Canad. Math. Bull. Vol. **58** (1), 2015 pp. 19–29
http://dx.doi.org/10.4153/CMB-2014-042-1

Compact Commutators of Rough Singular Integral Operators

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Abstract. Let $b \in \mathrm{BMO}(\mathbb{R}^n)$ and T_Ω be the singular integral operator with kernel $\Omega(x)/|x|^n$, where Ω is homogeneous of degree zero, integrable, and has mean value zero on the unit sphere S^{n-1}. In this paper, using Fourier transform estimates and approximation to the operator T_Ω by integral operators with smooth kernels, it is proved that if $b \in \mathrm{CMO}(\mathbb{R}^n)$ and Ω satisfies certain minimal size condition, then the commutator generated by b and T_Ω is a compact operator on $L^p(\mathbb{R}^n)$ for appropriate index p. The associated maximal operator is also considered.

1 Introduction

We will work on $\mathbb{R}^n$, $n \geq 2$. Let Ω be homogeneous of degree zero, integrable, and have mean value zero on the unit sphere S^{n-1}. Define the singular integral operator T_Ω by

$$T_\Omega f(x) = \mathrm{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)dy. \tag{1.1}$$

The maximal operator associated with T_Ω is defined by

$$T^\star_\Omega f(x) = \sup_{k \in \mathbb{Z}} \Big| \int_{|x-y|>2^k} \frac{\Omega(x-y)}{|x-y|^n} f(y)dy \Big|.$$

These operators were introduced by Calderón and Zygmund [3], and have been studied by many authors over the last sixty years. Calderón–Zygmund [4] proved that if $\Omega \in L\ln L(S^{n-1})$, then T_Ω and $T^\star_\Omega$ are bounded on $L^p(\mathbb{R}^n)$ for $p \in (1, \infty)$. Connett [9], Ricci and Weiss [20] improved the Calderón–Zygmund result and showed that $\Omega \in H^1(S^{n-1})$ guarantees the $L^p(\mathbb{R}^n)$ boundedness of T_Ω for $p \in (1, \infty)$. Seeger [21] showed that $\Omega \in L\ln L(S^{n-1})$ is a sufficient condition that T_Ω is bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$. Duoandikoetxea and Rubio de Francia [11], Duoandikoetxea [10], and Watson [23] considered independently the weighted estimates for T_Ω and $T^\star_\Omega$ when $\Omega \in L^q(S^{n-1})$ with $q \in (1, \infty]$. Grafakos and Stefanov [16] considered the L^p boundedness for T_Ω and $T^\star_\Omega$ when Ω satisfies the size condition that

$$\sup_{\zeta \in S^{n-1}} \int_{S^{n-1}} |\Omega(\eta)| \Big(\ln \frac{1}{|\eta \cdot \zeta|} \Big)^\theta d\eta < \infty, \tag{1.2}$$

Received by the editors April 22, 2014.
Published electronically October 20, 2014.
The research of the first author was supported by the NNSF of China under grant #11271330, and the research of the second (corresponding) author was supported by the NNSF of China under grant #11371370.
AMS subject classification: **42B20**.
Keywords: commutator,singular integral operator, compact operator, maximal operator.

and proved that if $\theta > 1$, then T_Ω is bounded on $L^p(\mathbb{R}^n)$ for $p \in ((1+\theta)/\theta, 1+\theta)$. Equation (1.2) can be regarded as a minimal size condition, since there exists function $\Omega \notin H^1(S^{n-1})$, but satisfies (1.2) for any $\theta \in (1, \infty)$ (see [16]). There are many other works about the mapping properties of T_Ω when Ω satisfies minimal size conditions. Among them, we mention the papers [5, 12, 13] and the references therein.

The commutator generated by T_Ω and $\mathrm{BMO}(\mathbb{R}^n)$ functions is also of interest. Let $b \in \mathrm{BMO}(\mathbb{R}^n)$, the space of functions of bounded mean oscillation introduced by John and Nirenberg. Define the commutator of T_Ω and b by

$$T_{\Omega,b}f(x) = b(x)T_\Omega f(x) - T_\Omega(bf)(x),$$

initially for $f \in \mathcal{S}(\mathbb{R}^n)$. As usual, the maximal operator associated with $T_{\Omega,b}$ is defined as

$$T^\star_{\Omega,b}f(x) = \sup_{j\in\mathbb{Z}}\Big|\int_{|x-y|>2^j}\big(b(x)-b(y)\big)\frac{\Omega(x-y)}{|x-y|^n}f(y)\,\mathrm{d}y\Big|.$$

Coifman, Rochberg, and Weiss [7] proved that if $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$ $(\alpha \in (0, 1))$, then $T_{\Omega,b}$ is bounded on $L^p(\mathbb{R}^n)$ $(p \in (1, \infty))$ if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$. Using the weighted estimates with $A_p(\mathbb{R}^n)$, weights of T_Ω, and the relation of A_p weights and $\mathrm{BMO}(\mathbb{R}^n)$ functions, Alvarez et al. [1] established the $L^p(\mathbb{R}^n)$ boundedness of $T_{\Omega,b}$ when $\Omega \in L^q(S^{n-1})$ for some $q \in (1, \infty]$. Hu [17] proved that $\Omega \in L(\ln L)^2(S^{n-1})$ is a sufficient condition such that $T_{\Omega,b}$ and $T^\star_{\Omega,b}$ are bounded on $L^p(\mathbb{R}^n)$ with bound $C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$ for all $p \in (1, \infty)$. For the case where Ω satisfies (1.2) for some $\theta > 2$, Hu, Sun, and Wang [19] showed that $T_{\Omega,b}$ is bounded on $L^p(S^{n-1})$ with bound $C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$ provided that $p \in \big(\theta/(\theta-1), \theta\big)$. Moreover, Hu [18] proved that if Ω satisfies (1.2) for $\theta > 5/2$, then $T^\star_{\Omega,b}$ is bounded on $L^p(\mathbb{R}^n)$ with bound $C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$ for $p \in \big(4\theta/(4\theta-5), 4\theta/5\big)$.

