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AN ATOMIC DECOMPOSITION FOR HARDY SPACES ASSOCIATED TO SCHRÖDINGER OPERATORS

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Abstract

Let $L = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^n$ where V is a nonnegative function in the space $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ of locally integrable functions on $\mathbb{R}^n$. In this paper we provide an atomic decomposition for the Hardy space $H^1_L(\mathbb{R}^n)$ associated to L in terms of the maximal function characterization. We then adapt our argument to give an atomic decomposition for the Hardy space $H^1_L(\mathbb{R}^n \times \mathbb{R}^n)$ on product domains.

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1. Introduction

Let V be a locally integrable nonnegative function on $\mathbb{R}^n$ (where $n \geq 1$), which is not identically zero. We define the form Q by

$$Q(u, v) := \int_{\mathbb{R}^n} \nabla u \nabla v \, dx + \int_{\mathbb{R}^n} V u v \, dx$$

with domain

$$\mathcal{D}(Q) := \left\{ u \in W^{1,2}(\mathbb{R}^n) \,\middle|\, \int_{\mathbb{R}^n} V |u|^2 \, dx < \infty \right\}.$$

The space $W^{1,2}(\mathbb{R}^n)$ which appears in the formula above is the Sobolev space consisting of those L^2 functions on $\mathbb{R}^n$ whose gradients are also square integrable. It is well known that this symmetric form is closed. We recall that it was shown by Simon [25] that this form coincides with the minimal closure of the form given by the same expression but defined on $C_0^\infty(\mathbb{R}^n)$, the space of C^∞ functions with compact support.

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Let L denote the self-adjoint operator associated with Q. The domain of L, written $\mathcal{D}(L)$, is defined to be the set of all $u \in \mathcal{D}(Q)$ for which there exists $v \in L^2$ such that

$$Q(u, \varphi) = \int_{\mathbb{R}^n} v\bar{\varphi}\, dx \quad \forall \varphi \in \mathcal{D}(Q).$$

Formally, we write $L = -\Delta + V$ as a Schrödinger operator with potential V. Since V is a locally integrable nonnegative function on $\mathbb{R}^n$, the Feynman–Kac formula implies that the kernel $p_t(x, y)$ of the semigroup e^{-tL} satisfies the estimate

$$0 \le p_t(x, y) \le (4\pi t)^{-n/2} e^{-|x-y|^2/(4t)} \tag{1.1}$$

for all $t > 0$ and $x, y \in \mathbb{R}^n$ (see [24, p. 195]).

Given a function $f \in L^2(\mathbb{R}^n)$, we consider the following nontangential maximal function associated with the Poisson semigroup generated by the operator L:

$$f_L^*(x) := \sup_{|y-x|<t} |e^{-t\sqrt{L}} f(y)| \quad \forall x \in \mathbb{R}^n.$$

The space $H_L^1(\mathbb{R}^n)$ is defined to be the completion of $L^2(\mathbb{R}^n)$ in the norm given by the L^1 norm of this maximal function, that is,

$$\|f\|_{H_L^1(\mathbb{R}^n)} := \|f_L^*\|_{L^1(\mathbb{R}^n)}.$$

See, for example, [1–3, 12, 17, 18] for the properties of $H_L^1(\mathbb{R}^n)$.

Note that if $L = -\Delta$, then the space $H_L^1(\mathbb{R}^n)$ is the classical Hardy space $H^1(\mathbb{R}^n)$ which is a natural substitute for $L^1(\mathbb{R}^n)$. Recall that the development of the theory of the classical Hardy spaces in $\mathbb{R}^n$ was initiated by Stein and Weiss [26] and was originally tied closely to the theory of harmonic functions. Real variable methods were introduced into this subject in the seminal paper of Fefferman and Stein [15], the evolution of whose ideas led eventually to a characterization of Hardy spaces via the so-called 'atomic decomposition' obtained by Coifman [7] when $n = 1$ and in higher dimensions by Latter [21].

An atomic decomposition for $H_L^1(\mathbb{R}^n)$ was given in [17] by combining the area S-function and the finite speed propagation property for the wave equation. Following [17], a function $a \in L^2(\mathbb{R}^n)$ is called a (1, 2)-atom associated to the operator L if there exists a function $b \in \mathcal{D}(L)$, the domain of an operator L and a ball B of $\mathbb{R}^n$ such that

$$a = Lb;$$
$$\operatorname{supp} L^k b \subseteq B;$$
$$\|(r_B^2 L)^k b\|_{L^2(\mathbb{R}^n)} \le r_B^2 |B|^{-1/2},$$

where $k = 0, 1$ and r_B denotes the radius of the ball B.

The aim of this paper is to get an atomic decomposition directly from the fact that $f_L^* \in L^1(\mathbb{R}^n)$ and then to provide a new proof of the atomic decomposition for $H_L^1(\mathbb{R}^n)$. Our first main result is the following theorem.

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Theorem 1.1. *Let $L = -\Delta + V$ where $V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is a nonnegative function on $\mathbb{R}^n$. Let $f \in H^1_L(\mathbb{R}^n)$. Then there exist $(1, 2)$-atoms a_j and real numbers λ_j for $j = 1, 2, 3, \ldots$ such that*

$$f = \sum_{j=1}^{\infty} \lambda_j a_j \tag{1.2}$$

in $H^1_L(\mathbb{R}^n)$. Furthermore, matters can be arranged so that the sequence λ_j satisfies the inequality

$$\sum_{j=1}^{\infty} |\lambda_j| \le C \|f\|_{H^1_L(\mathbb{R}^n)}$$

for some positive constant C, which may depend on n.

Conversely, any function f which is written in the form of (1.2), where the a_j are $(1, 2)$-atoms, satisfies the inequality

$$\|f\|_{H^1_L(\mathbb{R}^n)} \le C \sum_{j=1}^{\infty} |\lambda_j|.$$

We mention that the localized version of the atomic decomposition for $H^1_L(\mathbb{R}^n)$ when $L = -\Delta + V$ was given in [13], by using the properties of local Hardy spaces (see [16]), under the assumption that V was a fixed nonnegative function on $\mathbb{R}^n$ belonging to the reverse Hölder class B_q for some $q > 1$. That is, there exists a positive constant C, possibly depending on q and V, such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V^q \, dx\right)^{1/q} \le C\left(\frac{1}{|B|}\int_B V \, dx\right)$$

holds for every ball B in $\mathbb{R}^n$.

Let us now turn to the Hardy space on product domains. We note that the usual space $H^1(\mathbb{R}^n \times \mathbb{R}^n)$ on the product domain is now well understood (see, for instance, [4, 5, 14]). In this paper we shall be concerned with the space $H^1_L(\mathbb{R}^n \times \mathbb{R}^n)$ associated to the Schrödinger operator L (see [11] for more properties).

For any $(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n$ and $f \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$, define

$$f^*_L(x_1, x_2) = \sup_{\substack{|y_1 - x_1| < t_1 \\ |y_2 - x_2| < t_2}} |e^{-t_1\sqrt{L}} \otimes e^{-t_2\sqrt{L}} f(y_1, y_2)|$$

where

$$e^{-t_1\sqrt{L}} \otimes e^{-t_2\sqrt{L}} f(y_1, y_2) := \iint_{\mathbb{R}^n \times \mathbb{R}^n} p_{t_1}(y_1, z_1) p_{t_2}(y_2, z_2) f(z_1, z_2) \, dz_1 \, dz_2.$$

The space $H^1_L(\mathbb{R}^n \times \mathbb{R}^n)$ is defined to be the completion of $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ in the norm given by

$$\|f\|_{H^1_L(\mathbb{R}^n \times \mathbb{R}^n)} := \|f^*_L\|_{L^1(\mathbb{R}^n \times \mathbb{R}^n)}.$$

For our purposes, a product $(1, 2)$-atom is a function a on $\mathbb{R}^{2n}$, together with an associated open set Ω of finite measure satisfying the following two properties.

