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ANY 2-SPHERE IN E^3 WITH UNIFORM INTERIOR TANGENT BALLS IS FLAT

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Introduction. This paper addresses some flatness properties of an $(n-1)$-sphere Σ in Euclidean n-space E^n resulting from the presence of round balls in E^n tangent to Σ. The notion of tangency used here is geometric rather than differentiable, for a round n-cell B_p (that is, the set of points whose distance, in the standard metric, from some center point is less than or equal to a fixed positive number) is said to be *tangent to the* $(n-1)$*-sphere* Σ *in* E^n *at a point* $p \in \Sigma$ if $p \in B_p$ and $\text{Int } B_p \cap \Sigma = \emptyset$. The ball B_p is called an interior tangent ball at p if $\text{Int } B_p \subset \text{Int } \Sigma$; otherwise, it is called an *exterior tangent ball at* p. The sphere Σ is said to have *double tangent balls over a subset* K of Σ if Σ has both an interior and an exterior tangent ball at each point of K, and Σ is said to have *uniform interior* (exterior, double) *tangent balls over a subset* K if there exists a collection $\mathscr{B}$ of round n-cells of some fixed radius δ such that for each $p \in K \subset \Sigma$ there is an element of $\mathscr{B}$ that is an interior (exterior, double) tangent ball to Σ at p.

Questions concerning the implications such geometric structures have on the flatness of surfaces can be traced to the late 1950's. Perhaps suspecting that the existence of double tangent balls over a 2-sphere Σ in E^3 was somewhat analogous to Σ being smoothly embedded, Bing [**4**] asked if $\Sigma \subset E^3$ was necessarily flat when it has double tangent balls at each of its points. After Griffith [**16**] did the case where Σ has uniform double tangent balls, Loveland [**17**] and Bothe [**5**] independently gave an affirmative answer to Bing's question. Daverman and Loveland [**10**] cast doubt about higher dimensional analogues by exhibiting a wild $(n-1)$-sphere in E^n having uniform double tangent balls over its wild set, for it seems likely that this example could be smoothed out, away from the wild set, to one with (nonuniform) double tangent balls everywhere.

Studied here is the problem of whether the existence of uniform interior or exterior tangent balls implies flatness of the $(n-1)$-sphere Σ, a question first raised by Loveland [**17**], for the case $n = 3$, and the only one of four he mentioned there that remained open until now. Cannon's answer [**8**] to one of the others is of particular significance for this paper, because it follows from Cannon's *-taming set theory [**8**, Corollary 6] that a 2-sphere in E^3 is flat from its exterior if it has interior tangent balls.

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We show that such a 2-sphere Σ in E^3 is flat if it has uniform interior tangent balls; in light of Cannon's work, this amounts to showing that Σ is flat from its interior if it has uniform interior tangent balls. The argument provided works only for $\Sigma \subset E^3$, although portions of it shed light on certain special higher dimensional cases. We also describe an example of a wild $(n-1)$-sphere Σ in E^n $(n > 3)$ having uniform exterior tangent balls. It blocks the extension to higher dimensions of Cannon's result about flatness from the interior, because the wildness is found in its interior, not its exterior. As a result, the question asking whether an $(n-1)$-sphere in E^n $(n > 3)$ is flat from its interior if it has uniform interior tangent balls stands unsolved.

It follows directly from our Theorem 2.7 that the ϵ-boundary of a subset A of E^3 is locally flatly embedded in E^3 at each point where it is locally a 2-manifold, which answers a question raised by Weill [**23**, p. 248]. If ϵ is a positive number, $A \subset E^3$, and d denotes the usual metric for E^3, the ϵ-boundary $\partial(\epsilon, A)$ of A is defined as $\{x \in E^n | d(x, A) = \epsilon\}$. Weill called this the ϵ-envelope of A and asked about its flatness in E^3 when it is a 2-sphere [**23**]. In this situation Weill observed that the 2-sphere $\partial(\epsilon, A)$ would be flat from one side if A were contained entirely on the other side. Loveland [**18**, p. 359] pointed out that $\partial(\epsilon, A)$ is flat if it is an $(n-1)$-sphere in E^n, $n \neq 4$, and if A intersects both complementary domains of $\partial(\epsilon, A)$. The restriction $n \neq 4$ is superfluous in view of Loveland's work [**18**] in combination with Corollary 2.3 of [**10**].

Theorems known to apply for all Euclidean spaces are gathered together in Section 1 while the main 3-dimensional results are located in Section 2. Section 3 contains examples and Section 4 is devoted to ϵ-boundaries.

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1. Definitions, notation, and higher-dimensional theorems. Let Σ denote an $(n-1)$-dimensional sphere in E^n, $\mathscr{B}$ a uniform collection of interior tangent balls over a subset K of Σ, and δ the common radius of the elements of $\mathscr{B}$. The *sphere of directions* D_p at a point p in Σ is the $(n-1)$-sphere of radius δ whose center is p. Precisely stated, a *normal* to Σ at p is a vector from p to the center of some element of $\mathscr{B}$ that contains p, although the set N_p of normals at p is most often viewed as the set of terminal points of such vectors. In this latter sense N_p lies in D_p; in fact, as Lemma 1.1 verifies, N_p must lie in some hemisphere H_p in D_p.

LEMMA 1.1. *If an $(n-1)$-sphere Σ in E^n has a collection $\mathscr{B}$ of uniform interior tangent balls at a point p of Σ and N_p is the corresponding set of normals, then N_p lies in a closed hemisphere in the sphere D_p of directions at p.*

Proof. Let $\{p_i\}$ be a sequence of points of Ext Σ converging to p, and choose, for each i, the point r_i of D_p such that p_i lies between p and r_i. Let r be a point to which some subsequence of $\{r_i\}$ converges, and define H_p to be the closed hemisphere of D_p farthest from r. If there exists a point n of $N_p - H_p$, the ball B centered at n and tangent to Σ at p must contain a point p_i, which contradicts the fact that $B \cap \text{Ext } \Sigma = \phi$.

The next two geometrical lemmas are stated without proof.

LEMMA 1.2. *If $x \in E^n$ and $\mathscr{B}$ is any closed collection of round n-cells such that each element of $\mathscr{B}$ contains x and x lies in the interior of some element of $\mathscr{B}$, then the union of the elements of $\mathscr{B}$ is a star-like n-cell (hence, its boundary is bicollared).*

LEMMA 1.3. *Let D be an $(n - 1)$-sphere of radius δ centered at a point p in E^n, let H be a closed hemisphere in D, and let N be a compact subset of* Int H. *Then the intersection of the set of all round n-balls having p in their boundaries and centered at points of N is a convex n-cell with p in its boundary.*

See the proof of Lemma 2.2 for an argument that will establish Lemma 1.3.

Playing on the words "stable" and "unstable" used in [**10**], given a collection $\mathscr{B}$ of uniform tangent balls to Σ defined over K, we define

$S = \{p \in K |$ there exists a hemisphere H_p in D_p such that $N_p \subset \text{Int } H_p\}$,

$U = \{p \in K |$ for every hemisphere H_p of D_p containing N_p, $N_p \cap \text{Bd } H_p \neq \emptyset\}$.