The compactness of $T_{\Omega,b}$ was first considered by Uchiyama in his remarkable work [22]. Let $\mathrm{CMO}(\mathbb{R}^n)$ be the closure of $C_0^\infty(\mathbb{R}^n)$ in the $\mathrm{BMO}(\mathbb{R}^n)$ topology, which coincide with the space of functions of vanishing mean oscillation; see [2, 8]. For the case of $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$ $(\alpha \in (0, 1))$, Uchiyama proved that $T_{\Omega,b}$ is compact on $L^p(\mathbb{R}^n)$ if and only if $b \in \mathrm{CMO}(\mathbb{R}^n)$. Chen *et al.* [6] generalized the result in [22] and considered the compactness of $T_{\Omega,b}$ on Morrey space when Ω satisfies a certain regularity condition of L^q-Dini type. The purpose of this paper is to consider the compactness on $L^p(\mathbb{R}^n)$ for $T_{\Omega,b}$ and $T^\star_{\Omega,b}$ when Ω satisfies (1.2) for some $\theta > 2$. Our main results can be stated as follows.

Theorem 1.1 *Let Ω be homogeneous of degree zero, integrable, and have mean value zero on S^{n-1}. Suppose that $b \in \mathrm{CMO}(\mathbb{R}^n)$ and Ω satisfies (1.2) for some $\theta > 2$. Then for $p \in \big(\theta/(\theta-1), \theta\big)$, the operator $T_{\Omega,b}$ is compact on $L^p(\mathbb{R}^n)$.*

Theorem 1.2 *Let Ω be homogeneous of degree zero, integrable, and have mean value zero on S^{n-1}. Suppose that $b \in \mathrm{CMO}(\mathbb{R}^n)$ and Ω satisfies (1.2) for some $\theta > 5/2$. Then for $p \in \big(4\theta/(4\theta-5), 4\theta/5\big)$, $T^*_{\Omega,b}$ is compact on $L^p(\mathbb{R}^n)$.*

We establish some conventions. In what follows, C always denotes a positive constant that is independent of the main parameters involved but whose value may differ

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from line to line. We use the symbol $A \lesssim B$ to denote that there exists a positive constant C such that $A \leq CB$. For a set $E \subset \mathbb{R}^n$, χ_E denotes its characteristic function. For $p \in [1, \infty]$, we use p' to denote the dual exponent of p, namely, $p' = p/(p-1)$. For a suitable function f, let $\widehat{f}$ denote the Fourier transform of f.

2 Approximation

This section is devoted to approximations to the operators T_Ω and $T_\Omega^\star$ by some integral operators with smooth kernels. We remark that here we are very much motivated by the work of Watson [23].

For each $l \in \mathbb{Z}$, let $K_\Omega^l(y) = \frac{\Omega(y)}{|y|^n}\chi_{\{2^l<|y|\leq 2^{l+1}\}}(y)$. By integrability and the vanishing moment of Ω, it is easy to verify that

$$|\widehat{K_\Omega^l}(\xi)| \lesssim \min\{1, |2^l\xi|\}. \tag{2.1}$$

As proved in [16], if Ω satisfies (1.2) for some $\theta > 1$, then

$$|\widehat{K_\Omega^l}(\xi)| \lesssim \ln^{-\theta}\left(2 + |2^l\xi|\right). \tag{2.2}$$

Let $\phi \in C_0^\infty(\mathbb{R}^n)$ be a nonnegative function such that

$$\int_{\mathbb{R}^n} \phi(x)\mathrm{d}x = 1, \quad \operatorname{supp}\phi \subset \{x : |x| \leq 1/4\}.$$

For $l \in \mathbb{Z}$, let $\phi_l(y) = 2^{-nl}\phi(2^{-l}y)$. We then have

$$|\widehat{\phi_l}(\xi) - 1| = |\widehat{\phi}(2^l\xi) - 1| \lesssim \min\{1, |2^l\xi|\}. \tag{2.3}$$

for $\xi \in \mathbb{R}^n$. For a positive integer j, let

$$K^j(y) = \sum_{l=-\infty}^{\infty} K_\Omega^l * \phi_{l-j}(y), \tag{2.4}$$

and let T_Ω^j be the convolution operator be given by

$$T_\Omega^j f(x) = \text{p.v.}\int_{\mathbb{R}^n} K^j(x-y)f(y)\mathrm{d}y. \tag{2.5}$$

Set $K^{j,k}(y) = \sum_{l=k}^{\infty} K_\Omega^l * \phi_{l-j}(y)$. Define the maximal operator associated with T_Ω^j by

$$T_\Omega^{j,\star}f(x) = \sup_{k\in\mathbb{Z}}\Big|\int_{\mathbb{R}^n} K^{j,k}(x-y)f(y)\mathrm{d}y\Big|.$$