First, the function a can be further decomposed into the form $a = \sum_{R\in m(\Omega)} a_R$ where for each $R \in m(\Omega)$ there exists a function b_R such that $a_R = (L \otimes L)b_R$ and

$$\mathrm{supp}(L^i \otimes L^j)b_R \subseteq 10R, \quad i, j = 0, 1,$$

where $10R$ denotes the rectangle with the same center as R and 10 times the side lengths.

Second,

$$\|a\|_{L^2(\mathbb{R}^n\times\mathbb{R}^n)} \le |\Omega|^{-1/2}$$

and

$$\sum_{R\in m(\Omega)} \sum_{i,j=0}^{1} \ell(I)^{4i-4}\ell(J)^{4j-4}\|(L^i \otimes L^j)b_R\|^2_{L^2(\mathbb{R}^n\times\mathbb{R}^n)} \le |\Omega|^{-1}$$

where $R = I \times J$ denotes the dyadic rectangle of $\mathbb{R}^n \times \mathbb{R}^n$ whose side lengths are $\ell(I)$ and $\ell(J)$, $10R$ denotes the set $\{10x \mid x \in R\}$ and $m(\Omega)$ denotes the set of maximal dyadic subrectangles of Ω (see Section 4 below).

The second main result of this paper is the following theorem.

THEOREM 1.2. *Let $L = -\Delta + V$ where $V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is a nonnegative function on $\mathbb{R}^n$. Let $f \in H^1_L(\mathbb{R}^n \times \mathbb{R}^n)$. Then there exist product $(1, 2)$-atoms a_j and numbers λ_j, where $j = 0, 1, 2, \ldots,$ such that*

$$f = \sum_{j=1}^{\infty} \lambda_j a_j \tag{1.3}$$

in $H^1_L(\mathbb{R}^n \times \mathbb{R}^n)$, and the sequence λ_j satisfies the condition that

$$\sum_{j=1}^{\infty} |\lambda_j| \le C\|f\|_{H^1_L(\mathbb{R}^n\times\mathbb{R}^n)}.$$

Conversely, for any decomposition of f of the form in (1.3),

$$\|f\|_{H^1_L(\mathbb{R}^n\times\mathbb{R}^n)} \le C \sum_{j=1}^{\infty} |\lambda_j|.$$

The organisation of this paper is as follows. In Section 2 we introduce some notation and preliminary lemmas. Our main results, Theorems 1.1 and 1.2, are proved in Sections 3 and 4. The main contribution of this paper is to combine the Calderón reproducing formula, the finite propagation speed property and the methods of Wilson [27] to obtain an atomic decomposition of Hardy spaces and then to verify the required L^2 norm estimates of the atoms by using square function estimates.

Throughout this paper, the letters C and c denote (possibly different) constants that are independent of the essential variables.

2. Preliminaries

Recall that if L is a nonnegative, self-adjoint operator on $L^2(\mathbb{R}^n)$ and $E_L(\lambda)$ denotes a spectral decomposition associated with L, then for every bounded Borel

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function $F : [0, \infty) \to \mathbb{C}$ one defines the operator

$$F(L) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

by the formula

$$F(L) := \int_0^\infty F(\lambda)\, dE_L(\lambda). \tag{2.1}$$

In particular, the operator $\cos(t\sqrt{L})$ is well defined on $L^2(\mathbb{R}^n)$. Moreover, it follows from [9, Theorem 3] (see also [6]) that the integral kernel $K_{\cos(t\sqrt{L})}$ of $\cos(t\sqrt{L})$ satisfies

$$\operatorname{supp} K_{\cos(t\sqrt{L})} \subseteq \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid |x - y| \le t\}. \tag{2.2}$$

By the Fourier inversion formula, whenever F is an even bounded Borel function with Fourier transform $\hat{F}$ in $L^1(\mathbb{R})$, we can write $F(\sqrt{L})$ in terms of $\cos(t\sqrt{L})$. Specifically, using (2.1), we have

$$F(\sqrt{L}) = (2\pi)^{-1} \int_{-\infty}^{\infty} \hat{F}(t) \cos(t\sqrt{L})\, dt$$

which, when combined with (2.2), gives us that

$$K_{F(\sqrt{L})} = (2\pi)^{-1} \int_{|t| \ge |x-y|} \hat{F}(t) K_{\cos(t\sqrt{L})}\, dt.$$

Lemma 2.1. *Let $\varphi \in C_0^\infty(\mathbb{R})$ be an even function such that $\operatorname{supp} \varphi \subseteq [-1, 1]$. Let Φ denote the Fourier transform of φ. Then for every $\kappa = 0, 1, 2, \ldots$ and for every $t > 0$ the kernel $K_{(t^2L)^\kappa \Phi(t\sqrt{L})}$ of $(t^2L)^\kappa \Phi(t\sqrt{L})$ satisfies the condition*

$$\operatorname{supp} K_{(t^2L)^\kappa \Phi(t\sqrt{L})} \subseteq \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid |x - y| \le t\}.$$

Proof. We refer the reader to [17, Lemma 3.5] for the proof. □

In this paper we use $\mathbb{R}^{n+1}_+$ to denote the upper half space of $\mathbb{R}^{n+1}$. In the following lemma we shall assume that $\varphi \in C_0^1(\mathbb{R}^n)$ is nonnegative, radial and nonincreasing. We also assume that $\varphi = 1$ on $B(0, 1/2)$, $\operatorname{supp} \varphi \subseteq B(0, 1)$ and $\int \varphi(x)\, dx = 1$. We sometimes use capital letters to denote points of $\mathbb{R}^{n+1}_+$ (for example, $X = (x, t)$), and set

$$u(x, t) = e^{-t\sqrt{L}} f(x),$$
$$\nabla_X u(X) = (\nabla_x u, \partial_t u)$$
$$|\nabla_X u|^2 = |\nabla_x u|^2 + |\partial_t u|^2.$$

Lemma 2.2. *For every $f, g \in L^2(\mathbb{R}^n)$,*

$$\begin{aligned} &\iint_{\mathbb{R}^{n+1}_+} |t\nabla_X u(x, t)|^2 |\varphi_t * g(x)|^2 \frac{dx\, dt}{t} \\ &\quad \le \int_{\mathbb{R}^n} |f(x)|^2 |g(x)|^2\, dx + \iint_{\mathbb{R}^{n+1}_+} |u(x, t)|^2 |\psi_t * g(x)|^2 \frac{dx\, dt}{t} \end{aligned} \tag{2.3}$$

where ψ is a vector-valued function with the same support as φ and mean value 0.

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PROOF. The proof of Lemma 2.2 can be obtained by making minor modifications to the proof of [23, Lemma 3.1] in the case where $L = -\Delta$ is the Laplace operator on $\mathbb{R}^n$. For the sake of completeness and for the reader's convenience we give a brief sketch of the proof of this lemma.

Write $\nabla_X^2 = \nabla_X \nabla_X$. Since $u = e^{-t\sqrt{L}} f$, we have

$$\nabla_X^2 u^2 = (\partial_t^2 + \Delta)u^2 = 2|\nabla_X u|^2 + 2Vu^2.$$

This, together with the condition that $V \geq 0$, gives

$$\begin{aligned} 2 \iint_{\mathbb{R}^{n+1}_+} &|t\nabla_X u|^2 |\varphi_t * g|^2 \frac{dx\,dt}{t} \\ &= \iint_{\mathbb{R}^{n+1}_+} \nabla_X^2 u^2 |\varphi_t * g|^2 t\,dx\,dt - 2 \iint_{\mathbb{R}^{n+1}_+} Vu^2 |\varphi_t * g|^2 t\,dx\,dt \\ &\leq \iint_{\mathbb{R}^{n+1}_+} \nabla_X^2 u^2 |\varphi_t * g|^2 t\,dx\,dt. \end{aligned}$$

After an integration by parts we obtain

$$\begin{aligned} \iint_{\mathbb{R}^{n+1}_+} &\nabla_X^2 u^2 |\varphi_t * g|^2 t\,dx\,dt \\ &= -\iint_{\mathbb{R}^{n+1}_+} \nabla_X u^2 \nabla_X (t(\varphi_t * g)^2)\,dx\,dt \\ &= -2 \iint_{\mathbb{R}^{n+1}_+} (2u\nabla_X u(\varphi_t * g) t\nabla_X(\varphi_t * g) + u\partial_t u|\varphi_t * g|^2)\,dx\,dt. \end{aligned} \tag{2.4}$$

Note that the conditions $u(x, 0) \in L^2(\mathbb{R}^n)$ or $f \in L^2(\mathbb{R}^n)$ are sufficient to ensure that the boundary terms 'at ∞' for this integration by parts vanish, as does the boundary term for $t = 0$.