THEOREM 1.4. *If Σ is an $(n - 1)$-sphere in E^n having a collection $\mathscr{B}$ of uniform interior tangent balls over an $(n - 1)$-cell K in Σ and if $p \in S \cap$* Int K, *where S is defined above, then Σ is locally bicollared at p. Consequently, Σ is locally flat at p.*

Proof. Since K is closed we may assume $\mathscr{B}$ is closed and N_p is compact. From the definition of S, there exists a closed hemisphere H_p of D_p whose interior contains N_p. By Lemma 1.3 the intersection of all elements of $\mathscr{B}$ centered in N_p is a convex n-cell Z with p in its boundary. Choose $x \in \text{Int } Z$, and let E be an $(n - 1)$-cell in K such that $p \in \text{Int } E$ and E is so small that any ball of $\mathscr{B}$ that intersects E also contains x in its interior. The last condition is possible because $\mathscr{B}$ is closed. Let B^* denote the union of all balls in $\mathscr{B}$ having x in their interiors. From Lemma 1.2, Bd B^* is a bicollared $(n - 1)$-sphere. Since $E \subset \text{Bd } B^*$, Σ is locally bicollared at each point of Int E. The local flatness of Σ at p follows [**6**].

COROLLARY 1.5. *If an $(n - 1)$-sphere Σ in E^n has uniform interior tangent balls at each point, then Σ is locally flat modulo the set U (U is defined as above with $K = \Sigma$).*

The set W of points where an $(n-1)$-sphere Σ in E^n fails to be locally flat is called the *wild set* of Σ. Although not used in this paper, Theorem 1.6 gives a dimension restriction on W when Σ has interior tangent balls (not necessarily uniform) over W. A stronger restriction on the dimension of W is not even possible when the one-sided tangent balls are known to be uniform (see Section 3).

THEOREM 1.6. *If an $(n-1)$-sphere Σ in E^n has interior tangent balls over its wild set W, then W has codimension two in E^n.*

This is Corollary 2.4 of [**10**].

2. Flatness of 2-spheres with uniform tangent balls. In this section Σ denotes a 2-sphere in E^3, K is a subset of Σ usually representing either a 2-cell or Σ itself, and $\mathscr{B}$ is a collection of uniform interior tangent balls over K. The objective, to prove that Int K is locally flat, is accomplished in two steps. First the wildness is confined to a finite set F (see Proposition 2.4), and next the wild set is proven empty (Theorem 2.7). Subsets E and F of K are defined as

$$F = \{p \in K \mid \text{for every hemisphere } H_p \text{ of } D_p \text{ containing } N_p,\ \text{Bd } H_p \subset N_p\}, \quad \text{and}$$

$$E = \{p \in K \mid \text{there exists a hemisphere } H_p \text{ of } D_p \text{ containing } N_p \text{ such that Bd } H_p \not\subset N_p\}.$$

From Lemma 1.1 it is clear that $K = E \cup F$. In terms of the sets S and U of Section 1, S lies in E and F lies in U.

LEMMA 2.1. *If K is a compact subset of a 2-sphere Σ in E^3 such that Σ has uniform interior tangent balls over K and F is defined as above, then F is a finite set.*

Proof. It may be assumed that the common radius δ of the hypothesized set $\mathscr{B}$ of uniform tangent balls is ever so much smaller than the diameter of Σ and that $\mathscr{B}$ is closed. Then F is compact. If F were infinite, there would be a sequence $\{p(i)\}$ in F converging to a point $p(0) \in F$. By the definition of F each $p(i)$ carries with it a full great circle in $D_{p(i)}$ of centers of elements of $\mathscr{B}$. For each i, let V_i be the union of the balls from $\mathscr{B}$ centered somewhere in this circle. Since $\{V_i\}$ converges to V_0 it is clear that, for some i, $V_i \cup V_0$ would separate E^3 with Σ forced to lie in a component of diameter less than δ.

The next lemma is designed to apply to points of E; when coupled with Lemmas 2.1 and 2.3, it eventually leads to the conclusion that $W \cap \text{Int } K \subset F$, where W is the set of wild points of Σ.

LEMMA 2.2. *Let D be a 2-sphere with radius δ centered at a point $p \in E^3$, let H be a closed hemisphere of D, let N be a closed subset of H, let h be a*

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point of Bd $H - N$, *and let* J *be the equator of* D *that contains* h *and divides* H *into two congruent quarter-spheres* H_1 *and* H_2. *Then, for* $i = 1, 2$, *the intersection of all* 3*-cells of radius* δ *containing* p *whose centers lie in* $H_i \cap N$ *is a convex* 3*-cell* Z_i *with* p *in its boundary.*

Proof. There is an open ball V centered at h and not intersecting N. Let $G_i = H_i - V$ for $i = 1, 2$. For a subset X of G_i, X^* denotes the intersection of all balls of radius δ containing p with their centers in X. The basic fact of geometry from which Lemma 2.2 follows is that X^* is a (convex) 3-cell as long as X subtends a maximal angle at p of less than 180°. Since G_i subtends such an angle, the result follows. However, it may be instructive to know that if $G \in \{G_1, G_2\}$ then G^* is the intersection of the three balls centered at a, b, and c where c is the antipode of h on D and $\{a, b\}$ is the endpoint-set of the arc Q defined by (Bd V) $\cap G$. To see this, one should first verify the milder assertion that, for any great arc of D containing no pair of antipodal points, $A^* = \{e_1\}^* \cap \{e_2\}^*$, where e_1 and e_2 denote the endpoints of A. Writing G as the union of circular arcs A_x from points x of Q to c, one has expressed A_x^* as $\{x\}^* \cap \{c\}^*$, so that

$$\begin{aligned} G^* &= \cap A_x^* = \cap (\{x\}^* \cap \{c\}^*) \\ &= (\cap \{x\}^*) \cap \{c\}^* \\ &= Q^* \cap \{c\}^* = \{a\}^* \cap \{b\}^* \cap \{c\}^*. \end{aligned}$$

It follows that $(N \cap G)^*$ is a (convex) 3-cell containing the 3-cell G^*.

The next lemma is an immediate consequence of Theorem 4.1 of [**20**].

LEMMA 2.3. *If* K *is an* $(n - 1)$*-cell in the boundary* Σ *of a crumpled* n*-cell* C *in* E^n *and if* B_1 *and* B_2 *are flat* n*-cells in* C *such that* $K \subset$ Bd $B_1 \cup$ Bd B_2, *then* Int Σ *is* $1 - LC$ *at each point of* Int K.

PROPOSITION 2.4. *If* K *is a* 2*-cell in a* 2-sphere Σ *in* E^3 *such that* Σ *has a closed set of uniform interior tangent balls over* K, *then* Σ *is locally flat at each point of* Int $K - F$, *where* F *is the finite set defined above.*

Proof. If $p \in$ (Int K) $\cap E$, then the definition of E allows the application of Lemma 2.2, whose conclusion gives two 3-cells Z_1 and Z_2 with the properties stated there. Let z_1 and z_2 be points in the respective interiors of these cells. Since z_i lies in the interior of Z_i and the set $\mathscr{B}$ of uniform tangent balls is closed, there must be a subdisk M of K such that $p \in$ Int M and every element of $\mathscr{B}$ tangent to Σ at a point of M contains either z_1 or z_2 in its interior. Let B_i, $i = 1, 2$, be the union of all elements of $\mathscr{B}$ whose interiors contain z_i. Then B_i is a flat 3-cell (see Lemma 1.2) for each i, and M lies in the boundary of $B_1 \cup B_2$ which lies in $\Sigma \cup$ Int Σ. From Lemma 2.3, Int Σ is $1 - LC$ at p. Then from [**2**], Σ is locally flat from Int Σ at each point of Int K. Furthermore, by Cannon's *-taming

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set theory [**8**], Σ is also locally flat from Ext Σ at each point of Int K, so Proposition 2.4 follows, with Lemma 2.1 giving the finiteness of F.