Lemma 2.1 *Let Ω be homogeneous of degree zero and belong to $L^1(S^{n-1})$, and let K^j be the function defined as in (2.4). Then for any $y \in \mathbb{R}^n$ and $R > 0$ with $R > 4|y|$,*

$$\sum_{l\in\mathbb{Z}}\int_{|x|>R}\left|K_\Omega^l * \phi_{l-j}(x+y) - K_\Omega^l * \phi_{l-j}(x)\right|\mathrm{d}x \lesssim \min\{j, 2^j|y|/R\}. \tag{2.6}$$

Proof For fixed $R > 0$ and positive integer j, let l_0 be the integer such that $R < 2^{l_0+2} \leq 2R$. Observe that $\operatorname{supp} K_\Omega^l * \phi_{l-j} \subset \{x : 2^{l-1} \leq |x| \leq 2^{l+2}\}$ and

$$\|\phi_{l-j}(\cdot + y) - \phi_{l-j}(\cdot)\|_{L^1(\mathbb{R}^n)} \lesssim \min\{1, 2^{j-l}|y|\}.$$

It follows that

$$\sum_{l=l_0}^{\infty} \|\phi_{l-j}(\cdot+y)-\phi_{l-j}(\cdot)\|_{L^1(\mathbb{R}^n)} \le 2^j|y|\sum_{l=l_0}^{\infty} 2^{-l} \lesssim 2^j|y|/R,$$
$$\sum_{l=l_0}^{\infty} \|\phi_{l-j}(\cdot+y)-\phi_{l-j}(\cdot)\|_{L^1(\mathbb{R}^n)} \le j+\sum_{l=l_0+j}^{\infty} 2^{j-l}|y| \lesssim j.$$

Therefore,

$$\begin{aligned}\sum_{l\in\mathbb{Z}}\int_{|x|>R}\big|K_\Omega^l*\phi_{l-j}(x+y)-K_\Omega^l*\phi_{l-j}(x)\big|\,\mathrm{d}x &\lesssim \sum_{l=l_0}^{\infty}\big\|K_\Omega^l*\phi_{l-j}(\cdot+y)-K_\Omega^l*\phi_{l-j}(\cdot)\big\|_{L^1(\mathbb{R}^n)}\\ &\lesssim \sum_{l=l_0}^{\infty}\|K_\Omega^l\|_{L^1(\mathbb{R}^n)}\big\|\phi_{l-j}(\cdot+y)-\phi_{l-j}(\cdot)\big\|_{L^1(\mathbb{R}^n)}\\ &\lesssim \sum_{l=l_0}^{\infty}\big\|\phi_{l-j}(\cdot+y)-\phi_{l-j}(\cdot)\big\|_{L^1(\mathbb{R}^n)}\\ &\lesssim \min\{j,\,2^j|y|/R\}.\end{aligned}$$

This establishes (2.6). ■

Lemma 2.2 *Let Ω be homogeneous of degree zero and have mean value zero, and let Ω satisfy (1.2) for some $\theta\in(1,\infty)$. Then for $p\in(1,\infty)$, both of the operator T_Ω^j and the operator $T_\Omega^{j,\circ}$ defined by*

$$T_\Omega^{j,\circ}f(x)=\sup_{\epsilon>0}\Big|\int_{|x-y|>\epsilon}K^j(x-y)f(y)\mathrm{d}y\Big|$$

are bounded on $L^p(\mathbb{R}^n)$ with bound Cj.

Proof By the estimates (2.1) and (2.2), we see that for $\xi\in\mathbb{R}^n\backslash\{0\}$,

$$\sum_{l=-\infty}^{\infty}|\widehat{K_\Omega^l}(\xi)\widehat{\phi}(2^{l-j}\xi)|\lesssim\sum_{l=-\infty}^{\infty}|\widehat{K_\Omega^l}(\xi)|\lesssim 1.$$

Thus, by the Plancherel theorem, T_Ω^j is bounded on $L^2(\mathbb{R}^n)$ with bound depending only on n. This, via Lemma 2.1 and classical singular integral operator theory (see [15]), tells us that T_Ω^j is bounded on $L^p(\mathbb{R}^n)$ with bound Cj. To prove the $L^p(\mathbb{R}^n)$ boundedness of $T_\Omega^{j,\circ}$, note that for any $R>0$,

$$\int_{R<|y|\le 2R}|K^j(y)|\mathrm{d}y\lesssim\sum_{l\in\mathbb{Z}:\,2^l\approx R}\|K_\Omega^l\|_{L^1(\mathbb{R}^n)}\|\phi_{l-j}\|_{L^1(\mathbb{R}^n)}\lesssim 1. \tag{2.7}$$

Lemma 2.1 tells us that for any $y\in\mathbb{R}^n$ and $R>0$ with $R>4|y|$,

$$\int_{|x|>R}\big|K^j(x-y)-K^j(x)\big|\,\mathrm{d}x\lesssim j.$$

This, along with the $L^p(\mathbb{R}^n)$ boundedness of T_Ω^j and [14, Theorem 1], shows that $T_\Omega^{j,\circ}$ is bounded on $L^p(\mathbb{R}^n)$ with bound Cj. ■

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The following result plays an important role in the proofs of our theorems and is of independent interest.

