 = 0$ $\left(\text{see (2.21), (2.33)}\right)$ and the inner spheroidal
boundary reduces to the sphere $R = b$ $(\zeta = -1)$. The boundary condition on
the sphere is now simply $W = 0$ on $\zeta = -1$. When $\gamma = 0$, on putting
$m = 1$, (2.34) reduces to

$$(3.1) \quad \left[2+\tfrac{1}{6}F\left(1-\mu^2\right)\right]\left\{\omega_1\left(1-\mu^2\right)\left(\omega_1^2-4\mu^2\right)\frac{d^2\chi_1}{d\mu^2} - 4\mu\omega_1\left(\omega_1^2-2-2\mu^2\right)\frac{d\chi_1}{d\mu}\right.$$

$$\left. - 2\left(\omega_1^3+\omega_1^2+4\mu^2\right)\chi_1\right\}$$

$$= F\left\{\omega_1\mu\left(1-\mu^2\right)\left(\omega_1^2-4\mu^2\right)\frac{d\chi_1}{d\mu} - \left(\omega_1^2-4\mu^2\right)\left(\omega_1^3+2\mu^2-3\omega_1\mu^2\right)\chi_1\right\} \ .$$

It is clear that one solution of (3.1) is given by

$$(3.2) \qquad\qquad \chi_1(\mu) = \mu \ , \quad \omega_1 = 1 \ ,$$

which is true for all values of F . If we assume that the ω_1 are
discrete for a given F , it follows from (2.35), (2.36), (3.2), that

$$(3.3) \qquad\qquad 0 < \omega_1 \leq 1 \quad \text{for all} \quad F \ .$$

Further, from (2.38) it follows that we have regions of 'dry ocean' when
$F \leq -4$. However, it must be remembered that physically the parameter F
is restricted to positive values $\big($see (2.10), (2.16)$\big)$ and hence in this
case, when $\gamma = 0$, there will be no regions of the globe which are not
water covered. Nevertheless the determination of the eigenvalues of (3.1)
poses an interesting mathematical problem for all F , and in the following
discussion we include $F < 0$. (Our analysis breaks down in the vicinity
of $F = -4$, as expected; see analogous discussion in Rickard [4].)

We shall now consider the solutions and eigenvalues of (3.1).

4. Solution of (3.1) for small values of F

When the parameter F is small, χ_1, ω_1 can be expressed as power
series in F of the form

$$(4.1) \qquad\qquad \chi_1(\mu) = \bar{\chi}_0(\mu) + F\bar{\chi}_1(\mu) + F^2\bar{\chi}_2(\mu) + \ldots \ ,$$

$$(4.2) \qquad\qquad \omega_1 = \bar{\omega}_0 + F\bar{\omega}_1 + F^2\bar{\omega}_2 + \ldots \ .$$

On substituting (4.1), (4.2) into (3.1), and equating like powers of
F we can calculate successively better approximations to the solution and

the eigenvalues ω_1 . Considering the zeroth order terms in F , we have

$$(4.3) \quad \bar{\omega}_0\left(1-\mu^2\right)\left(\bar{\omega}_0^2-4\mu^2\right)\frac{d^2\bar{\chi}_0}{d\mu^2} - 4\mu\bar{\omega}_0\left(\bar{\omega}_0^2-2-2\mu^2\right)\frac{d\bar{\chi}_0}{d\mu} - 2\left(\bar{\omega}_0^3+\bar{\omega}_0^2+4\mu^2\right)\bar{\chi}_0 = 0 \quad ,$$

which has solution

$$(4.4) \qquad\qquad \bar{\chi}_0 = \bar{\omega}_0\left(1-\mu^2\right)\frac{d\Psi}{d\mu} + 2\left(2-\bar{\omega}_0\right)\Psi \quad ,$$

where Ψ satisfies the differential equation

$$(4.5) \qquad\qquad \left(1-\mu^2\right)\frac{d^2\Psi}{d\mu^2} - 4\mu\frac{d\Psi}{d\mu} + \frac{2\left(1-\bar{\omega}_0\right)}{\bar{\omega}_0}\Psi = 0 \quad .$$

Further it follows that $\bar{\omega}_0$ must be one of a discrete set of real values given by

$$(4.6) \qquad\qquad \bar{\omega}_0 = \frac{2}{n(n+1)} \quad ,$$

where n is a positive integer. For a more complete discussion of (4.3) see Stewartson and Rickard [5].