We use a further integration by parts to obtain

$$\begin{aligned} 2 \iint_{\mathbb{R}^{n+1}_+} &u\partial_t u(\varphi_t * g)^2\,dx\,dt \\ &= -\lim_{t\to 0} \int_{\mathbb{R}}^{n} u^2(\varphi_t * g)^2\,dx\,dt - 2 \iint_{\mathbb{R}^{n+1}_+} u^2(\varphi_t * g)(\partial_t(\varphi_t * g))\,dx\,dt \\ &= -\int_{\mathbb{R}}^{n} f^2 g^2\,dx - 2 \iint_{\mathbb{R}^{n+1}_+} u^2(\varphi_t * g)(\partial_t(\varphi_t * g))\,dx\,dt. \end{aligned}$$

When combined with (2.4), integration by parts and the Cauchy–Schwarz inequality, this gives (2.3) provided that

$$|\psi_t * f|^2 = 9|(t\nabla_x \varphi_t) * g)|^2 + 9|\vec{\rho}_t * g|^2.$$

Here $\vec{\rho} = (x_1\varphi, \ldots, x_n\varphi)$. For the details, we refer the reader to [23, (3.8)]. This completes our proof. □

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Finally for $s > 0$ we define the set of measurable functions

$$\mathbb{F}(s) := \Big\{ \psi : \mathbb{C} \to \mathbb{C} \,\Big|\, |\psi(z)| \le C \frac{|z|^s}{(1 + |z|^{2s})} \Big\}.$$

Then for any nonzero function $\psi \in \mathbb{F}(s)$,

$$\Big\{ \int_0^\infty |\psi(t)|^2 \, dt/t \Big\}^{1/2} < \infty.$$

We write $\psi_t(z) = \psi(tz)$. It follows from spectral theory (see [10]) that, if $f \in L^2(\mathbb{R}^n)$, then

$$\begin{aligned} \Big\{ \int_0^\infty \|\psi(t\sqrt{L})f\|^2_{L^2(\mathbb{R}^n)} \frac{dt}{t} \Big\}^{1/2} &= \Big\{ \int_0^\infty \langle \bar{\psi}(t\sqrt{L})\psi(t\sqrt{L})f, f \rangle \, \frac{dt}{t} \Big\}^{1/2} \\ &= \Big\{ \Big\langle \int_0^\infty |\psi|^2(t\sqrt{L}) \, \frac{dt}{t} f, f \Big\rangle \Big\}^{1/2} \\ &= \kappa \|f\|_{L^2(\mathbb{R}^n)}, \end{aligned} \tag{2.5}$$

where $\kappa = \{\int_0^\infty |\psi(t)|^2 \, dt/t\}^{1/2}$.

3. Proof of Theorem 1.1

We shall use $\mathcal{M}$ to denote the Hardy–Littlewood maximal function with respect to the balls of $\mathbb{R}^n$. We use the notation

$$\Gamma(x) = \{(y, t) \in \mathbb{R}^{n+1}_+ \mid |x - y| < t\}$$

to denote the standard cone (of aperture 1) with vertex $x \in \mathbb{R}^n$.

For any closed subset F of $\mathbb{R}^n$, we denote by $\mathcal{R}(F)$ the union of all cones with vertices in F, that is,

$$\mathcal{R}(F) = \bigcup_{x \in F} \Gamma(x).$$

If O is an open subset of $\mathbb{R}^n$, then the 'tent' over O, denoted by $\widehat{O}$, is defined to be

$$\widehat{O} = {}^c[\mathcal{R}({}^cO)].$$

Proof of Theorem 1.1. Let $f \in H^1_L(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. We shall prove that f has an atomic decomposition as in (1.2). We start with a suitable version of the Calderón reproducing formula.

Let φ and Φ be as in Lemma 2.1 and set $\Psi(x) := x^4\Phi(x)$ for all $x \in \mathbb{R}$. By the L^2-functional calculus (see [22]) for every $f \in L^2(\mathbb{R}^n)$ we can write

$$\begin{aligned} f &= c_\Psi \int_0^\infty \Psi(t\sqrt{L}) t\sqrt{L} e^{-t\sqrt{L}} f \frac{dt}{t} \\ &= \lim_{N \to \infty} c_\Psi \int_{1/N}^N \Psi(t\sqrt{L}) t\sqrt{L} e^{-t\sqrt{L}} f \frac{dt}{t} \end{aligned} \tag{3.1}$$

with the integral converging in $L^2(\mathbb{R}^n)$.

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For $i \in \mathbb{Z}$ we define the sets

$$O_i := \{x \in \mathbb{R}^n \mid f_L^*(x) > 2^i\}$$

and consider

$$O_i^* = \{x \in \mathbb{R}^n \mid \mathcal{M}(\chi_{O_i})(x) > 2^{-(n+1)}\}.$$

Then $O_i \subseteq O_i^*$ and $|O_i^*| \le C|O_i|$ for every $i \in \mathbb{Z}$.

Now let $\{Q_i^j\}_j$ be a Whitney decomposition of O_i^* and let $\widehat{O_i^*}$ be a tent region, that is,

$$\widehat{O_i^*} := \{(x, t) \in \mathbb{R}^n \times (0, \infty) \mid \mathrm{dist}(x, {}^cO_i^*) \ge t\}.$$

For every $i, j \in \mathbb{Z}$ we define

$$T_i^j = (Q_i^j \times (0, +\infty)) \cap \widehat{O_i^*} \setminus \widehat{O_{i+1}^*},$$

and $\lambda_i^j = 2^i|Q_i^j|$. Using formula (3.1), we write

$$\begin{aligned} f &= \sum_{j,i\in\mathbb{Z}} c_\Psi \int_0^\infty \Psi(t\sqrt{L})(\chi_{T_i^j} t\sqrt{L}e^{-t\sqrt{L}})f \frac{dt}{t} \\ &=: \sum_{j,i\in\mathbb{Z}} \lambda_i^j a_i^j \end{aligned}$$

where $a_i^j = Lb_i^j$ and

$$b_i^j = (\lambda_i^j)^{-1} c_\Psi \int_0^\infty t^4 L\Phi(t\sqrt{L})(\chi_{T_i^j} t\sqrt{L}e^{-t\sqrt{L}})f \frac{dt}{t}.$$

We claim that, up to normalization by a multiplicative constant, the a_i^j are $(1, 2)$-atoms. Once this claim is established, we shall have

$$\begin{aligned} \sum_{j,i} |\lambda_i^j| = \sum_{j,i} 2^i|Q_i^j| &\le C \sum_i 2^i|O_i^*| \\ &\le C \sum_i 2^i|O_i| \le C\|f\|_{H_L^1(\mathbb{R}^n)}, \end{aligned}$$

as desired.