Theorem 2.3 *Let Ω be homogeneous of degree zero and have mean value zero, and let T_Ω, T_Ω^j be the operators defined by (1.1) and (2.5) respectively.*

(i) *If Ω satisfies (1.2) for some $\theta \in (1, \infty)$, then for any $p \in (2\theta/(2\theta-1), 2\theta)$ and $\varepsilon \in (0, \infty)$,*

$$\|T_\Omega f - T_\Omega^{2^j} f\|_{L^p(\mathbb{R}^n)} \lesssim 2^{j(-2\theta \min\{1/p, 1/p'\}+1+\varepsilon)} \|f\|_{L^p(\mathbb{R}^n)}. \tag{2.8}$$

(ii) *If Ω satisfies (1.2) for some $\theta \in (3/2, \infty)$, then for any $p \in \big(\theta/(\theta-1), \theta\big)$, there exists a constant $\sigma = \sigma_{p,\theta} > 0$ such that*

$$\Big\| \sup_{k\in\mathbb{Z}} \Big| \sum_{l=k}^{\infty} S_l^j * f \Big| \Big\|_{L^p(\mathbb{R}^n)} \lesssim j^{-\sigma} \|f\|_{L^p(\mathbb{R}^n)}, \tag{2.9}$$

where and in the following, for $l \in \mathbb{Z}$ and $j \in \mathbb{N}$,

$$S_l^j(y) = K_\Omega^l * \phi_{l-j}(y) - K_\Omega^l(y).$$

Proof For each $\xi \in \mathbb{R}^n \backslash \{0\}$ and positive integer j, let l_0 be the integer such that $2^{j/2-1} < |2^{l_0}\xi| \le 2^{j/2}$. A trivial computation involving the Fourier transform estimates (2.1)–(2.3) leads to

$$\sum_{l=-\infty}^{\infty} \big| \widehat{K_\Omega^l}(\xi)\widehat{\phi}(2^{j-l}\xi) - \widehat{K_\Omega^l}(\xi) \big| \lesssim \sum_{l\in\mathbb{Z}:\, l\le l_0} |2^{l-j}\xi| + \sum_{l\in\mathbb{Z}:\, l>l_0} \ln^{-\theta}(|2^l\xi|) \lesssim j^{-\theta+1}.$$

This, via the Plancherel theorem, leads to

$$\|T_\Omega f - T_\Omega^j\|_{L^2(\mathbb{R}^n)} \lesssim j^{-\theta+1} \|f\|_{L^2(\mathbb{R}^n)}$$

directly. Therefore,

$$\|T_\Omega^j f - T_\Omega^{j+1} f\|_{L^2(\mathbb{R}^n)} \lesssim j^{-\theta+1} \|f\|_{L^2(\mathbb{R}^n)}, \tag{2.10}$$

and so

$$T_\Omega - T_\Omega^{2^j} = \sum_{m=j}^{\infty} \big(T_\Omega^{2^{m+1}} - T_\Omega^{2^m} \big) \tag{2.11}$$

converges in the $L^2(\mathbb{R}^n)$ operator norm. On the other hand, Lemmas 2.1 and 2.2 tell us that for any positive integer m and $q \in (1, \infty)$,

$$\|T_\Omega^m f - T_\Omega^{m+1} f\|_{L^q(\mathbb{R}^n)} \lesssim m \|f\|_{L^q(\mathbb{R}^n)}. \tag{2.12}$$

Interpolation of inequalities (2.10) and (2.12) then shows that if $p \in (1, \infty)$, then for any $\varepsilon \in (0, \infty)$,

$$\|T_\Omega^m f - T_\Omega^{m+1} f\|_{L^p(\mathbb{R}^n)} \lesssim m^{-2\theta \min\{1/p, 1/p'\}+1+\varepsilon} \|f\|_{L^p(\mathbb{R}^n)},$$

which, along with (2.11), yields (2.8).

We turn our attention to the estimate (2.9). We will employ the ideas used in [11], with appropriate modifications. Let $\psi \in C_0^\infty$ such that

$$\operatorname{supp} \psi \subset \{x \in \mathbb{R}^n : |x| \le 2^{j/2+1}\}, \quad \psi(x) \equiv 1 \text{ if } |x| \le 2^{j/2}.$$

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For each integer k, let $\Psi_k \in \mathcal{S}(\mathbb{R}^n)$ such that $\widehat{\Psi}_k(\xi) = \psi(2^k\xi)$. For each fixed $k \in \mathbb{Z}$, write

$$\begin{aligned}\sum_{l=k}^{\infty} S_l^j * f(x) &= \Psi_k * \big(T_\Omega f - T_\Omega^j f\big)(x) - \Psi_k * \Big(\sum_{l=-\infty}^{k-1} S_l^j * f\Big)(x)\\ &\quad + \sum_{l=k}^{\infty}(\delta - \Psi_k) * S_l^j * f(x)\\ &= \mathrm{I}_k^j f(x) + \mathrm{II}_k^j f(x) + \mathrm{III}_k^j f(x),\end{aligned}$$

with δ the Dirac distribution. It is obvious that

$$\big|\mathrm{I}_k^j f(x)\big| \lesssim M\big(T_\Omega f - T_\Omega^j f\big)(x),$$

with M the Hardy–Littlewood Maximal operator, and so by (2.10)

$$\big\|\sup_{k\in\mathbb{Z}} |\mathrm{I}_k^j f|\big\|_{L^2(\mathbb{R}^n)} \lesssim \|T_\Omega f - T_\Omega^j f\|_{L^2(\mathbb{R}^n)} \lesssim j^{-\theta+1}\|f\|_{L^2(\mathbb{R}^n)}.$$