Remark. An initial study of the crucial ingredients in the proof of Proposition 2.4 might lead one to believe that a proof of the flatness of Σ at points of F can be constructed along similar lines. If p belongs to F it is clear that a closed hemisphere H_p containing N_p is the union of three sets G_1, G_2, and G_3, no one of which contains antipodal points of D_p. An extension of Lemma 2.2 would yield three convex cells Z_1, Z_2, and Z_3 such that each ball of $\mathscr{B}$ containing p contains one of these convex cells. As in the proof of Proposition 2.4 the existence of a 2-cell M in Σ and three star-like 3-cells B_1, B_2, and B_3 in $\Sigma \cup \text{Int } \Sigma$ could be established such that $p \in \text{Int } M$ and $M \subset \bigcup_{i=1}^{3} \text{Bd } B_i$. Then a generalization of Lemma 2.3, to cover three flat cells rather than two, would yield the desired flatness at p. However, this proposed approach is doomed to fail; Lemma 2.3 cannot be generalized from two to three flat cells (see [**20**]).

The following geometric proof that Σ is also flat at points of F was developed long after we had established Proposition 2.4. The difficulties in the proof can be captured by attempting to prove that the Fox-Artin [**13**] sphere does not have uniform tangent balls on its wild side. Before continuing to the proof of Theorem 2.7 the reader might find it interesting to verify that the next two results, which are simple consequences of Theorem 2.7, can also be deduced from Proposition 2.4.

COROLLARY 2.5. *If Σ is a 2-sphere in E^3 and $\mathscr{B}$ is a countable closed set of uniform interior tangent balls over Σ, then Σ is flat.*

COROLLARY 2.6. *If a crumpled cube C in E^3 contains a finite set of round 3-cells whose union contains* Bd C, *then C is a 3-cell.*

Notice that Corollary 2.5 is false when "uniform" is removed from its hypothesis. Corollary 2.6 is worth mentioning because of its connection with previous work by Loveland [**19**] and Pixley [**22**] concerning the flatness of the boundary of a crumpled cube C when it is the union of various sorts of 3-cells.

THEOREM 2.7. *If a 2-sphere Σ in E^3 has uniform interior tangent balls over Σ, then Σ is flat.*

Proof. Here $\mathscr{B}_\delta$ will denote the hypothesized set of uniform interior tangent balls, having common radius δ. Expanding $\mathscr{B}_\delta$ to include all balls of radius δ tangent to Σ from the interior, we regard $\mathscr{B}_\delta$ as a closed collection.

It is convenient to improve this to a new closed collection $\mathscr{B}$ of interior balls such that $B \cap \Sigma$ is a single point and the radius of B equals 2, for each $B \in \mathscr{B}$. To achieve this, we scale measurements so that $\delta > 2$ and then name $\mathscr{B}$ as the set of all balls B having radius 2 such that there exist $p \in \Sigma$ and $c \in D_p$ (the sphere of directions about p, with radius 2), where c is the center of B and lies in a segment from p to the center of some ball from $\mathscr{B}_\delta$ tangent to Σ at p.

According to Proposition 2.4, Σ is locally flat modulo the finite set F defined earlier. Let $p \in F$ and let $\mathscr{B}_p$ denote the set of all balls in $\mathscr{B}$ containing p. By the definition of F, in the sphere D_p of directions there is a hemisphere H_p whose boundary is covered by the centers of elements from $\mathscr{B}_p$.

We impose coordinates on E^3 with origin at p so that Bd H_p lies in the horizontal xy-plane and that the part of $\Sigma - \{p\}$ near p lies vertically above the xy-plane. Letting U denote the union of the elements from $\mathscr{B}_p$ whose centers are in Bd H_p, we choose an interval $[0, u]$ on the z-axis such that the horizontal plane P_t, defined by $z = t$, intersects U for each $t \in [0, u]$. (As the proof progresses, we shall restrict u in other ways as well, but we prefer to set forth these restrictions as the needs arise.) Then the component of $(\bigcup\{P_t | t \in [0, u]\} - U)$ having p in its closure is a bugle-shaped open 3-cell G that contains a neighborhood of p (but with p deleted) in Σ. For t $\in (0, u]$ we let G_t denote the open circular 2-cell $G \cap P_t$ in P_t. Figure 1 may help identify some of this structure.

When L_t is a straight line in P_t intersecting the z-axis at $(0, 0, t)$, we say that L_t is *projective* if no line in P_t parallel to L_t meets two components of $G_t \cap$ Ext Σ. Later in this section we prove that either Σ is

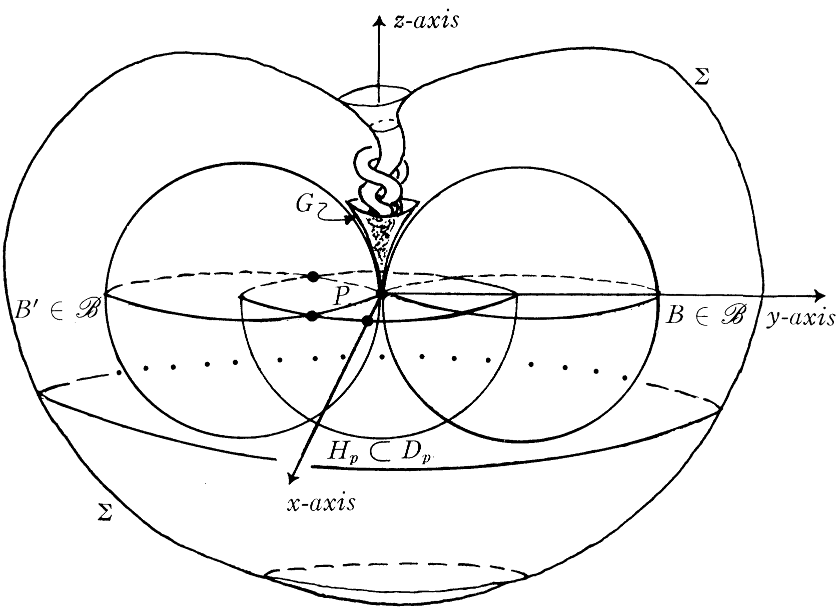

FIGURE 1.

locally flat at p or, after further restrictions on u, there exists a continuous collection $\{L_t|0 < t \leqq u\}$ of projective lines, where continuity results from a continuous function f of $(0, u]$ into $E^2 - (0, 0)$ and each line L_t passes through both $(0, 0, t)$ and $(f(t), t)$.

For now we presume the existence of such a continuous collection $\{L_t\}$ of projective lines; in Lemma 2.13 later on we establish this existence. It is a simple matter to find a homeomorphism h of E^3 to itself that takes each plane P_t onto itself, that fixes points of the z-axis, and that isometrically rotates the various circular sections G_t so that the segments $G_t \cap L_t$ are all parallel. We shall suppress h and simply assume that each L_t in the family of projective lines lies in the vertical xz-plane.

Let A be an arc in $(G \cap \text{Ext } \Sigma) \cup \{p\}$ that contains p as an endpoint and that is locally polyhedral modulo p. This arc will serve as a guide for constructing a flat arc T in $(G \cap \text{Ext } \Sigma) \cup \{p\}$ having p as an endpoint, and the existence of such a flat arc T will imply that Σ is locally flat at p [**21**], proving the theorem at hand. In what follows, the flatness of the arc T to be constructed will be deduced from the flatness of an equivalently embedded arc R (not necessarily contained in $\{p\} \cup \text{Ext } \Sigma$), also constructed with A as its guide.