Let us now prove the claim. We shall show that for every $j, i \in \mathbb{Z}$, the function $C^{-1}a_i^j$ is a $(1, 2)$-atom associated with the cube $10\sqrt{n}Q_i^j$ for some constant C (independent of i and j). Observe that if $(x, t) \in T_i^j$, then $B(x, t) \in O_i^*$. This, together with the fact that Q_i^j is the Whitney cube of O_i^*, allows us to deduce that

$$t \le 6\sqrt{n}\ell(Q_i^j).$$

By Lemma 2.1 the integral kernel $K_{\Phi_t(\sqrt{L})}$ of the operator $\Phi_t(\sqrt{L})$ satisfies the condition that

$$\mathrm{supp}\, K_{\Phi_t(\sqrt{L})} \subseteq \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid |x - y| \le t\}.$$

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This enables us to deduce that, whenever $k = 0, 1$,

$$\operatorname{supp}(L^k b_i^j) \subseteq 10\sqrt{n}Q_i^j.$$

To continue, for each cube Q_i^j we consider some $h \in L^2(Q_i^j)$ such that $\|h\|_{L^2(Q_i^j)} = 1$. Then for $k = 0, 1$,

$$\begin{aligned}
&\Big|\lambda_i^j \int_{\mathbb{R}^n} (\ell(Q_i^j)^2 L)^k b_i^j(x) h(x)\, dx\Big| \\
&\quad = c_\Psi \Big|\int_{\mathbb{R}^{n+1}_+} t^4 (\ell(Q_i^j)^2 L)^k L\Phi(t\sqrt{L})(\chi_{T_i^j} t\sqrt{L}e^{-t\sqrt{L}}) f(x) h(x) \frac{dx\, dt}{t}\Big| \\
&\quad \le C\ell(Q_i^j)^2 \int_{\mathbb{R}^{n+1}_+} |(\chi_{T_i^j} t\sqrt{L}e^{-t\sqrt{L}}) f(x) t^{2(k+1)} L^{k+1}\Phi(t\sqrt{L})(h)(x)| \frac{dx\, dt}{t} \\
&\quad \le C\ell(Q_i^j)^2 \Big(\iint_{T_i^j} |t\sqrt{L}e^{-t\sqrt{L}} f(x)|^2 \frac{dx\, dt}{t}\Big)^{1/2} \\
&\qquad \times \Big(\iint_{\mathbb{R}^{n+1}_+} |(t^2 L)^{k+1}\Phi(t\sqrt{L})(h)(x)|^2 \frac{dx\, dt}{t}\Big)^{1/2} \\
&\quad \le C\ell(Q_i^j)^2 \Big(\iint_{T_i^j} |t\sqrt{L}e^{-t\sqrt{L}} f(x)|^2 \frac{dx\, dt}{t}\Big)^{1/2}.
\end{aligned}$$

Note that the first inequality is obtained from the fact that $0 < t < 6\sqrt{n}\ell(Q_i^j)$ and the third inequality follows from (2.5).

Therefore, in order to prove our claim, it suffices to show that

$$\int_{T_i^j} |t\sqrt{L}e^{-t\sqrt{L}} f(y)|^2 \frac{dy\, dt}{t} \le C2^{2i}|Q_i^j|. \tag{3.2}$$

Let us show that (3.2) is satisfied. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be as in Lemma 2.2 and set

$$F_i^j = 10\sqrt{n}Q_i^j \setminus O_{i+1}.$$

We first show that, for all $(y, t) \in T_i^j$,

$$|\varphi_t * \chi_{F_i^j}(y)| \ge C. \tag{3.3}$$

Indeed, for any $(y, t) \in T_i^j$, we can obtain

$$B(y, t) \subseteq 10\sqrt{n}Q_i^j$$

and

$$B(y, t) \cap {}^c O_{i+1}^* \neq \emptyset.$$

This shows that there exists $x_0 \in B(y, t) \cap {}^c O_{i+1}^*$ such that

$$\mathcal{M}(\chi_{O_{i+1}})(x_0) \le 2^{-(n+1)}.$$

It then follows that

$$|B(y, t) \cap O_{i+1}| \le 2^{-(n+1)}|B(y, t)|.$$

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This implies that

$$\begin{aligned}|B(y,t/2)\cap F_i^j| &\geq |B(y,t/2)\cap 10\sqrt{n}Q_i^j| - |B(y,t/2)\cap O_{i+1}|\\ &\geq |B(y,t/2)| - 2^{-(n+1)}|B(y,t)|\\ &= 2^{-(n+1)}|B(y,t)|\end{aligned}$$

and then, for any $(y,t)\in T_i^j$,

$$|\varphi_t * \chi_{F_i^j}(y)| = \left|\int \varphi_t(y-z)\chi_{F_i^j}(z)\,dz\right| \geq t^{-n}|B(y,t/2)\cap F_k^j| \geq C$$

which proves estimate (3.3).

By Lemma 2.2, we have

$$\begin{aligned}&\int_{T_i^j} |t\sqrt{L}e^{-t\sqrt{L}}f(y)|^2\frac{dy\,dt}{t}\\ &\leq C\int_{\mathbb{R}^{n+1}_+} |t\nabla_X e^{-t\sqrt{L}}f(y)|^2|\varphi_t * \chi_{F_i^j}(y)|^2\frac{dy\,dt}{t}\\ &\leq C\Big(\int_{\mathbb{R}^{n+1}_+} |e^{-t\sqrt{L}}f(y)|^2|\psi_t * \chi_{F_i^j}(y)|^2\frac{dy\,dt}{t} + \int_{\mathbb{R}^n}|f(x)|^2|\chi_{F_i^j}(x)|^2\,dx\Big)\\ &=: \mathrm{T}_1 + \mathrm{T}_2.\end{aligned}$$

Observe that if $\psi_t * \chi_{F_i^j}(y) \neq 0$, then $F_i^j \cap B(y,t) \neq \varnothing$ and there exists an element

$$x_0 \in B(y,t)\cap (10\sqrt{n}Q_i^j)\cap {}^cO_{i+1}.$$

This gives us that

$$|e^{-t\sqrt{L}}f(y)| \leq f_L^*(x_0) \leq 2^{i+1}.$$

Hence

$$\mathrm{T}_1 \leq C2^{2i+2}\int_{\mathbb{R}^{n+1}_+} |\psi_t * \chi_{F_i^j}(y)|^2\frac{dy\,dt}{t} \leq C2^{2i}|Q_i^j|.$$

Also

$$\mathrm{T}_2 \leq C2^{2i+2}\int_{\mathbb{R}^n} |\chi_{F_i^j}(x)|^2\,dx \leq C2^{2i}|Q_i^j|$$

and the estimate (3.2) follows readily.

We have shown that, up to normalization by a multiplicative constant, the a_i^j are $(1,2)$-atoms associated with the ball $B(x_i^j, c_1\ell(Q_i^j))$ for some constant c_1 where x_i^j is the center of the cube Q_i^j. This proves that f has an atomic decomposition as in (1.2).

To prove the converse we assume that $f = \sum_j \lambda_j a_j$ where the a_j are $(1,2)$-atoms and $\sum_j |\lambda_j| < \infty$. In this case, it was proved in [17, Theorem 7.4] that $f \in H^1_L(\mathbb{R}^n)$. We omit the details here. The proof of Theorem 1.1 is now complete. □

4. Proof of Theorem 1.2

In this section we shall work exclusively with the domain $\mathbb{R}^{n+1}_+ \times \mathbb{R}^{n+1}_+$ and its distinguished boundary $\mathbb{R}^n\times\mathbb{R}^n$. If $x = (x_1, x_2) \in \mathbb{R}^n\times\mathbb{R}^n$, then we shall denote

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by $\Gamma(x)$ the product cone $\Gamma(x) = \Gamma(x_1) \times \Gamma(x_2)$, where

$$\Gamma(x_i) = \{(y_i, t_i) \in \mathbb{R}^{n+1}_+ \mid |x_i - y_i| < t_i\}$$

for $i = 1, 2$. If $(x, t) := ((x_1, t_1), (x_2, t_2)) \in \mathbb{R}^{n+1}_+ \times \mathbb{R}^{n+1}_+$, then we shall write

$$B_{x,t} := B(x_1, t_1) \times B(x_2, t_2)$$

for the product ball.