To give the desired estimate for $\sup_{k\in\mathbb{Z}} |\mathrm{II}_k^j f|$, write

$$\sup_{k\in\mathbb{Z}} |\mathrm{II}_k^j f(x)| \lesssim \Big(\sum_{u=-\infty}^{\infty}\Big|\Psi_u * \sum_{l=-\infty}^{u-1} S_l^j * f(x)\Big|^2\Big)^{1/2}.$$

Note that for any $\xi \in \mathbb{R}^n$,

$$\Big|\psi(2^u\xi)\sum_{l=-\infty}^{u-1}\widehat{K}_\Omega^l(\xi)\big(\widehat{\phi}(2^{l-j}\xi) - 1\big)\Big| \lesssim \Big|\psi(2^u\xi)\sum_{l=-\infty}^{u-1}|2^{l-j}\xi|\Big| \lesssim 2^{-j}\psi(2^u\xi)|2^u\xi|.$$

Therefore, we have by the Plancherel theorem that

$$\begin{aligned}\big\|\sup_{k\in\mathbb{Z}} |\mathrm{II}_k^j f|\big\|^2_{L^2(\mathbb{R}^n)} &= \sum_{u=-\infty}^{\infty}\Big\|\Psi_u * \sum_{l=-\infty}^{u-1} S_l^j * f\Big\|^2_{L^2(\mathbb{R}^n)}\\ &= \sum_{u=-\infty}^{\infty}\int_{\mathbb{R}^n}\Big|\sum_{l=-\infty}^{u-1}\widehat{K_\Omega^l}(\xi)\big(\widehat{\phi}(2^{l-j}\xi) - 1\big)\Big|^2|\psi(2^u\xi)\widehat{f}(\xi)|^2 d\xi\\ &\lesssim 2^{-2j}\int_{\mathbb{R}^n}\sum_{u=-\infty}^{\infty}|\psi(2^u\xi)|^2|2^u\xi|^2\big|\widehat{f}(\xi)\big|^2 d\xi.\end{aligned}$$

Recalling that $\operatorname{supp}\psi \subset \{x : |x| \le 2^{j/2+1}\}$, we thus get that

$$\big\|\sup_{k\in\mathbb{Z}} |\mathrm{II}_k^j f|\big\|_{L^2(\mathbb{R}^n)} \lesssim 2^{-j/2}\|f\|_{L^2(\mathbb{R}^n)}.$$

As for the term $\sup_{k\in\mathbb{Z}} |\mathrm{III}_k^j f|$, write

$$\begin{aligned}\sup_{k\in\mathbb{Z}} |\mathrm{III}_k^j f(x)| &\le \sum_{l=0}^{\infty}\sup_{k\in\mathbb{Z}}\big|(\delta - \Psi_k) * S_{l+k}^j * f(x)\big|\\ &\lesssim \sum_{l=0}^{\infty}\Big(\sum_{u=-\infty}^{\infty}\big|(\delta - \Psi_{u-l}) * S_u^j * f(x)\big|^2\Big)^{1/2}.\end{aligned}$$

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An application of (2.2) and (2.3) tells us that

$$\begin{aligned}
&\Big\| \Big(\sum_{u=-\infty}^{\infty} \Big|(\delta - \Psi_{u-l}) * S_u^j * f(x)\Big|^2 \Big)^{1/2} \Big\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad = \sum_{u=-\infty}^{\infty} \int_{\mathbb{R}^n} \big|1 - \psi(2^{u-l}\xi)\big|^2 \Big|\widehat{K}_\Omega^u(\xi)\big(\widehat{\phi}(2^{u-j}\xi) - 1\big)\Big|^2 |\widehat{f}(\xi)|^2 d\xi \\
&\quad \lesssim \int_{\mathbb{R}^n} \sum_{u=-\infty}^{\infty} |1 - \psi(2^{u-l}\xi)|^2 \ln^{-2\theta}\big(2 + |2^u\xi|\big) \big|\widehat{f}(\xi)\big|^2 d\xi \\
&\quad \lesssim (l+j)^{-2\theta+1} \|f\|_{L^2(\mathbb{R}^n)}^2,
\end{aligned}$$

since for each $\xi \in \mathbb{R}^n \backslash \{0\}$,

$$\begin{aligned}
\sum_{u=-\infty}^{\infty} |1 - \psi(2^{u-l}\xi)|^2 \ln^{-2\theta}\big(2 + |2^u\xi|\big) &\lesssim \sum_{u:\, |2^u\xi| \geq 2^{l+j/2}} \ln^{-2\theta}\big(2 + |2^u\xi|\big) \\
&\lesssim (l+j)^{-2\theta+1}.
\end{aligned}$$

Thus,

$$\big\| \sup_{k\in\mathbb{Z}} |\mathrm{III}_k^j f| \big\|_{L^2(\mathbb{R}^n)} \lesssim j^{-\theta+3/2} \|f\|_{L^2(\mathbb{R}^n)}.$$