A small adjustment of A allows the assumption that the projection $\pi: A \to \{yz\text{-plane}\}$ is regular in the sense that $\pi^{-1}(\pi(a))$ contains at most two points for each $a \in A$ and the "double" points of A lie in a decreasing sequence $\{P_{t(i)}\}$ of horizontal planes where no $P_{t(i)}$ contains two pairs of these double points. After giving A an order with p as last point, we will construct arcs R and T by inductively performing countably many "arc transplants" in A. The inductive procedure should be clear from the description of the first step and the brief summary of the second step that follow.

Let x_1 and y_1 be the two points of $P_{t(1)} \cap A$ such that $\pi(x_1) = \pi(x_2)$. Insist that x_1 precede y_1 in A, and let μ_1 denote the subarc of A with endpoints in $\{x_1, y_1\}$. Because the line L through x_1 and y_1 parallels a projective line in $P_{t(1)}$, x_1 and y_1 belong to the same component W_1 of $P_{t(1)} \cap \text{Ext } \Sigma$. This fact about L is also used in choosing a geometric rectangle J_1 in $P_{t(1)}$ with two sides parallel to L such that Bd W_1 touches all four sides of J_1, $W_1 \subset \text{Int } J_1$, and if L' is a line parallel to L that intersects Int J_1, then L' intersects only the component W_1 of $G_{t(1)} \cap \text{Ext } \Sigma$. Let γ be an arc in W_1 with endpoints x_1 and y_1 such that $\gamma \cap [x_1, y_1]$ is the finite set $\{x_1 = s_1, s_2, \ldots, s_m = y_1\}$, and let γ_i denote the subarc of γ from s_{i-1} to s_i. The object is to adjust each γ_i slightly to an arc β_i with the same endpoints such that $[s_{i-1}, s_i] \cup \beta_i$ bounds a disk D_i in Int J_1 where $A \cap \text{Int } D_i = \phi$. Then an arc T_1' from x_1 to y_1 could be defined as $\cup \beta_i$ with the assurance that T_1' and $[x_1, y_1]$ are isotopic in Int J_1 via an isotopy with support missing all of $A \cap P_{t(1)}$ except $\{x_1, y_1\}$, which is fixed. Such an isotopy could then be extended to one of E^3 to itself, fixed on A and outside a slight thickening of Ext J_1. At the end of the first inductive step, define

$$T_1 = (A - \mu_1) \cup T_1' \quad \text{and} \quad R_1 = (A - \mu_1) \cup [x_1, y_1].$$

To achieve the required adjustment of γ_i to β_i, let V_i be the component of $W_1 - L$ containing $\gamma_i - \{s_{i-1}, s_i\}$ and choose β_i in $\mathrm{cl}(V_i)$ so that no point of $A \cap V_i$ lies in the interior of the disk D_i bounded by $\beta_i \cup [s_{i-1}, s_i]$ in J_1. The definition of J_1 insures that $D_i \cap A \subset W_1$, but the possibility of there being a point a of $A \cap \mathrm{Int}\, D_i$, where a belongs to a component V_i' of $W_1 - L$ different from V_i, still exists. To rule this out we assume u is less than the number u_1 promised by Lemma 2.8 (stated after this proof) so that if $B \in \mathscr{B}$ contains a point of $\Sigma \cap G_{t(1)}$, then the radius of the circular 2-cell $B \cap P_{t(1)}$ is at least 1. Now suppose such a point a exists in a component V_i' of $W_1 - L$ where $V_i' \neq V_i$. Since $a \in \mathrm{Int}\, D_i$, $V_i' \subset \mathrm{Int}\, D_i$ and there is a point Q_1 of $[s_{i-1}, s_i] \cap$ Ext Σ in the boundary of V_i'. Let C denote the point on the non-D_1-side of L at a distance 1 from both Q_1 and s_i, and let α be the smaller open arc from Q_1 to s_i on the circle centered at C. Since α intersects V_1' near Q_1 and $V_1' \subset \mathrm{Int}\, D_i$, there must be a point Z of $\Sigma \cap D_i$ in α. See Figure 2. By Lemma 2.8 there is a ball B in $\mathscr{B}$ such that $B \cap P_{t(1)}$ contains a 2-cell K of radius 1, centered at a point X, such that $K \cap (\Sigma \cup \mathrm{Ext}\, \Sigma) = \{Z\}$. Since the diameter of D_i is smaller than the length of ZX and $\beta_i \subset \mathrm{Ext}\, \Sigma$, X must lie on the C-side of L and ZX must intersect the open segment $s_{i-1}s_i$. But K does not intersect $\{s_{i-1}, s_i, Q_1\}$ and $d(X, Q_1) > 1$, so Bd K contains an arc β on the D_i-side of L whose endpoints lie in the open segment $Q_1 s_i$. Since α and β intersect at Z, this contradicts Lemma 2.9. Thus D_i contains no point of A.

In the second step of the inductive construction the integer $n(2)$ is identified as the least of all integers i larger than $n(1) = 1$ for which $P_{t(i)}$ contains a double point $\{x_2, y_2\}$ of R_1 under π, and μ_2 is the subarc of A from x_2 to y_2. Now the construction of R_2 as $(R_1 - \mu_2) \cup [x_2, y_2]$ and T_2 as $(T_1 - \mu_2) \cup T_2'$ proceeds just as in Step 1. Again R_2 and T_2 are ambient isotopic fixed on A and fixed outside a small thickening of a small rectangle J_2 in $G_{n(2)}$.

Assuming the inductive construction completed, define

$$R = \left(A - \bigcup_{i=1}^{\infty} \mu_i\right) \cup \left(\bigcup_{i=1}^{\infty} [x_i, y_i]\right) \quad \text{and}$$

$$T = \left(A - \bigcup_{i}^{\infty} \mu_i\right) \cup \left(\bigcup_{i}^{\infty} T_i'\right).$$

Then R and T are equivalently embedded arcs since they are ambiently isotopic. Since $T \subset (\mathrm{Ext}\, \Sigma) \cup \{p\}$, the flatness of Σ at p will follow from [**21**] once T is known to be flat, but the flatness of R is more easily detected than that of T. Since $\pi | R : R \to \{yz\text{-plane}\}$ takes R to an arc, it

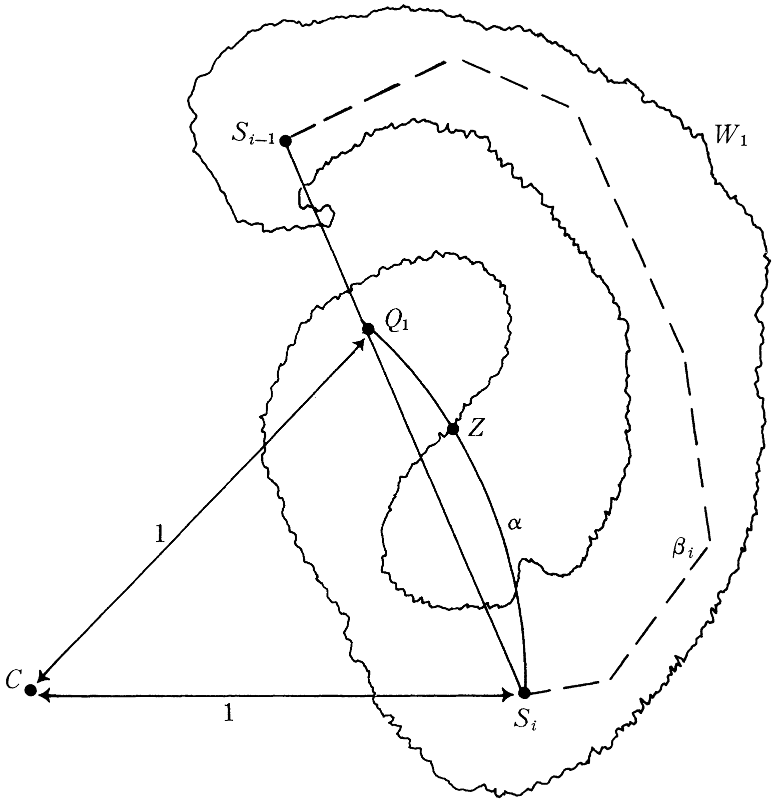

FIGURE 2.

is easy to use $\pi(R)$ as a guide to see that R is locally peripherally unknotted at p and hence flat [**15**].