For any open set $\Omega \subseteq \mathbb{R}^{2n}$, the tent over Ω, denoted by $\hat{\Omega}$, is the set

$$\{(x, t) \in \mathbb{R}^{n+1}_+ \times \mathbb{R}^{n+1}_+ \mid B_{x,t} \subseteq \Omega\}.$$

Let $m(\Omega)$ denote the set of maximal dyadic subrectangles of Ω. Let $m_1(\Omega)$ denote the subset of those dyadic subrectangles $R = I \times J$ of Ω that are maximal in the x_1 direction. In other words, if $S = I' \times J \supseteq R$ is a dyadic subrectangle of Ω, then $I = I'$. Similarly, define $m_2(\Omega)$ to be the collection of those dyadic subrectangles of Ω that are maximal in the x_2 direction. Let $\mathcal{M}_s$ denote the strong maximal operator, that is, for any $x \in \mathbb{R}^{2n}$ let

$$\mathcal{M}_s f(x) = \sup_{t_1>0, t_2>0} \frac{1}{|B_{x,t}|} \int_{B_{x,t}} |f(y_1, y_2)| \, dy_1 \, dy_2. \tag{4.1}$$

In order to prove Theorem 1.2 we need some auxiliary results. The first one is Journé's covering lemma (see [20]).

Lemma 4.1. *Let Ω be an open subset of $\mathbb{R}^n \times \mathbb{R}^n$ and let $R = I \times J \in m_2(\Omega)$ where I, J are dyadic cubes of $\mathbb{R}^n$. Suppose that $\hat{I}$ is the biggest dyadic cube of $\mathbb{R}^n$ containing I such that $\hat{I} \times J \subseteq \Omega^*$ where*

$$\Omega^* = \{x \in \mathbb{R}^{2n} \mid \mathcal{M}\chi_\Omega(x) > 1/2\}.$$

We set $\gamma_1(R) = |\hat{I}|/|I|$ and define γ_2 similarly. Then for any $\delta > 0$,

$$\sum_{R\in m_2(\Omega)} |R|\gamma_1^{-\delta}(R) \le c_\delta |\Omega|$$

and

$$\sum_{R\in m_1(\Omega)} |R|\gamma_2^{-\delta}(R) \le c_\delta |\Omega|$$

where c_δ is a constant depending only on δ and not on Ω.

For every $i = 1, 2$ we let $\nabla_{X_i} = (\nabla_{x_i}, \partial_{t_i})$. In the following lemma we assume that $\varphi \in C^1_0(\mathbb{R}^n)$ is nonnegative, radial and nonincreasing. We also assume that $\varphi = 1$ on $B(0, 1/2)$, $\operatorname{supp} \varphi \subseteq B(0, 1)$ and $\int \varphi(x)\, dx = 1$.

Lemma 4.2. *For every $f, g \in L^2(\mathbb{R}^{2n})$ and $i = 1, 2$ there exist vector-valued functions $\psi^{(i)} \in C^\infty_0(\mathbb{R}^n)$ satisfying the conditions $\operatorname{supp} \psi^{(i)} \subseteq B(0, 1)$, $\int_{\mathbb{R}^n} \psi^{(i)}(x)\, dx = 0$ and*

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such that

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}_+\times\mathbb{R}^{n+1}_+} |t_1\nabla_{X_1}e^{-t_1\sqrt{L}}\otimes t_2\nabla_{X_2}e^{-t_2\sqrt{L}}f(y_1,y_2)|^2|(\varphi_{t_1}\otimes\varphi_{t_2})*g(y_1,y_2)|^2\frac{dy\,dt}{t_1t_2}\\
&\quad\le \int_{\mathbb{R}^{n+1}_+\times\mathbb{R}^{n+1}_+} |e^{-t_1\sqrt{L}}\otimes e^{-t_2\sqrt{L}}f(y_1,y_2)|^2|(\psi^{(1)}_{t_1}\otimes\psi^{(2)}_{t_2})*g(y_1,y_2)|^2\frac{dy\,dt}{t_1t_2}\\
&\qquad+\int_{\mathbb{R}^{n+1}_+\times\mathbb{R}^{n}} |e^{-t_1\sqrt{L}}f(y_1,x_2)|^2|\psi^{(1)}_{t_1}*g(y_1,x_2)|^2\frac{dy_1\,dt_1}{t_1}\,dx_2\\
&\qquad+\int_{\mathbb{R}^{n}\times\mathbb{R}^{n+1}_+} |e^{-t_2\sqrt{L}}f(x_1,y_2)|^2|\psi^{(2)}_{t_2}*g(x_1,y_2)|^2\frac{dy_2\,dt_2}{t_2}\,dx_1\\
&\qquad+\int_{\mathbb{R}^{n}\times\mathbb{R}^{n}} |f(x_1,x_2)|^2|g(x_1,x_2)|^2\,dx_1\,dx_2.
\end{aligned}$$

PROOF. Repeated applications of Lemma 2.2 can be used to prove Lemma 4.2. □

Finally we state the following lemma whose proof we omit since it is similar to that of [17, Lemma 4.3].

LEMMA 4.3. *Suppose that T is a bounded sublinear operator on $L^2(\mathbb{R}^{2n})$ and that for every product $(1,2)$-atom a on product domains we have*

$$\|Ta\|_{L^1(\mathbb{R}^{2n})}\le C$$

where the constant C is independent of a. Then for any decomposition of the form given in (1.3) of f we have

$$\|Tf\|_{L^1(\mathbb{R}^{2n})}\le C\sum_{j=1}^{\infty}|\lambda_j|.$$

PROOF OF THEOREM 1.2 By condition (1.1) for every $K=0,1,\ldots$ there exists a constant C_K such that the kernel $p_{t,K}$ of the operator $(t\sqrt{L})^{2K}e^{-t\sqrt{L}}$ satisfies the condition that

$$|p_{t,K}(x,y)|\le C_K\frac{t}{(t+|x-y|)^{n+1}}\quad\forall t>0 \tag{4.2}$$

and almost every $x,y\in\mathbb{R}^n$ (see, for instance, [17, Lemma 7.2]).

Step 1. Let $f=\sum_j\lambda_ja_j$ where the a_j are product $(1,2)$-atoms and $\sum_{j=1}^{\infty}|\lambda_j|<\infty$. Recall that the strong maximal operator $\mathcal{M}_s$ defined in (4.1) is bounded on $L^2(\mathbb{R}^{2n})$ (see [14]). This, together with condition (1.1), gives us that

$$\|f^*_L\|_{L^2(\mathbb{R}^{2n})}\le C\|\mathcal{M}_sf\|_{L^2(\mathbb{R}^{2n})}\le C\|f\|_{L^2(\mathbb{R}^{2n})}.$$

By Lemma 4.3, it is enough to show that $\|a^*_L\|_{L^1(\mathbb{R}^{2n})}\le C$ for every product $(1,2)$-atom a, for some constant C which is independent of a.

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Suppose that

$$a = \sum_{R \in m(\Omega)} a_R = \sum_{R \in m(\Omega)} (L \otimes L) b_R$$

is a product (1, 2)-atom supported on some open subset Ω of $\mathbb{R}^{2n}$. For any maximal dyadic subrectangle $R = I \times J \in m(\Omega)$ let $\ell(I)$, $\ell(J)$ be the side-lengths of cubes I and J and let I' be the longest dyadic interval containing I so that

$$I' \times J \subseteq \Omega^* = \{x \in \mathbb{R}^{2n} \mid \mathcal{M}_s(\chi_\Omega)(x) > 1/2\}.$$

Then $I' \times J$ is in $m_1(\Omega^*)$. Let S be the longest dyadic interval so that $S \supseteq J$ and $I' \times S \subseteq \Omega^{**}$ where

$$\Omega^{**} = \{x \in \mathbb{R}^{2n} \mid \mathcal{M}_s(\chi_{\Omega^*})(x) > 1/2\}.$$

Let $\widetilde{R}$ be the 10-fold dilate of $I' \times S$ concentric with $I' \times S$. Clearly, an application of the strong maximal theorem (see [8, 19] for the proof) shows that

$$\Big|\bigcup \widetilde{R}\Big| \leq c|\Omega^{**}| \leq c|\Omega^*| \leq c|\Omega|.$$