Combining the estimates for $\sup_{k\in\mathbb{Z}} |\mathrm{I}_k^j f|$, $\sup_{k\in\mathbb{Z}} |\mathrm{II}_k^j f|$ and $\sup_{k\in\mathbb{Z}} |\mathrm{III}_k^j f|$ leads to

$$\Big\| \sup_{k\in\mathbb{Z}} \Big| \sum_{l=k}^{\infty} S_l^j * f \Big| \Big\|_{L^2(\mathbb{R}^n)} \lesssim j^{-\theta+3/2} \|f\|_{L^2(\mathbb{R}^n)}. \tag{2.13}$$

Recall that $\|T_\Omega^\star f\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^q(\mathbb{R}^n)}$ when $q \in ((2\theta-1)/(2\theta-2),\, 2\theta-1)$; see [13]. By the estimate

$$\sup_{k\in\mathbb{Z}} \Big| \sum_{l=k}^{\infty} S_l^j * f(x) \Big| \lesssim T_\Omega^\star f(x) + T_\Omega^{j,\star} f(x),$$

we deduce from Lemma 2.2 that

$$\Big\| \sup_{k\in\mathbb{Z}} \Big| \sum_{l=k}^{\infty} S_l^j * f \Big| \Big\|_{L^q(\mathbb{R}^n)} \lesssim j \|f\|_{L^q(\mathbb{R}^n)} \tag{2.14}$$

when $q \in \big((2\theta-1)/(2\theta-2),\, 2\theta-1\big)$. Interpolating the inequalities (2.13) and (2.14) leads to that for $p \in (1, \infty)$ and $\varepsilon > 0$,

$$\Big\| \sup_{k\in\mathbb{Z}} \Big| \sum_{l=k}^{\infty} S_l^j * f \Big| \Big\|_{L^p(\mathbb{R}^n)} \lesssim j^{-\delta_{\theta,p}+\varepsilon} \|f\|_{L^p(\mathbb{R}^n)}.$$

with $\delta_{\theta,p} = (\theta - 3/2)t - (1-t)$, and $1/p = t/2 + (1-t)/(2\theta-1)$ if $p \in (2,\, 2\theta-1)$, or $1/p = t/2 + (1-t)(2\theta-2)/(2\theta-1)$ if $p \in \big(2\theta-1)/(2\theta-2),\, 2\big)$. A straightforward computation shows that when $p \in (\theta/(\theta-1),\, \theta)$, $\delta_{\theta,p} > 0$. This gives (2.9) and completes the proof of Theorem 2.3. ■

3 Proof of Theorems

We only prove Theorem 1.2; the proof of Theorem 1.1 is similar and simpler.

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Proof of Theorem 1.2 At first, we claim that if $q \in (1, \infty)$ and $b \in C_0^\infty(\mathbb{R}^n)$, then for each $\epsilon > 0$, there exists a positive constant A independent of f such that

$$\big\| \big(T^{\star}_{\Omega,b} f\big) \chi_{\{|x|>A\}} \big\|_{L^q(\mathbb{R}^n)} \lesssim \epsilon \|b\|_{L^\infty(\mathbb{R}^n)} \|f\|_{L^q(\mathbb{R}^n)}. \tag{3.1}$$

To see this, let $R > 0$ be large enough such that $\operatorname{supp} b \subset B(0, R)$. Without loss of generality, we assume that $\|b\|_{L^\infty(\mathbb{R}^n)} = 1$. It follows from the Hölder inequality that for $x \in \mathbb{R}^n$ with $|x| > 4R$,

$$|T^{\star}_{\Omega,\,b} f(x)|^q \lesssim |x|^{-nq} \int_{|y|<R} |\Omega(x-y)||f(y)|^q \mathrm{d}y \Big(\int_{|y|<R} |\Omega(x-y)| \mathrm{d}y \Big)^{q/q'}.$$

On the other hand, a trivial computation shows that

$$\int_{|y|<R} |\Omega(x-y)| \mathrm{d}y \le \int_{|x|-R<|y|<|x|+R} |\Omega(y)| \mathrm{d}y \lesssim R|x|^{n-1}.$$

Our claim (3.1) then follows from

$$\begin{aligned}
\int_{|x|>A} \big| T^{\star}_{\Omega,b} f(x) \big|^q \mathrm{d}x &\lesssim R^{q/q'} \int_{|x|>A} \int_{|y|<R} |\Omega(x-y)||f(y)|^q \mathrm{d}y \frac{\mathrm{d}x}{|x|^{n+q/q'}} \\
&\lesssim R^{q/q'} \|f\|^q_{L^q(\mathbb{R}^n)} \int_{|x|>A/2} |\Omega(x)| \frac{\mathrm{d}x}{|x|^{n+q/q'}} \\
&\lesssim \Big(\frac{R}{A}\Big)^{q/q'} \|f\|^q_{L^q(\mathbb{R}^n)}.
\end{aligned}$$

Now we prove that if $b \in C_0^\infty(\mathbb{R}^n)$ such that $\|b\|_{L^\infty(\mathbb{R}^n)} + \|\nabla b\|_{L^\infty(\mathbb{R}^n)} = 1, \theta > 3/2$, and $p \in \big((2\theta-1)/(2\theta-2),\, 2\theta-1\big)$, then for each $t \in \mathbb{R}^n$ with $|t| < 1$,

$$\big\| T^{j,\star}_{\Omega,\,b} f(\,\cdot\,) - T^{j,\star}_{\Omega,\,b} f(\,\cdot\, + t) \big\|_{L^p(\mathbb{R}^n)} \lesssim 2^j |t|^{1/2} \|f\|_{L^p(\mathbb{R}^n)}. \tag{3.2}$$