Next we place further restrictions on the positive number u introduced during the proof of Theorem 2.7.

LEMMA 2.8. *There exists $u_1 \in (0, u]$ such that for any $B \in \mathscr{B}$ containing a point of $\Sigma \cap G_t$, $t \in (0, u_1]$, the radius of the circular 2-cell $B \cap P_t$ is at least* 1.

Proof. The closedness of the collection $\mathscr{B}$ of radius 2 interior tangent balls implies that, if b denotes the center of $B \in \mathscr{B}$ tangent to Σ at $x \in \Sigma \cap G_t$, then b approaches H_p as $t \to 0$. Furthermore, elementary geometry reveals that b actually must approach Bd H_p as $t \to 0$. Consequently, when t is sufficiently small, the radius of $B \cap P_t$ is approximately 2 cos (arc sin$(t/2)$). See Figure 3.

The next five lemmas are directed toward a proof of the existence of the continuous collection of projective lines, as claimed early in the proof

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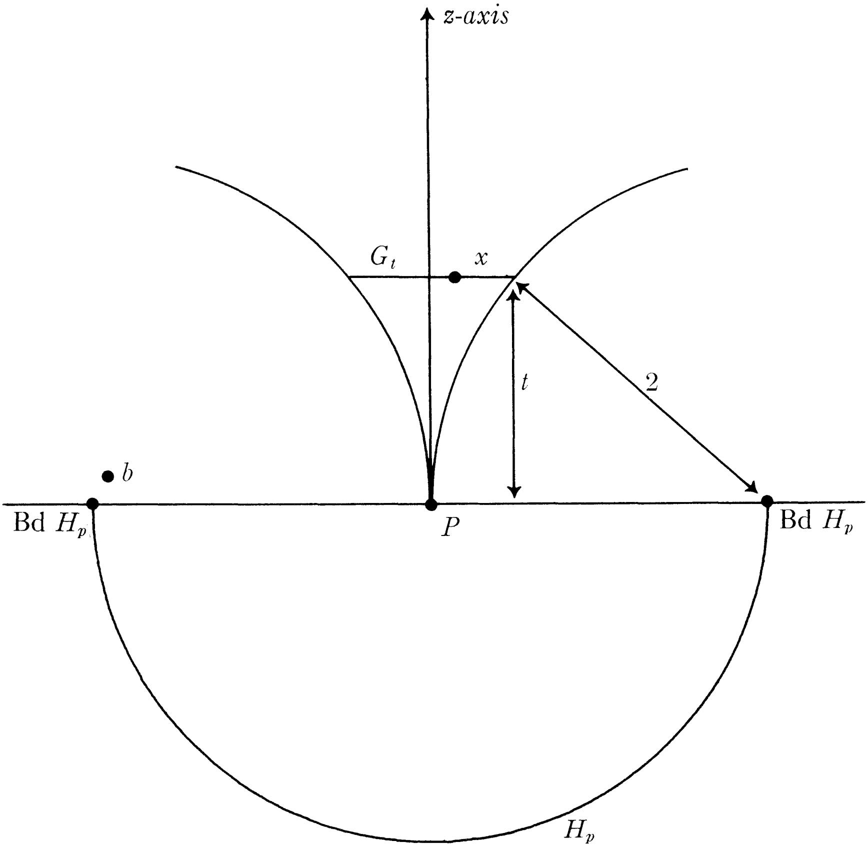

FIGURE 3.

of Theorem 2.7, and the first of these lemmas serves as well to guarantee important properties of the isotopy described near the end of that proof. In addition to the hypotheses of Theorem 2.7 and to the notation introduced there, we shall use $L(A, B)$ to denote the line determined by two points A and B, and S_A the circle of radius 1 centered at A. As before, the common radius δ of the set $\mathscr{B}$ of uniform interior tangent balls is taken as 2, for convenience in writing. A minor arc of a unit circle is an arc of a circle of radius 1 whose length is less than π.

LEMMA 2.9. *Suppose two minor open arcs α and β of unit circles in the plane lie above the x-axis and have their endpoints on the x-axis, and suppose the endpoints of β lie between those of α on the x-axis. Then β lies in the interior of the unit circle containing α.*

Proof. Suppose there is a point of β on or above α. Then α must intersect β at two points X and Y, and the centers C_α and C_β of α and β, respectively, must lie on the perpendicular bisector L of the chord XY below the x-axis. Then C_β lies above C_α on L, so a diameter of the circle S_β containing β chosen parallel to XY has its endpoints outside the other circle S_α. As a consequence, S_α and S_β intersect in at least four points, an impossibility.

If L and L' are intersecting lines in E^n, we use $\theta(L, L')$ to denote the (radian) measure of the small angle between L and L'.

LEMMA 2.10. *There exists $u_2 \in (0, u_1]$ such that if the points Q_1, Q_2 and Q_3 of $\Sigma \cup \operatorname{Ext} \Sigma$ are the vertices of a triangle T in $G_t (t \in (0, u_2])$ with C the point where the angle bisectors of T intersect, and if each of the broken segments Q_1CQ_2 and Q_1CQ_3 intersects Σ in* Int T, *then*

$$\theta(L(Q_1, Q_2), L(Q_1, Q_3)) < \pi/12.$$

Proof. Choose $u_2 > 0$ such that $u_2 < u_1$ and if S_A is a unit circle in P_t $(t \in (0, u_2])$ centered at A and Q and Q' are points of $S_A \cap G_t$, then $\angle QAQ' < \pi/24$. Choose $t < u_2$ and let Q_1, Q_2, Q_3, C and T be as hypothesized. Let Z be a point of Σ in Q_1CQ_3 such that $Z \notin \{Q_1, Q_3\}$. By Lemma 2.8 there is a circular 2-cell K in P_t with radius 1 such that

$$Z \in \operatorname{Bd} K \quad \text{and} \quad K \cap (\Sigma \cup \operatorname{Ext} \Sigma) = \{Z\}.$$

The size of K compared to G_t insures that Bd K contains a minor arc β that intersects Int T and whose boundary lies in one of the three open segments Q_1Q_2, Q_1Q_3, Q_2Q_3.