We then have

$$\begin{aligned}\int_{\bigcup \widetilde{R}} a_L^*(x)\,dx &\leq C\Big|\bigcup \widetilde{R}\Big|^{1/2} \|a_L^*\|_{L^2(\mathbb{R}^{2n})} \leq C\Big|\bigcup \widetilde{R}\Big|^{1/2} \|\mathcal{M}_s(a)\|_{L^2(\mathbb{R}^{2n})}\\ &\leq C|\Omega|^{1/2}\|a\|_{L^2(\mathbb{R}^{2n})} \leq C|\Omega|^{1/2}|\Omega|^{-1/2} \leq C.\end{aligned}$$

We now find an estimate for

$$\int_{(\bigcup \widetilde{R})^c} a_L^*(x)\,dx \leq C.$$

We can write

$$\begin{aligned}\int_{(\bigcup \widetilde{R})^c} a_L^*(x)\,dx &\leq \sum_{R \in m(\Omega)} \int_{(\widetilde{R})^c} (a_R)_L^*(x)\,dx\\ &\leq \sum_{R \in m(\Omega)} \int_{x_1 \notin 10I'} (a_R)_L^*(x)\,dx + \sum_{R \in m(\Omega)} \int_{x_2 \notin 10S} (a_R)_L^*(x)\,dx.\end{aligned}$$

We only need to calculate the estimate for the first term above since the proof of the estimate for the second term is similar.

Observe that

$$\begin{aligned}\sum_{R \in m(\Omega)} \int_{x_1 \notin 10I'} (a_R)_L^*(x)\,dx &= \sum_{R \in m(\Omega)} \Big(\int_{x_1 \notin 10I'} \int_{x_2 \in 10J} + \int_{x_1 \notin 10I'} \int_{x_2 \notin 10J}\Big)(a_R)_L^*(x)\,dx\\ &=: \mathrm{E}_1 + \mathrm{E}_2.\end{aligned}$$

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By Hölder's inequality, we have

$$
\begin{aligned}
\mathrm{E}_1 &\le \sum_{R\in m(\Omega)} |J|^{1/2} \int_{x_1\notin 100I'} \|(a_R)^*_L(x_1,\cdot)\|_{L^2(dx_2)}\,dx_1\\
&\le C \sum_{R\in m(\Omega)} |J|^{1/2} \int_{x_1\notin 100I'} \Big\{\int_{\mathbb{R}^n} \sup_{|x_1-y_1|<t_1} |e^{-t_1\sqrt{L}}a_R(y_1,x_2)|^2\,dx_2\Big\}^{1/2}\,dx_1\\
&\le C \sum_{R\in m(\Omega)} |J|^{1/2} \int_{x_1\notin 100I'} \Big\{\int_{\mathbb{R}^n} \sup_{\substack{|x_1-y_1|<t_1\\ t_1<\ell(I)}} |e^{-t_1\sqrt{L}}a_R(y_1,x_2)|^2\,dx_2\Big\}^{1/2}\,dx_1\\
&\quad + C \sum_{R\in m(\Omega)} |J|^{1/2} \int_{x_1\notin 100I'} \Big\{\int_{\mathbb{R}^n} \sup_{\substack{|x_1-y_1|<t_1\\ t_1\ge\ell(I)}} |e^{-t_1\sqrt{L}}a_R(y_1,x_2)|^2\,dx_2\Big\}^{1/2}\,dx_1\\
&=: \mathrm{E}_{11} + \mathrm{E}_{12}.
\end{aligned}
$$

We consider the term E_{11} above. Let x_I denote the center of cube I. Note that $x_1 \notin 100I'$ and $|x_1 - y_1| < t_1 < \ell(I)$. It follows from the estimate (4.2) that

$$
\begin{aligned}
|e^{-t_1\sqrt{L}}a_R(\cdot,x_2)(y_1)| &\le C \int_{\mathbb{R}^n} \frac{t_1}{(t_1+|y_1-z_1|)^{n+1}}|a_R(z_1,x_2)|\,dz_1\\
&\le C\frac{\ell(I)}{|x_1-x_I|^{n+1}}\|a_R(\cdot,x_2)\|_{L^1(\mathbb{R}^n)}\\
&\le C|I|^{1/2}\frac{\ell(I)}{|x_1-x_I|^{n+1}}\|a_R(\cdot,x_2)\|_{L^2(\mathbb{R}^n)}
\end{aligned}
\tag{4.3}
$$

which, in combination with Lemma 4.1, gives us that

$$
\begin{aligned}
\mathrm{E}_{11} &\le C \sum_{R\in m(\Omega)} |J|^{1/2}|I|^{1/2}\Big\{\int_{x_1\notin 100I'} \frac{\ell(I)}{|x_1-x_I|^{n+1}}\,dx_1\Big\}\|a_R\|_{L^2(\mathbb{R}^{2n})}\\
&\le C \sum_{R\in m(\Omega)} |R|^{1/2}\|a_R\|_{L^2(\mathbb{R}^{2n})}\gamma_1(R)^{-1}\\
&\le C\Big(\sum_{R\in m(\Omega)} \|a_R\|^2_{L^2(\mathbb{R}^{2n})}\Big)^{1/2}\Big(\sum_{R\in m(\Omega)} |R|\gamma_1(R)^{-2}\Big)^{1/2}\\
&\le C.
\end{aligned}
$$

For the term E_{12} above, we apply the definition of the product $(1,2)$-atom to obtain

$$
\begin{aligned}
&|e^{-t_1\sqrt{L}}a_R(\cdot,x_2)(y_1)|\\
&\quad\le \Big(\frac{\ell(I)}{t_1}\Big)^2 |t_1^2 L e^{-t_1\sqrt{L}}\ell(I)^{-2}(L^0\otimes L^1)b_R(\cdot,x_2)(y_1)|\\
&\quad\le C\Big(\frac{\ell(I)}{t_1}\Big)^2 \int_{\mathbb{R}^n} \frac{t_1}{(t_1+|y_1-z_1|)^{n+1}}|\ell(I)^{-2}(L^0\otimes L^1)b_R(\cdot,x_2)(z_1)|\,dz_1.
\end{aligned}
$$

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Note that

$$x_1 \notin 100I', |x_1 - y_1| < t_1, \ell(I) \le t_1, z_1 \in I.$$

We can obtain the estimate

$$t_1 + |y_1 - z_1| \ge |x_1 - x_I|/2$$

and deduce that

$$|e^{-t_1\sqrt{L}} a_R(\cdot, x_2)(y_1)| \le C \frac{\ell(I)}{|x_1 - x_I|^{n+1}} \|\ell(I)^{-2}(L^0 \otimes L^1) b_R(\cdot, x_2)\|_{L^1(\mathbb{R}^n)}. \tag{4.4}$$

It follows from (4.4) and Hölder's inequality that

$$\begin{aligned}
\mathrm{E}_{12} &\le C \sum_{R \in m(\Omega)} |J|^{1/2} |I|^{1/2} \int_{x_1 \notin 100I'} \frac{\ell(I)}{|x_1 - x_I|^{n+1}}\, dx_1 \\
&\qquad \times \|\ell(I)^{-2}(L^0 \otimes L^1) b_R\|_{L^2(\mathbb{R}^{2n})} \\
&\le C \sum_{R \in m(\Omega)} |R|^{1/2} \gamma_1(R)^{-1} \|\ell(I)^{-2}(L^0 \otimes L^1) b_R\|_{L^2(\mathbb{R}^{2n})} \\
&\le C \Big(\sum_{R \in m(\Omega)} |R| \gamma_1(R)^{-2} \Big)^{1/2} \Big(\sum_{R \in m(\Omega)} \ell(I)^{-4} \|(L^0 \otimes L^1) b_R\|^2_{L^2(\mathbb{R}^{2n})} \Big) \\
&\le C
\end{aligned}$$

which, together with the estimate of E_{11}, gives us that $\mathrm{E}_1 \le C$.