Continuing in a formal way we find that the equation satisfied by $\bar{\chi}_1$ is

$$(4.7) \quad \bar{\omega}_0\left(1-\mu^2\right)\left(\bar{\omega}_0^2-4\mu^2\right)\frac{d^2\bar{\chi}_1}{d\mu^2} - 4\mu\bar{\omega}_0\left(\bar{\omega}_0^2-2-2\mu^2\right)\frac{d\bar{\chi}_1}{d\mu} - 2\left(\bar{\omega}_0^3+\bar{\omega}_0^2+4\mu^2\right)\bar{\chi}_1$$

$$= \frac{2\bar{\omega}_1}{\bar{\omega}_0}\left[\mu\left(\bar{\omega}_0^3+6\bar{\omega}_0+4\mu^2\bar{\omega}_0-8\mu^2\right)\Psi-\bar{\omega}_0\left(1-\mu^2\right)\left(\bar{\omega}_0^2+4\mu^2\right)\frac{d\Psi}{d\mu}\right]$$

$$+ \tfrac{1}{2}\left(\bar{\omega}_0^2-4\mu^2\right)^2\left[\mu\left(\bar{\omega}_0-1\right)^2\Psi-\bar{\omega}_0^2\left(1-\mu^2\right)\frac{d\Psi}{d\mu}\right] \quad .$$

It is clear from (4.4), (4.5) that

$$(4.8) \qquad\qquad \Psi = \frac{dP_n(\mu)}{d\mu} \quad ,$$

where $P_n(\mu)$ is the Legendre polynomial of order n . On using the relationship

(4.9) $$\left(1-\mu^2\right) \frac{d^2 P_n}{d\mu^2} = 2\mu \frac{dP_n}{d\mu} - \frac{2}{\bar{\omega}_0} P_n \ ,$$

the right hand side of (4.7) may be written in the form

(4.10) $$\frac{2\bar{\omega}_1}{\bar{\omega}_0} \left[\mu\left(-\bar{\omega}_0^3 + 6\bar{\omega}_0^2 - 4\mu^2\bar{\omega}_0 - 8\mu^2\right) \frac{dP_n}{d\mu} + 2\left(\bar{\omega}_0^2 + 4\mu^2\right)P_n \right]$$
$$+ \tfrac{1}{2}\left(\bar{\omega}_0^2 - 4\mu^2\right)^2 \left[\mu\left(-\bar{\omega}_0^2 - 2\bar{\omega}_0 + 1\right) \frac{dP_n}{d\mu} + 2\bar{\omega}_0 P_n \right] \ .$$

Let us write

(4.11) $$\bar{\chi}_1 = \left(\alpha + b\mu^2\right)P_n + \mu\left(\beta + d\mu^2\right)\frac{dP_n}{d\mu} \ ,$$

where α, β, b, d are constants to be determined. On substituting (4.11) into (4.7), and bearing in mind (4.10), we can determine these constants and the value of $\bar{\omega}_1$ by comparing coefficients of corresponding terms.

Specifically, proceed as follows: in order that terms in $\mu^5 \dfrac{dP_n}{d\mu}$ and $\mu^4 P_n$ shall match we must have

$$\bar{\omega}_0(d+2b) = -\bar{\omega}_0^2 - 2\bar{\omega}_0 + 1 \ , \quad 3\bar{\omega}_0 b + 4d = 2\bar{\omega}_0 \ ,$$

respectively, which gives

(4.12) $$b = \frac{2\left(3\bar{\omega}_0^2 + 4\bar{\omega}_0 - 2\right)}{\bar{\omega}_0\left(3\bar{\omega}_0 - 8\right)} \ , \quad d = \frac{-3\bar{\omega}_0^2 - 10\bar{\omega}_0 + 3}{\left(3\bar{\omega}_0 - 8\right)} \ .$$