We now consider the case where the point Z lies on Q_1C, where two subcases result. Case (1a) deals with Bd $\beta \subset Q_1Q_2$ which is the same situation as Bd $\beta \subset Q_1Q_3$. In this case let A be the point of P_t at a distance 1 from both Q_1 and Q_2 and on the non-Q_3-side of $L(Q_1, Q_2)$. By Lemma 2.9, $Z \in \operatorname{Int} S_A$. This means a subsegment Q_1Y of Q_1C, the bisector of the angle of T at Q_1, lies in Int S_A. Let ϕ denote the angle between $L(Q_1, Q_2)$ and the line tangent to S_A at Q_1. Then

$$\angle Q_2Q_1Q_3 < 2\phi = 2(\angle Q_1AM)$$

where M is the midpoint of Q_1Q_2. By the choice of u_2,

$$2(\angle Q_1AM) = \angle Q_1AQ_2 < \pi/24.$$

These facts imply $\angle Q_2Q_1Q_3 < \pi/24$. Thus in Case (1a),

$$\theta(L(Q_1, Q_2), L(Q_1, Q_3)) = \angle Q_2Q_1Q_3 < \pi/24 < \pi/12,$$

as desired.

In Case (1b), let Z belong to Q_1C as in Case (1a), but require that Bd $\beta \subset Q_3Q_2$. Proceed just as in Case (1a) with A now at distance 1 from Q_3 and Q_2. However, in this case subsegments Q_3Y and Q_2Y' of Q_3C and Q_2C can be found to lie in Int S_A. Then, as before, both $\angle Q_1Q_2Q_3$ and

$\angle Q_1Q_3Q_2$ of T are less than $\pi/24$. By the Exterior Angle Theorem it follows that

$$\theta(L(Q_1, Q_2), L(Q_1, Q_3)) = \angle Q_1Q_2Q_3 + \angle Q_1Q_3Q_2 < \pi/24 + \pi/24 = \pi/12.$$

In Case 2 we assume Q_1C does not contain a point of Σ different from Q_1. The hypothesis then requires that both Q_2C and Q_3C intersect Σ at their interiors. Let Z and Z' be points of Σ in the interiors of Q_2C and Q_3C, respectively, and use Lemma 2.8 to obtain two unit circular 2-cells K and K' such that $Z \in \text{Bd}\, K$, $Z' \in \text{Bd}\, K'$, and $(K \cup K') \cap (\Sigma \cup \text{Ext}\, \Sigma) = \{Z, Z'\}$. Then $\text{Bd}\, K$ and $\text{Bd}\, K'$ contain minor arcs β and β', respectively, each intersecting $\text{Int}\, T$ and each having endpoints in the interior of exactly one of the intervals Q_1Q_2, Q_1Q_3, Q_2Q_3. If $\text{Bd}\, \beta \subset Q_3Q_2$ or $\text{Bd}\, \beta \subset Q_1Q_2$, then Case (1a) applies to show $\angle Q_1Q_2Q_3 < \pi/24$. Similarly if $\text{Bd}\, \beta' \subset Q_2Q_3$ or $\text{Bd}\, \beta' \subset Q_1Q_3$, then $\angle Q_1Q_3Q_2 < \pi/24$. In these situations the Exterior Angle Theorem applies to show

$$\theta(L(Q_1, Q_2), L(Q_1, Q_3)) < \pi/12.$$

Otherwise, either $\text{Bd}\, \beta \subset Q_1Q_3$ or $\text{Bd}\, \beta' \subset Q_1Q_2$. These cases are alike, so assume $\text{Bd}\, \beta \subset Q_1Q_2$. As in Case (1b), the angle at Q_1 is less than $\pi/24$. The result follows.

LEMMA 2.11. *If* $0 < t < u_2$ *and* L *and* L' *represent straight lines in* P_t *that each touch two components of* $G_t \cap \text{Ext}\, \Sigma$, *then* $\theta(L, L') < \pi/4$.

Proof. Choose points Q_1 and Q_2 on L and lying in distinct components of $G_t \cap \text{Ext}\, \Sigma$. Let R_1 and R_2 be points of L' in distinct components of $G_t \cap \text{Ext}\, \Sigma$. As a special case, let $R_1 = Q_1$, and let C be the intersection of the angle bisectors of the triangle T determined by Q_1, Q_2, R_2 in P_t. The hypothesis of Lemma 2.10 that both Q_1CR_2 and Q_1CQ_2 intersect Σ is clear, so $\theta(L, L') < \pi/12$ in this case.

In the general case where Q_1, Q_2, R_1, R_2 are distinct, we may assume Q_1 and R_2 are in distinct components of $G_t \cap \text{Ext}\, \Sigma$. Define $L'' = L(Q_1, R_2)$. Then $\theta(L, L'') < \pi/12$, by the special case, and for the same reason $\theta(L'', L') < \pi/12$. Then

$$\theta(L, L') \leqq \theta(L, L'') + \theta(L'', L') < 2(\pi/12) = \pi/6 < \pi/4.$$

LEMMA 2.12. *If* $0 < t < u_2$, L *is a straight line in* P_t *that touches at least two components of* $G_t \cap \text{Ext}\, \Sigma$, *and* L^* *is a line in* P_t *such that* $\theta(L, L^*) < \pi/4$, *then the line* L_t *through* $(0, 0, t)$ *and perpendicular to* L^* *in* P_t *is projective.*

Proof. If L_t were not projective, there would be a line L_t' parallel to L_t such that L_t' touches at least two components of $G_t \cap \text{Ext}\, \Sigma$. Then by

Lemma 2.11, $\theta(L, L_t) < \pi/4$, so that one would deduce

$$\pi/2 = \theta(L^*, L_t) \leqq \theta(L^*, L) + \theta(L, L_t) < 2(\pi/4) = \pi/2,$$

which is impossible.

LEMMA 2.13. *Either Σ is locally flat at p or there exists a continuous family $\{L_t | t \in (0, u_2]\}$ of projective lines.*

Proof. Let A be an arc in $\{p\} \cup \text{Ext } \Sigma$ containing p as an endpoint such that A is locally polyhedral modulo p and no two of its vertices lie in a common plane P_t. Since Σ is locally flat modulo p (to be accurate, Σ is locally flat modulo F, but the points of $F - \{p\}$ are ignored), there exists a "taming" homeomorphism h of E^3 to itself, fixing points of $A \cup (\bigcup\{\text{Bd } G_t | t \in (0, u_1]\})$, such that $h(\Sigma)$ is locally polyhedral modulo p, no two vertices of $h(\Sigma - p)$ lie in a common plane P_t, no vertex of $h(\Sigma - p)$ lies in the same plane as a vertex of A, and $h(\Sigma - p) \subset \text{Int } \Sigma$. Turned inside out this last condition states, of course, that $\text{Ext } h(\Sigma) \supset \text{Ext } \Sigma$ and guarantees that any two points of $\text{Ext } \Sigma$ from distinct components of $G_t \cap \text{Ext } h(\Sigma)$ are found in distinct components of $G_t \cap \text{Ext } \Sigma$. We shift the focus from Σ to $h(\Sigma)$, a small matter since certainly one of these spheres is locally flat at p if and only if the other is.

To construct the desired family of projective lines it is convenient to know that A intersects at least three components of $G_t \cap \text{Ext } h(\Sigma)$ whenever t is sufficiently close to 0. We now show that unless this is true, the desired conclusion that Σ is locally flat at p follows. Let $t(i)$ be a decreasing sequence in $(0, u_1]$ converging to 0 such that A intersects at most two components of $G_{t(i)} \cap \text{Ext } h(\Sigma)$, for each i. The proof of the flatness of Σ at p follows from the construction of a new arc A' such that $p \in \text{Bd } A'$, $A' - \{p\} \subset \text{Ext } \Sigma$, and $A' \cap P_{t(i)}$ is a single point for each i. Such a locally peripherally unknotted arc is flat [**15**] and is sufficient to insure that Σ is locally flat at p [**21**].