Consider the term E_2. We first estimate the maximal function $(a_R)^*_L$. Now

$$\begin{aligned}
(a_R)^*_L(x) &= \sup_{|y_2 - x_2| < t_2} \sup_{|y_1 - x_1| < t_1} |e^{-t_1\sqrt{L}} \otimes e^{-t_2\sqrt{L}} a_R(y_1, y_2)| \\
&\le \sup_{\substack{|y_2 - x_2| < t_2 \\ t_2 < \ell(J)}} \sup_{\substack{|y_1 - x_1| < t_1 \\ t_1 < \ell(I)}} |e^{-t_1\sqrt{L}} \otimes e^{-t_2\sqrt{L}} a_R(y_1, y_2)| \\
&\qquad + \sup_{\substack{|y_2 - x_2| < t_2 \\ t_2 < \ell(J)}} \sup_{\substack{|y_1 - x_1| < t_1 \\ t_1 \ge \ell(I)}} |e^{-t_1\sqrt{L}} \otimes e^{-t_2\sqrt{L}} a_R(y_1, y_2)| \\
&\qquad + \sup_{\substack{|y_2 - x_2| < t_2 \\ t_2 \ge \ell(J)}} \sup_{\substack{|y_1 - x_1| < t_1 \\ t_1 < \ell(I)}} |e^{-t_1\sqrt{L}} \otimes e^{-t_2\sqrt{L}} a_R(y_1, y_2)| \\
&\qquad + \sup_{\substack{|y_2 - x_2| < t_2 \\ t_2 \ge \ell(J)}} \sup_{\substack{|y_1 - x_1| < t_1 \\ t_1 \ge \ell(I)}} |e^{-t_1\sqrt{L}} \otimes e^{-t_2\sqrt{L}} a_R(y_1, y_2)| \\
&=: \mathrm{E}_{21} + \mathrm{E}_{22} + \mathrm{E}_{23} + \mathrm{E}_{24}.
\end{aligned}$$

We only need to estimate the term E_{22} since the estimates of the remaining terms are similar.

Applying (4.4) with $a_R(\cdot, x_2)$ replaced by $e^{-t_2\sqrt{L}}a_R(\cdot, y_2)$, we obtain

$$\begin{aligned}
\mathrm{E}_{22} &\le C \sup_{\substack{|y_2-x_2|<t_2\\ t_2<\ell(J)}} \frac{\ell(I)}{|x_1-x_I|^{n+1}} \|\ell(I)^{-2}(L^0 \otimes e^{-t_2\sqrt{L}}L^1)b_R(\cdot, y_2)\|_{L^1(\mathbb{R}^n)} \\
&\le C\frac{\ell(I)}{|x_1-x_I|^{n+1}} \Big\| \sup_{\substack{|y_2-x_2|<t_2\\ t_2<\ell(J)}} |(\ell(I)^{-2}L^0 \otimes e^{-t_2\sqrt{L}}L^1)b_R(\cdot, y_2)|\Big\|_{L^1(\mathbb{R}^n)}.
\end{aligned}$$

Applying (4.3) with $a_R(\cdot, x_2)$ replaced by $(\ell(I)^{-2}L^0 \otimes L^1)b_R(x_1, \cdot)$, together with Hölder's inequality, we obtain

$$\begin{aligned}
\mathrm{E}_{22} &\le C\frac{\ell(I)}{|x_1-x_I|^{n+1}}\frac{\ell(J)}{|x_2-x_J|^{n+1}} \|(\ell(I)^{-2}L^0 \otimes L^1)b_R\|_{L^1(\mathbb{R}^n\times\mathbb{R}^n)} \\
&\le C\frac{\ell(I)}{|x_1-x_I|^{n+1}}\frac{\ell(J)}{|x_2-x_J|^{n+1}} |R|^{1/2}\|(\ell(I)^{-2}L^0 \otimes L^1)b_R\|_{L^2(\mathbb{R}^n\times\mathbb{R}^n)}.
\end{aligned}$$

A similar argument to that given for E_{22} shows that

$$\begin{aligned}
(a_R)^*_L(x) \le C&\frac{\ell(J)}{|x_2-x_J|^{n+1}}\frac{\ell(I)}{|x_1-x_I|^{n+1}}|R|^{1/2} \\
&\times \sum_{i,j=0}^{1} \ell(I)^{2i-2}\ell(J)^{2j-2}\|(L^i \otimes L^j)b_R\|_{L^2(\mathbb{R}^n\times\mathbb{R}^n)}.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathrm{E}_2 &= \sum_{R\in m(\Omega)} \int_{x_1\notin 10I'}\int_{x_2\notin 10J} (a_R)^*_L(x)\,dx \\
&\le \sum_{R\in m(\Omega)} \ell(I)/\ell(I')|R|^{1/2} \sum_{i,j=0}^{1} \ell(I)^{2i-2}\ell(J)^{2j-2}\|(L^i \otimes L^j)b_R\|_{L^2(\mathbb{R}^n\times\mathbb{R}^n)} \\
&\le C\sum_{R\in m(\Omega)} |R|^{1/2}\gamma_1(R)^{-1} \sum_{i,j=0}^{1} \ell(I)^{2i-2}\ell(J)^{2j-2}\|(L^i \otimes L^j)b_R\|_{L^2(\mathbb{R}^n\times\mathbb{R}^n)}.
\end{aligned}$$

Applying Lemma 4.1 and the definition of product (1, 2)-atom, together with Hölder's inequality, we obtain the estimate $\mathrm{E}_2 \le C$. We thus obtain the required estimate $\|a^*_L\|_{L^1(\mathbb{R}^{2n})} \le C$ and can deduce that $f \in H^1_L(\mathbb{R}^n \times \mathbb{R}^n)$.

Step 2. Let

$$f \in H^1_L(\mathbb{R}^n \times \mathbb{R}^n) \cap L^2(\mathbb{R}^n \times \mathbb{R}^n).$$

We begin with a version of the Calderón reproducing formula. Let $\Psi(x) = x^4\Phi(x)$ be the function in Lemma 2.1. Since $f \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$, applying the L^2-functional calculus gives us that

$$f = c_\Psi \int_0^\infty\int_0^\infty \Psi_{t_1}(\sqrt{L})t_1\sqrt{L}e^{-t_1\sqrt{L}} \otimes \Psi_{t_2}(\sqrt{L})t_2\sqrt{L}e^{-t_2\sqrt{L}} f\frac{dt}{t_1t_2}. \tag{4.5}$$

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For $k = 0, \pm 1, \ldots$ we set

$$E_k = \{x \mid f_L^*(x) > 2^k\},$$
$$E_k^* = \{x \mid \mathcal{M}_s \chi_{E_k}(x) > 2^{-(2n+1)}\}$$

and

$$E_k^{**} = \{x \mid \mathcal{M}_s \chi_{E_k^*}(x) > (4n)^{-n}\}.$$

Then

$$E_k \subseteq E_k^* \subseteq E_k^{**} \quad \text{and} \quad |E_k^{**}| \le C|E_k^*| \le C'|E_k|.$$

We define $T_k := \widehat{E_k^*} \setminus \widehat{E_{k+1}^*}$ and apply formula (4.5) to write

$$f = \sum_{k\in\mathbb{Z}} \lambda_k a_k$$

where $\lambda_k = 2^k|E_k^*|$ and

$$a_k = \lambda_k^{-1} c_\Psi \int_0^\infty \int_0^\infty \Psi_{t_1}(\sqrt{L})\Psi_{t_2}(\sqrt{L})(\chi_{T_k} t_1\sqrt{L}e^{-t_1\sqrt{L}} \otimes t_2\sqrt{L}e^{-t_2\sqrt{L}}) f \frac{dt}{t_1t_2}.$$

It is clear that

$$\sum_k |\lambda_k| \le C \sum_k 2^k|E_k^*| \le C\|f_L^*\|_{L^1(\mathbb{R}^{2n})}.$$

We claim that for each $k \in \mathbb{Z}$ the term a_k is a product $(1, 2)$-atom associated with the open set E_k^{**} for some constant C.