The terms in $\mu^3 \dfrac{dP_n}{d\mu}$ correspond provided we choose

(4.13) $$\bar{\omega}_1 = \frac{\bar{\omega}_0\left(\bar{\omega}_0 - 1\right)\left[-3\bar{\omega}_0^3 - 19\bar{\omega}_0^2 - 12\bar{\omega}_0 + 8\right]}{4\left(3\bar{\omega}_0 - 8\right)}$$

and with this choice of $\bar{\omega}_1$ the terms in $\mu^2 P_n$ also agree. Finally, in order that terms in $\mu \dfrac{dP_n}{d\mu}$ and P_n shall agree we find that

$$4\bar{\omega}_0\beta\left(2-\bar{\omega}_0\right) - 2\bar{\omega}_0\left(\bar{\omega}_0^2-4\right)\alpha = 2\bar{\omega}_1\left(-\bar{\omega}_0^2+6\bar{\omega}_0\right) + \tfrac{1}{2}\bar{\omega}_0^4\left(-\bar{\omega}_0^2-2\bar{\omega}_0+1\right) - 6d\bar{\omega}_0^3 - 4b\bar{\omega}_0^3 \ ,$$

$$-2\bar{\omega}_0\left(\bar{\omega}_0+2\right)\alpha - 4\bar{\omega}_0\beta = 4\bar{\omega}_1 + \bar{\omega}_0^4 - 2b\bar{\omega}_0^2 \ ,$$

respectively. If we substitute for b, d, $\bar{\omega}_1$ from (4.12), (4.13), it becomes clear that the above equations are in fact identical and that α, β are connected by the equation

$$(4.14) \qquad\qquad 2\beta + \left(\bar{\omega}_0+2\right)\alpha = \frac{\bar{\omega}_0\left(24\bar{\omega}_0^2+5\bar{\omega}_0-4\right)}{2\left(3\bar{\omega}_0-8\right)} \ .$$

The constants α, β are not uniquely determined because they determine, in part, the complimentary function in the differential equation for $\bar{\chi}_1$.

From (4.1), (4.2), (4.11)-(4.14) it follows that

$$(4.15) \qquad \omega_1 = \bar{\omega}_0 + \frac{\bar{\omega}_0\left(\bar{\omega}_0-1\right)\left(-3\bar{\omega}_0^3-19\bar{\omega}_0^2-12\bar{\omega}_0+8\right)}{4\left(3\bar{\omega}_0-8\right)} F + O\left(F^2\right) \ ,$$

$$(4.16) \quad \chi_1 = \left[\mu\left(\bar{\omega}_0+2\right)\frac{dP_n}{d\mu} - 2P_n\right] + \left[\left\{\left(\frac{-\left(\bar{\omega}_0+2\right)}{2}\ \alpha\ +\ \frac{\bar{\omega}_0\left(24\bar{\omega}_0^2+5\bar{\omega}_0-4\right)}{4\left(3\bar{\omega}_0-8\right)}\right)\mu\right.\right.$$
$$\left.+ \left(\frac{-3\bar{\omega}_0^2-10\bar{\omega}_0+3}{\left(3\bar{\omega}_0-8\right)}\right)\mu^3\right\}\frac{dP_n}{d\mu} + \left\{\alpha + \frac{2\left(3\bar{\omega}_0^2+4\bar{\omega}_0-2\right)}{\bar{\omega}_0\left(3\bar{\omega}_0-8\right)}\ \mu^2\right\}P_n\Bigg] F + O\left(F^2\right) \ ,$$

where $\bar{\omega}_0$ is given by (4.6) and α is now an arbitrary constant (which, without loss of generality, could be taken as zero). Clearly the term of $O(F)$ in (4.15) is zero when $\bar{\omega}_0 = 1$; it can be shown that the higher order terms also vanish when $\bar{\omega}_0 = 1$, a result in agreement with (3.2).

The expansion procedure could be continued formally and $\bar{\chi}_2$, $\bar{\omega}_2$ calculated in the same way as above. Before leaving this discussion of solutions for small F , it is noted that (4.15), (4.16) hold for small F , whether F is positive or negative.