For each fixed $t \in \mathbb{R}^n$, let $A_t = 4|t|^{1/2}$ and write

$$\begin{aligned}
&\big| T^{j,\star}_{\Omega,b} f(x) - T^{j,\star}_{\Omega,b} f(x+t) \big| \\
&\quad \lesssim \big| b(x+t) - b(x) \big| \sup_{k\in\mathbb{Z}} \Big| \int_{|x-y|>A_t} K^{j,\,k}(x-y) f(y) \mathrm{d}y \Big| \\
&\qquad + \sup_{k\in\mathbb{Z}} \Big| \int_{|x-y|>A_t} U_{j,\,k}(x,\, y;\, t) \big(b(y) - b(x+t) \big) f(y) \mathrm{d}y \Big| \\
&\qquad + \sup_{k\in\mathbb{Z}} \Big| \int_{|x-y|\le A_t} K^{j,\,k}(x-y) \big(b(y) - b(x) \big) f(y) \mathrm{d}y \Big| \\
&\qquad + \sup_{k\in\mathbb{Z}} \Big| \int_{|x-y|\le A_t} K^{j,\,k}(x+t-y) \big(b(y) - b(x+t) \big) f(y) \mathrm{d}y \Big| \\
&\quad = \mathrm{J}^j_1 f(x,\, t) + \mathrm{J}^j_2 f(x,\, t) + \mathrm{J}^j_3 f(x,\, t) + \mathrm{J}^j_4 f(x,\, t),
\end{aligned}$$

where $U_{j,k}(x,\ y;\ t) = K^{j,k}(x-y) - K^{j,k}(x+t-y)$. By the fact that supp $K^l_\Omega * \phi_{j-l} \subset \{2^{l-1} \le |x| \le 2^{l+2}\}$, a trivial computation leads to the fact that for any $k \in \mathbb{Z}$,

$$\begin{aligned}
\Big| \int_{|x-y|>A_t} & K^{j,k}(x-y)f(y)\mathrm{d}y\Big| \\
&\le \Big| \int_{|x-y|>A_t} K^j(x-y)\chi_{\{|x-y|>2^k\}}(x-y)f(y)\mathrm{d}y\Big| \\
&\quad + \Big| \int_{|x-y|>A_t} \Big(K^{j,k}(x-y) - K^j(x-y)\chi_{\{|x-y|>2^k\}}(x-y)\Big) f(y)\mathrm{d}y\Big| \\
&\lesssim T^{j,\circ}_\Omega f(x) + \sum_{l=k-1}^{k+1} \int_{|x-y|>A_t} \Big|K^l_\Omega * \phi_{l-j}(x-y)\Big|\,|f(y)|\mathrm{d}y \\
&\lesssim T^{j,\circ}_\Omega f(x) + M_\Omega M f(x),
\end{aligned}$$

where M_Ω is the maximal operator defined by

$$M_\Omega f(x) = \sup_{r>0} r^{-n} \int_{|x-y|<r} |\Omega(x-y)f(y)|\mathrm{d}y.$$

Therefore,

$$|\mathrm{J}^j_1 f(x,\ t)| \lesssim |t|\|\nabla b\|_{L^\infty(\mathbb{R}^n)}\big(T^{j,\circ}_\Omega f(x) + M_\Omega M f(x)\big).$$

It is well known that M_Ω is bounded on $L^p(\mathbb{R}^n)$. This, together with Lemma 2.2, gives us

$$\|\mathrm{J}^j_1 f(\,\cdot\,,\ t)\|_{L^p(\mathbb{R}^n)} \lesssim j|t|\|f\|_{L^p(\mathbb{R}^n)}. \tag{3.3}$$

We now turn our attention to the terms J^j_k for $k = 2,\ 3,\ 4$. Lemma 2.1 tells us that

$$\begin{aligned}
\Big\| \sum_{l\in\mathbb{Z}} \Big| & K^l_\Omega * \phi_{l-j}(\,\cdot\,+t) - K^l_\Omega * \phi_{l-j}(\,\cdot\,)\Big|\chi_{\{|\,\cdot\,|>A_t\}}(\,\cdot\,)\Big\|_{L^1(\mathbb{R}^n)} \\
&\lesssim \sum_{l\in\mathbb{Z}} \Big\| \Big(K^l_\Omega * \phi_{l-j}(\,\cdot\,+t) - K^l_\Omega * \phi_{l-j}(\,\cdot\,)\Big)\chi_{\{|\,\cdot\,|>A_t\}}(\,\cdot\,)\Big\|_{L^1(\mathbb{R}^n)} \lesssim 2^j \frac{|t|}{A_t}.
\end{aligned}$$

Since

$$\mathrm{J}^j_2 f(x,t) \lesssim \sum_{l\in\mathbb{Z}} \int_{|x-y|>A_t} \Big|K^l_\Omega * \phi_{l-j}(x+t-y) - K^l_\Omega * \phi_{l-j}(x-y)\Big|\,|f(y)|\mathrm{d}y,$$

we deduce by the Young inequality that

$$\|\mathrm{J}^j_2 f(\,\cdot\,,\ t)\|_{L^p(\mathbb{R}^n)} \lesssim 2^j \frac{|t|}{A_t}\|f\|_{L^p(\mathbb{R}^n)} \lesssim 2^j|t|^{1/2}\|f\|_{L^p(\mathbb{R}^n)}. \tag{3.4}$$