Let us prove the claim. First, it follows by Lemma 2.1 that the integral kernel $K_{\Psi_t(\sqrt{L})}$ of the operator $\Psi_t(\sqrt{L})$ has its support contained in

$$\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid |x_1 - y_1| \le t_1, |x_2 - y_2| \le t_2\}.$$

This, together with the definition of T_k, shows that $\operatorname{supp} a_k \subseteq E_k^{**}$. Second, for any dyadic rectangle $R = I \times J$ of $\mathbb{R}^n \times \mathbb{R}^n$, we define

$$R^+ = \left\{(y_1, y_2, t_1, t_2) \mid y_1 \in I, y_2 \in J, \frac{\ell(I)}{2} < t_1 \le \ell(I), \frac{\ell(J)}{2} < t_1 \le \ell(J)\right\}.$$

It can be verified that if $T_k \cap R^+ \ne \emptyset$, then $R \subseteq E_k^{**}$. Applying the definition of R^+, we obtain $T_k = \bigcup_R (T_k \cap R^+)$ where the R are all dyadic rectangles of $\mathbb{R}^n \times \mathbb{R}^n$. We can further decompose a_k as follows:

$$\begin{aligned} a_k &= \sum_{\bar{R}\in m(E_k^{**})} \sum_{R\subseteq\bar{R}} \lambda_k^{-1} \\ &\quad \times \int_0^\infty \int_0^\infty \Psi_{t_1}(\sqrt{L})\Psi_{t_2}(\sqrt{L})(\chi_{T_k} t_1\sqrt{L}e^{-t_1\sqrt{L}} \otimes t_2\sqrt{L}e^{-t_2\sqrt{L}}) f \frac{dt}{t_1t_2} \\ &=: \sum_{\bar{R}\in m(E_k^{**})} a_{k,\bar{R}} =: \sum_{\bar{R}\in m(E_k^{**})} (L \otimes L) b_{k,\bar{R}} \end{aligned}$$

where $m(E_k^{**})$ denotes the set of all maximal dyadic rectangles of E_k^{**}.

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By Lemma 2.1, if $i, j = 0, 1$, then

$$\mathrm{supp}((L^i \otimes L^j) b_{k,\bar{R}}) \subseteq 2\bar{R}.$$

To continue, for each $\bar{R}$ we consider some $h \in L^2(\bar{R})$ such that $\|h\|_{L^2(\bar{R})} = 1$. Then for every $k \in \mathbb{Z}$, we have

$$\begin{aligned}\|a_{k,\bar{R}}\|_{L^2} &= \sup_{\|h\|_{L^2} \le 1} |\langle a_{k,\bar{R}}, h\rangle| \\ &\le C2^{-k}|E_k^*|^{-1} \times \bigg(\sum_{R \subseteq \bar{R}} \int_{T_k \cap R^+} |t_1 \nabla_{X_1} e^{-t_1\sqrt{L}} \otimes t_2 \nabla_{X_2} e^{-t_2\sqrt{L}} f(y_1, y_2)|^2 \frac{dy\, dt}{t_1 t_2}\bigg)^{1/2}.\end{aligned}$$

In order to verify that

$$\sum_{\bar{R} \in m(E_k^{**})} \|a_{k,\bar{R}}\|_2^2 \le C|E_k^{**}|^{-1},$$

it is enough to prove that

$$\int_{T_k} |t_1 \nabla_{X_1} e^{-t_1\sqrt{L}} \otimes t_2 \nabla_{X_2} e^{-t_2\sqrt{L}} f(y_1, y_2)|^2 \frac{dy\, dt}{t_1 t_2} \le C2^{2k}|E_k^*|. \tag{4.6}$$

We now prove inequality (4.6). Let $\varphi \in C_0^\infty$ be the function in Lemma 4.2 and set $F_k = E_k^* \setminus E_{k+1}$. The argument given to prove formula (3.3) shows that

$$|(\varphi_{t_1} \otimes \varphi_{t_2}) * \chi_F(y)| \ge C$$

for all $(y, t) \in T_k$. This, together with Lemma 4.2, gives us that

$$\begin{aligned}&\int_{T_k} |t_1 \nabla_{X_1} e^{-t_1\sqrt{L}} \otimes t_2 \nabla_{X_2} e^{-t_2\sqrt{L}} f(y_1, y_2)|^2 \frac{dy\, dt}{t_1 t_2} \\ &\quad\le \int_{\mathbb{R}_+^{n+1} \times \mathbb{R}_+^{n+1}} |t_1 \nabla_{X_1} e^{-t_1\sqrt{L}} \otimes t_2 \nabla_{X_2} e^{-t_2\sqrt{L}} f(y_1, y_2)|^2 |(\varphi_{t_1} \otimes \varphi_{t_2}) * \chi_{F_k}(y)|^2 \frac{dy\, dt}{t_1 t_2} \\ &\quad\le \int_{\mathbb{R}_+^{n+1} \times \mathbb{R}_+^{n+1}} |e^{-t_1\sqrt{L}} \otimes e^{-t_2\sqrt{L}} f(y_1, y_2)|^2 |(\psi_{t_1}^{(1)} \otimes \psi_{t_2}^{(2)}) * \chi_{F_k}(y_1, y_2)|^2 \frac{dy\, dt}{t_1 t_2} \\ &\qquad + \int_{\mathbb{R}_+^{n+1} \times \mathbb{R}^n} |e^{-t_1\sqrt{L}} f(y_1, x_2)|^2 |\psi_{t_1}^{(1)} * \chi_{F_k}(y_1, x_2)|^2 \frac{dy_1\, dt_1}{t_1}\, dx_2 \\ &\qquad + \int_{\mathbb{R}^n \times \mathbb{R}_+^{n+1}} |e^{-t_2\sqrt{L}} f(x_1, y_2)|^2 |\psi_{t_2}^{(2)} * \chi_{F_k}(x_1, y_2)|^2 \frac{dy_2\, dt_2}{t_2}\, dx_1 \\ &\qquad + \int_{\mathbb{R}^n \times \mathbb{R}^n} |f(x_1, x_2)|^2 |\chi_{F_k}(x_1, x_2)|^2\, dx_1\, dx_2 \\ &\quad =: \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4.\end{aligned}$$

In order to estimate the term I_1, we note that if

$$(\psi_{t_1}^{(1)} \otimes \psi_{t_2}^{(2)}) * \chi_{F_k}(y_1, y_2) \ne 0,$$

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then $F_k \cap B_{y,t} \neq \emptyset$. Moreover, there exists

$$x^0 = (x_1^0, x_2^0) \in B_{y,t} \cap E_k^* \cap {}^cE_{k+1}$$

and we have

$$|e^{-t_1\sqrt{L}} \otimes e^{-t_2\sqrt{L}} f(y_1, y_2)| \leq f_L^*(x^0) \leq 2^{k+1}$$

which gives us that $\mathrm{I}_1 \leq C2^{2k}|E_k^*|$. We similarly have

$$\mathrm{I}_2 + \mathrm{I}_3 \leq C2^{2k}|E_k^*|.$$

We now obtain an estimate for the term I_4. It follows from the inequality $f(x_1, x_2) \leq f_L^*(x_1, x_2)$ that

$$\mathrm{I}_4 \leq C2^{2k}|E_k^*|.$$

The desired estimate (4.6) follows easily and then

$$\sum_{\bar{R} \in m(E_k^{**})} \|a_{k,\bar{R}}\|^2_{L^2(\mathbb{R}^{2n})} \leq C|E_k^{**}|^{-1}.$$

A similar argument to the one given above shows that

$$\sum_{\bar{R} \in m(E_k^{**})} \sum_{i,j=0}^{1} \ell(I)^{4i-4}\ell(J)^{4j-4}\|(L^i \otimes L^j)b_{k,\bar{R}}\|^2_{L^2(\mathbb{R}^{2n})} \leq C|E_k^{**}|^{-1}.$$

We have shown that for every $k \in \mathbb{Z}$ the expression $C^{-1}a_k$ is a product $(1, 2)$-atom associated with the open set E_k^{**} for some constant C. This shows that f has an decomposition of the form given in (1.3). The proof of Theorem 1.2 is complete. □

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