The validity of the linear theory derived here will be discussed at the end of the following section.

5. Numerical solutions of (3.1)

Picken [3] derived a numerical method for solving ordinary differential equations in terms of Chebyshev series. A major advantage of the method is that boundary value problems can be solved with no more difficulty than initial value problems.

The solution of (3.1) is assumed to possess a convergent Chebyshev series expansion of the form

$$(5.1) \qquad \chi_1(\mu) = \sum_{r=0}^{\infty} a_r T_r(\mu) \; ;$$

the method computes the coefficients of a polynomial approximation to $\chi_1(\mu)$, such that

$$(5.2) \qquad \chi_1(\mu) = \sum_{r=0}^{n} a_r T_r(\mu) \; ,$$

where n is an integer chosen by the user.

When calculating eigenvalues associated with even eigenfunctions it is convenient to impose boundary conditions

$$(5.3) \qquad \frac{d\chi_1}{d\mu} = 0 \; , \quad \chi_1 = 1 \; , \text{ when } \mu = 0 \; .$$

For odd eigenfunctions we may specify

$$(5.4) \qquad \frac{d\chi_1}{d\mu} = 1 \; , \quad \chi_1 = 0 \; , \text{ when } \mu = 0 \; ,$$

as suitable boundary conditions.

The series (5.2) fails to converge except at an eigenvalue, and the procedure employed consists of varying ω_1 until higher a_n's become very small. In this way those eigenvalues which are equal to 1/3, 1/6 , when $F = 0$, were computed, and the results obtained are tabulated in Table 1 and illustrated graphically in Figure 1 (p. 138). The results here were obtained with $n = 32$. To extend the results to values of F
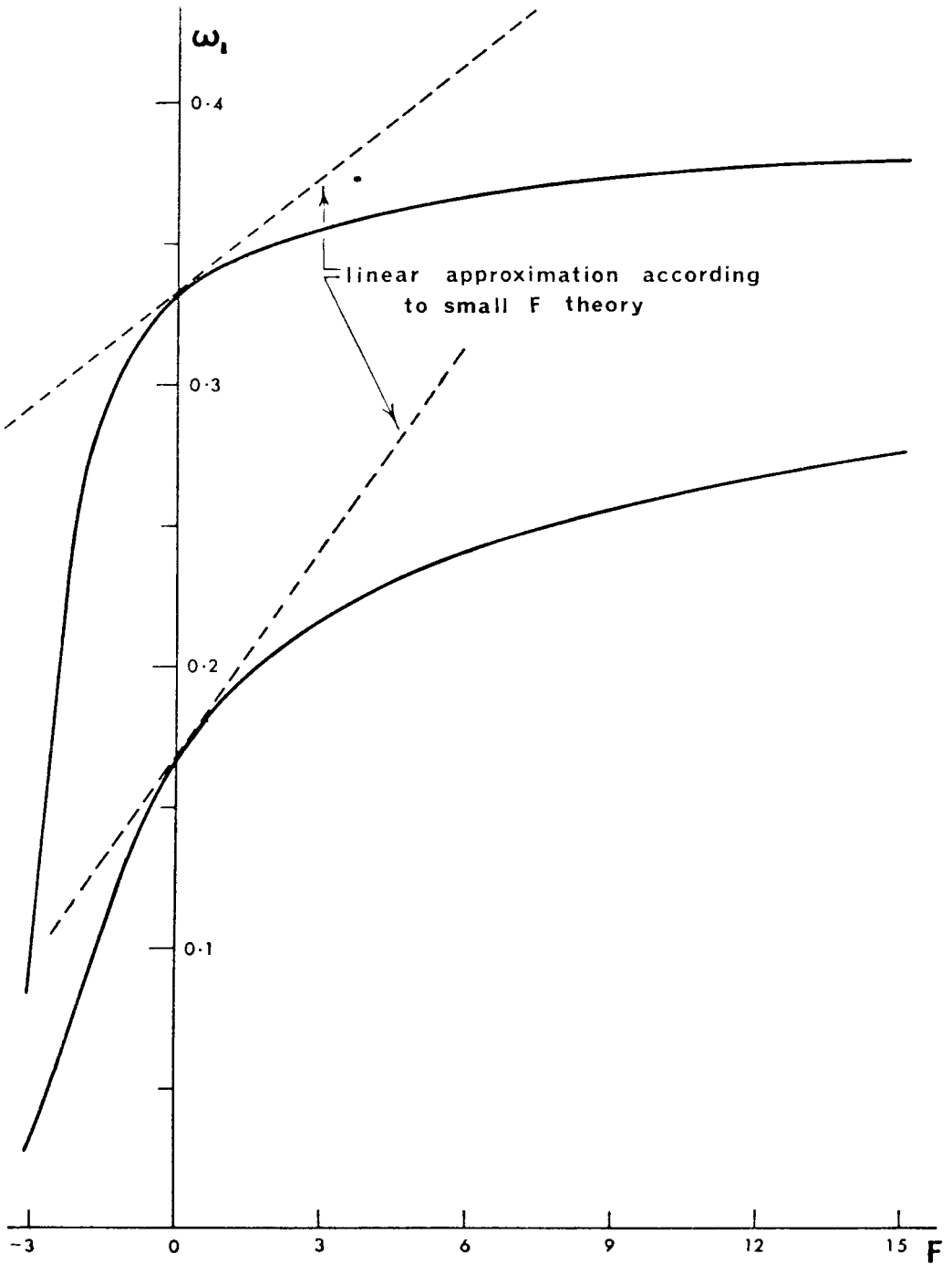
J.A. Rickard