The arc A' is constructed inductively using A as a guide. First we may assume that A pierces each plane $P_{t(i)}$ at each intersection and that A is ordered with p as its last point. Let x_1 be the first point of A in $P_{t(1)}$, let V_1 be the component of $G_{t(1)} \cap \text{Ext } \Sigma$ containing x_1, let y_1 be the last point of A in V_1, let μ_1 be the subarc of A bounded by $\{x_1, y_1\}$, and let γ_1 be an arc in V_1 from x_1 to y_1. Then the arc $(A - \mu_1) \cap \gamma_1$ can be adjusted near γ_1 to an arc A_1' whose intersection with V_1 is at most one point. If $A_1' \cap P_{t(1)}$ is a single point, let $A_1' = A_1$. Otherwise, A_1' intersects a second component of $G_{t(1)} \cap \text{Ext } \Sigma$, and the same process can be applied again to adjust A_1'. After a finite number of steps, we produce an arc A_1 meeting $P_{t(1)}$ in a singleton set. This ends the first step of the inductive construction of A', which will be realized as $\{p\} \cup \{\lim A_i\}$, and the remaining steps of the construction should now be clear.

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Restrict u_2 further to be smaller than the number promised by Lemma 2.10.

Next we show that for each $t \in (0, u_2]$ there exist disjoint straight line segments α_t and β_t in A and a neighborhood M_t of t such that $\alpha_t \cap P_s$ and $\beta_t \cap P_s$ are points in distinct components of $G_s \cap \text{Ext } \Sigma$ $(s \in M_t)$. There are several cases to consider, the most obvious one occurring when the level P_t contains no vertex of $A \cup h(\Sigma)$. In case P_t contains a vertex v_t of A, we use the fact that $G_t \cap \text{Ext } h(\Sigma)$ has three components intersecting A to choose points a_t and b_t of $A \cap P_t$ from components W_t and V_t of $G_t \cap \text{Ext } h(\Sigma)$ where $W_t \neq V_t$ and $v_t \notin W_t \cup V_t$, and name disjoint line segments α_t and β_t containing these points in their interiors; if M_t is a neighborhood of t close enough to t that no point of $\text{Bd } \alpha_t \cup \text{Bd } \beta_t$ and no vertex of $h(\Sigma)$ lies in P_s, $s \in M_t$, one can readily show that $\alpha_t \cap P_s$ and $\beta_t \cap P_s$ lie in distinct components of $G_s \cap \text{Ext } h(\Sigma)$ and thus the desired property holds. In case P_t contains a vertex of $h(\Sigma)$, choose two points x and y from distinct components of $G_t \cap \text{Ext } h(\Sigma)$ and name disjoint segments α_t and β_t in A with x and y in their respective interiors. Then a neighborhood M_t of t must exist so that $\alpha_t \cap P_s$ and $\beta_t \cap P_s$ lie in distinct components of $G_s \cap \text{Ext } \Sigma$ for each $s \in M_t$. To verify the existence of M_t, suppose a sequence $\{t(i)\}$ of numbers exists converging to t such that $\alpha_t \cap P_{t(i)}$ and $\beta_t \cap P_{t(i)}$ lie in the same component of $G_{t(i)} \cap \text{Ext } \Sigma$, for each i. Then, for each i, there would be an arc in $P_{t(i)} \cap \text{Ext } \Sigma$ joining $\alpha_t \cap P_{t(i)}$ and $\beta_t \cap P_{t(i)}$. A subsequence of these arcs would converge to a continuum M in $(\Sigma \cup \text{Ext } \Sigma) \cap P_t$ containing x and y. Since

$$M \subset (\Sigma \cup \text{Ext } \Sigma) \cap P_t \subset (\text{Ext } h(\Sigma)) \cap P_t,$$

this contradicts the fact that x and y were chosen in different components of $G_t \cap \text{Ext } h(\Sigma)$.

Mark off a sequence $\{t(i)|i = 0, 1, 2, \ldots\}$ of numbers from $(0, u_2]$, starting with $t(0) = u_2$ and decreasing to 0, so that each interval $[t(i + 1), t(i)]$ is a subset of some M_t. Consequently, for $i = 1, 2, \ldots$ the choice of some M_t containing $[t(i), t(i - 1)]$ secures straight line segments α_i and β_i for which $\alpha_i \cap P_s$ and $\beta_i \cap P_s$ are points from different components of $G_s \cap \text{Ext } \Sigma$ when $s \in [t(i), t(i - 1)]$. By Lemma 2.12, the line L_s in P_s through $(0, 0, s)$ perpendicular to $L(\alpha_i \cap P_s, \beta_i \cap P_s)$ is projective. It should be obvious that for $i = 1, 2, \ldots$ these perpendiculars $\{L_s|s \in [t(i), t(i - 1)]\}$ form a continuous family. The only trouble results from ambiguity at the levels $t(i)$ of overlap. The ambiguity is small in scale, for according to Lemma 2.11,

$$\theta(L(\alpha_i \cap P_{t(i)}, \beta_i \cap P_{t(i)}), L(\alpha_{i+1} \cap P_{t(i)}, \beta_{i+1} \cap P_{t(i)}) < \pi/4.$$

Consequently, for $s \in [t(i), t(i + 1))$ near $t(i)$ we can twist the lines L_s, limiting to $\pi/4$ the angular modification in any one line, until the twisted bottom line is perpendicular to $L(\alpha_{i+1} \cap P_{t(i)}, \beta_{i+1} \cap P_{t(i)})$, thereby forming a well-defined, continuous family $\{L_s | s \in (0, u_2]\}$. Lemma 2.12 supports the final claim that such lines are projective.

The flatness of a 2-sphere or a 2-manifold Σ in E^3 can be detected with less stringent conditions on the uniform tangent balls. In the following theorem and its corollary the tangent balls are not required to all lie on the same side of Σ.

THEOREM 2.14. *If K is a 2-cell in a 2-manifold Σ in E^3 and δ is a positive number such that, for each $p \in K$, there is a ball of radius δ tangent to Σ at p, then Σ is locally flat at each point of* Int K.

Proof. Local separation properties imply that there is a connected neighborhood N of K such that $N - \Sigma$ has two components U and V, each with limit points in K. Restricting δ, if necessary, we can suppose each $B \in \mathscr{B}$ tangent to Σ at a point $p \in K$ lies either in Cl U or in Cl V. Define subsets I and E of Int K by placing p in I or E, respectively, depending on the existence of a ball B of $\mathscr{B}$ tangent to Σ at p with $B \subset$ Cl U or with $B \subset$ Cl V. Then Int $K \subset I \cup E$, and Σ has uniform double tangent balls over $I \cap E$. From an obvious local version of Theorem 2.7, Σ is locally flat at each point of Int $K - (I \cap E)$. By Corollary 6 of [**8**], it is also locally flat at points of $I \cap E$.

COROLLARY 2.15. *A 2-manifold Σ in E^3 is locally flat if there exists a positive number δ such that every point of Σ lies in a round 3-cell of radius δ whose interior misses Σ.*

3. Wild spheres in $E^n (n > 3)$ with uniform exterior tangent balls everywhere. These examples are obtained by the technique of inflating crumpled $(n - 1)$-cubes, as first described by Daverman [**9**, Section 11]. The basis for this procedure is a crumpled $(n - 1)$-cube C in E^{n-1} (that is, C is the closure of the bounded complementary domain of an $(n - 2)$-sphere in E^{n-1}) such that the interior of C is not 1-LC at any point of Bd C and that $C \cup_{\text{Id}} C$ is homeomorphic to S^{n-1}. Perhaps the simplest example to cite is the $(n - 4)$-fold suspension C of a crumpled 3-cube C', like that of Bing [**3**], where $C' \cup_{\text{Id}} C'$ is topologically S^3 [**11**, Corollary 2]; then $C \cup_{\text{Id}} C$ is naturally homeomorphic to the $(n - 4)$-fold suspension of $C' \cup_{\text{Id}} C'$, which is S^{n-1}. For technical reasons we require the diameter of $C \subset E^{n-1}$ to be less than 1.