To consider the term $\mathrm{J}^j_3 f(x,\ t)$, let $k_0 \in \mathbb{Z}$ such that $2^{k_0-1} < A_t \le 2^{k_0}$. As in the inequality (2.7), we can verify that

$$\sum_{l\in\mathbb{Z}} \int_{|x|\le A_t} |K^l_\Omega| * \phi_{l-j}(x)|x|\mathrm{d}x \lesssim \sum_{l=-\infty}^{k_0} 2^l \|K^l_\Omega\|_{L^1(\mathbb{R}^n)}\|\phi_{l-j}\|_{L^1(\mathbb{R}^n)} \lesssim A_t.$$

Noticing that

$$\mathrm{J}^j_3 f(x,\ t) \lesssim \sum_{l\in\mathbb{Z}} \int_{|x-y|\le A_t} |K^l_\Omega| * \phi_{l-j}(x-y)|x-y||f(y)|\mathrm{d}y.$$

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we then apply the Young inequality and deduce that

$$\|\mathrm{J}_3^j f(\,\cdot\,,\,t)\|_{L^p(\mathbb{R}^n)} \lesssim A_t \|f\|_{L^p(\mathbb{R}^n)}. \tag{3.5}$$

Observe that,

$$|\mathrm{J}_4^j f(x,t)| \lesssim \sum_{l\in\mathbb{Z}} \int_{|x+t-y|\le A_t+t} |K_\Omega^l| * \phi_{l-j}(x+t-y)|x+t-y||f(y)|\mathrm{d}y;$$

another application of the Young inequality yields

$$\|\mathrm{J}_4^j f(\,\cdot\,,\,t)\|_{L^p(\mathbb{R}^n)} \lesssim (A_t+|t|)\|f\|_{L^p(\mathbb{R}^n)} \lesssim |t|^{1/2}\|f\|_{L^p(\mathbb{R}^n)}. \tag{3.6}$$

Combining the estimates (3.3)–(3.6) leads to (3.2).

We can now conclude the proof of Theorem 1.2. Let $\theta \in (3/2,\,\infty)$ and $p \in \big(\theta/(\theta-1),\,\theta\big)$. For $b \in C_0^\infty(\mathbb{R}^n)$, it is easy to see that

$$\begin{aligned}\big|\, T_{\Omega,\,b}^{j,\star} f(x) - T_{\Omega,\,b}^{\star} f(x)\big| &\lesssim \sup_{k\in\mathbb{Z}} \Big| \sum_{l=k}^{\infty} \int_{\mathbb{R}^n} \big(b(x)-b(y)\big)\, S_l^j(x-y) f(y)\mathrm{d}y \Big| \\ &\lesssim \|b\|_{L^\infty(\mathbb{R}^n)} \sup_{k\in\mathbb{Z}} \Big| \sum_{l=k}^{\infty} S_l^j * f(x)\Big| + \sup_{k\in\mathbb{Z}} \Big| \sum_{l=k}^{\infty} S_l^j * (bf)(x)\Big|,\end{aligned}$$

Thus, by Theorem 2.3

$$\| T_{\Omega,\,b}^{j,\star} f - T_{\Omega,\,b}^{\star} f\|_{L^p(\mathbb{R}^n)} \lesssim j^{-\sigma}\|b\|_{L^\infty(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)}.$$

For fixed $\epsilon > 0$, we choose an integer j_0 such that $j_0^{-\sigma} \le \epsilon$. Let $\varrho = \min\{1,\ 2^{-2j_0}\epsilon^2\}$. Then for any $t \in \mathbb{R}^n$ with $0 < |t| < \varrho$,

$$\begin{aligned}\big\|\, T_{\Omega,\,b}^{\star} f(\,\cdot\,) - T_{\Omega,\,b}^{\star} f(\,\cdot\, + t)\big\|_{L^p(\mathbb{R}^n)} &\le 2\big\|\, T_{\Omega,\,b}^{j_0,\star} f - T_{\Omega,\,b}^{\star} f\big\|_{L^p(\mathbb{R}^n)} \\ &\quad + \big\|\, T_{\Omega,\,b}^{j_0,\star} f(\,\cdot\,) - T_{\Omega,\,b}^{j_0,\star}(\,\cdot\, + t)\big\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \epsilon\big(\|\nabla b\|_{L^\infty(\mathbb{R}^n)} + \|b\|_{L^\infty(\mathbb{R}^n)}\big)\,\|f\|_{L^p(\mathbb{R}^n)}.\end{aligned}$$

This, along with (3.1) and the Fréchet–Kolmogorov theorem characterizing the precompactness of a set in $L^p(\mathbb{R}^n)$ (see [24, p. 275]), implies that $T_{\Omega,\,b}^{\star}$ is compact on $L^p(\mathbb{R}^n)$ when $p \in \big(\theta/(\theta-1),\,\theta\big)$ and $b \in C_0^\infty(\mathbb{R}^n)$. Recall that for $b \in \mathrm{BMO}(\mathbb{R}^n)$, $\theta > 5/2$ and $p \in \big(4\theta/(4\theta-5),\,4\theta/5\big)$, $T_{\Omega,\,b}^{\star}$ is bounded on $L^p(\mathbb{R}^n)$ with bound $C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$. The conclusion in Theorem 1.2 now follows immediately. ∎

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https://doi.org/10.4153/CMB-2014-042-1 Published online by Cambridge University Press