Figure 1.   Variation of ω_1 with F.

greater than those given an increased number of Chebyshev coefficients are required; that is, the value of n must be increased.

TABLE 1

F	ω_1	
0	1/3	1/6
-3	.090	.026
-2.5	.193	.052
-2	.263	.085
-1	.314	.136
1	.344	.188
5	.365	.235
10	.376	.262
15	.382	.278

Good agreement exists between the linear approximation for small F (see (4.15)) and the numerical results obtained here provided $|F| < 1$ approximately. In fact, we may use the numerical results to estimate the next term in the expansion (4.2). When $\bar{\omega}_0 = 1/3$ we may take

(5.5)
$$\omega_1 = 0.333 + .014F - .004F^2 ,$$

which gives values of .343 and .315 when $F = \pm 1$ respectively. These values compare favourably with the numerical values obtained (.344, .314), especially considering that here we are only working to three decimal places.

6. Discussion

It will be observed from Figure 1 that ω_1 shows considerable variation as F increases. As an illustration we will compare the values of ω_1 when $F = 0$ and $F = 12$ (a typical value for the Pacific Ocean). The eigenvalue which is equal to 1/3 at $F = 0$ is approximately 0.38 at $F = 12$, while that which is 1/6 at $F = 0$ has increased to approximately 0.27. We also note that when $\gamma = 0$, ω_1 increases as F

increases $\left(\text{except for } \omega_1 = 1 \text{ , which remains true for all } F \text{ ; see}\right.$ (3.2)$\left.\right)$. In a subsequent paper we shall show that ω_1 decreases as F increases for certain γ .

Longuet-Higgins [1] showed that for the free surface oscillations of a global ocean it was justified to treat the motion as horizontally non-divergent when $F \ll 1$. The results obtained here entirely agree with this conclusion, but they show quite clearly that for the Pacific Ocean, when $F \simeq 12$, this simplification is not valid.

Stewartson and Rickard [5] found that singularities were present in the higher order terms U_n, V_n, χ_{n+1}, W_{n+2} $(n > 1)$ in the expansion of velocity components and pressure $\left(\text{see } (2.23a\text{-}2.23e)\right)$ at the two critical circles where the characteristic cones of the governing equation touched the shell boundaries. A superficial examination suggests that corresponding singularities occur here and that the properties of the fluid in the neighbourhood of the circles $\mu = \pm\frac{1}{2}\omega_1$ are essentially the same as those discussed in Stewartson and Rickard [5]. It is hoped to examine this aspect of the problem in further detail in a later paper, and also to consider the effects of density stratification.

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