Now we describe a kind of rolled inflation of C in terms of the inflation function

$$f(x) = 1 - [1 - (d(x, \text{Bd } C))^2]^{1/2}$$

defined on C. Typically with this procedure, the $(n - 1)$-sphere Σ we want is

$$\Sigma = \{(x, t) \in E^{n-1} \times E^1 | x \in C, t = \pm f(x)\}.$$

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Since Σ decomposes into upper and lower sections corresponding to the graphs of f and $-f$, respectively, intersecting in the set Bd $C \times \{0\}$, one can identify Σ with $C \cup_{\mathrm{Id}} C$ to see that it is an $(n-1)$-sphere. Furthermore, it is wildly embedded at every point of the $(n-2)$-sphere Bd $C \times \{0\}$, because Int $C \times \{0\}$ is a strong deformation retract of Int Σ via vertical deformation and, therefore, Int Σ is not $1 - LC$ at any point of Bd $C \times \{0\}$. With the particular inflation function $f(x)$ named here, one can easily show that the set $\mathscr{B}$ of balls having radius 1 and centered at points of Bd $C \times \{\pm 1\}$ constitutes a collection of uniform exterior tangent balls over Σ.

4. Applications to ϵ-boundaries in E^3. Following earlier discoveries by Brown [**7**] and Gariepy and Pepe [**14**], Ferry [**12**] proved that ϵ-boundaries of sets in E^3 are 2-manifolds for almost all ϵ. He also proved a collaring theorem [**12**, Theorem 4.1] that implies the flatness of an ϵ-boundary $\partial(\epsilon, A)$ of a subset A of E^n if it is a codimension-one manifold and ϵ lies in an interval containing none of his "critical values". Theorem 2.7 of Section 2 applies in case $n = 3$ to yield the local flatness of $\partial(\epsilon, A)$ at each point where it is a 2-manifold. This answers in the affirmative a question by Weill [**23**, p. 248].

THEOREM 4.1. *If A is a subset of E^3, $\epsilon > 0$, D is an open subset of $\partial(\epsilon, A)$, and D is a 2-manifold, then D is locally flat in E^3.*

Proof. Let $p \in D$. Choose a 2-cell M' in D such that $p \in$ Int M' and M' lies in a 2-sphere Σ [**2**, Theorem 5]. For each $x \in M'$ there is a point σ_x in the closure of A such that $d(x, \sigma_x) = \epsilon$. The ball B_x centered at σ_x with radius ϵ intersects M' only in its boundary. Let $\mathscr{B}$ be the collection of all such balls as x varies over M'. Let M be a 2-cell in Int M' with $p \in$ Int M, and let $\delta = \min\{\epsilon, d(M, \Sigma - M')\}$. For each $x \in M$ choose a ball B_x' in some element of $\mathscr{B}$ such that $x \in$ Bd B_x' and B_x' has radius δ, and let $\mathscr{B}'$ be the resulting collection of such balls. Then $\mathscr{B}'$ is a uniform collection of tangent balls to Σ over M, and by Theorem 2.14 Σ is locally flat at p. The result follows.

COROLLARY 4.2. *If $A \subset E^3$ and ϵ is a positive number such that the ϵ-boundary $\partial(\epsilon, A)$ of A is a 2-manifold, then $\partial(\epsilon, A)$ is locally flat in E^3.*

The status of the higher dimensional analogue to Corollary 4.2 remains an open question; however, a partial solution is worth noting.

THEOREM 4.3. *If $A \subset E^n$, $\epsilon > 0$, and $\partial(\epsilon, A)$ is a connected $(n-1)$-manifold such that A intersects both complementary domains of $\partial(\epsilon, A)$ in E^n, then $\partial(\epsilon, A)$ is flatly embedded in E^n.*

Proof. The hypothesis easily implies that $\partial(\epsilon, A)$ has uniform double tangent balls (see [**18**, Theorem 5.1]), so Theorem 4.3 follows from Corollary 2.3 of [**10**].

References

1. R. H. Bing, *Approximating surfaces with polyhedral ones*, Ann. of Math. *65* (1957), 456–483.
2. ——— *A surface is tame if its complement is* 1-ULC, Trans. Amer. Math. Soc. *101* (1961), 294–305.
3. ——— *A wild surface each of whose arcs is tame*, Duke Math. J. *28* (1961), 1–15.
4. ——— *Spheres in* E^3, Amer. Math. Monthly *71* (1964), 353–364.
5. H. G. Bothe, *Differenzierbare flächen sind sahm*, Math. Nachr. *43* (1970), 161–180.
6. M. Brown, *Locally flat imbeddings of topological manifolds*, Ann. of Math. *75* (1962), 331–341.
7. ——— *Sets of constant distance from a planar set*, Michigan Math. J. *19* (1972), 321–323.
8. J. W. Cannon, **-taming sets for crumpled cubes. II: Horizontal sections in closed sets*, Trans. Amer. Math. Soc. *161* (1971), 441–446.
9. R. J. Daverman, *Sewings of closed n-cell-complements*, Trans. Amer. Math. Soc., to appear.
10. R. J. Daverman and L. D. Loveland, *Wildness and flatness of codimension one spheres having double tangent balls*, Rocky Mountain J. Math., to appear.
11. W. T. Eaton, *The sum of solid spheres*, Michigan Math. J. *19* (1972), 193–207.
12. S. Ferry, *When* ϵ*-boundaries are manifolds*, Fund. Math. *90* (1976), 199–210.
13. R. H. Fox and E. Artin, *Some wild cells and spheres in three-dimensional space*, Ann. of Math. *49* (1948), 979–990.
14. R. Gariepy and W. D. Pepe, *On the level sets of a distance function in a Minkowski space*, Proc. Amer. Math. Soc. *31* (1972), 255–259.
15. O. G. Harrold, Jr., *Locally peripherally unknotted surfaces in* E^3, Ann. of Math. *69* (1959), 276–290.
16. H. C. Griffith, *Spheres uniformly wedged between balls are tame in* E^3, Amer. Math. Monthly *75* (1968), 767.
17. L. D. Loveland, *A surface is tame if it has round tangent balls*, Trans. Amer. Math. Soc. *152* (1970), 389–397.
18. ——— *When midsets are manifolds*, Proc. Amer. Math. Soc. *61* (1976), 353–360.
19. ——— *Crumpled cubes that are finite unions of cells*, Houston J. Math. *4* (1978), 223–228.
20. ——— *Unions of cells with applications to visibility*, Proc. Amer. Math. Soc., to appear.
21. D. R. McMillan, Jr., *Some topological properties of piercing points*, Pacific J. Math. *22* (1967), 313–322.
22. C. P. Pixley, *On crumpled cubes in* S^3 *which are the finite union of tame* 3*-cells*, Houston J. Math. *4* (1978), 105–112.
23. L. R. Weill, *A new characterization of tame* 2*-spheres in* E^3, Trans. Amer. Math. Soc. *190* (1974), 243–252.

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