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On the Monodromy of Milnor Fibers of Hyperplane Arrangements

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Abstract. We describe a general setting where the monodromy action on the first cohomology group of the Milnor fiber of a hyperplane arrangement is the identity.

1 Introduction

Let $\mathcal{A} = \{H_1, \dots, H_d\} \subset \mathbb{C}^{n+1}$ be a central arrangement of d hyperplanes with Milnor fiber $F_{\mathcal{A}}$ and intersection lattice $L(\mathcal{A})$. For any edge $X \in L(\mathcal{A})$, we denote the corresponding subarrangement by $\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subset H\}$. We associate with $\mathcal{A}$ the projective arrangement $\mathcal{A}' \subset \mathbb{P}^n_{\mathbb{C}}$ obtained by associating with a hyperplane $H \in \mathcal{A}$, given by $\ell_H = 0$, the hyperplane $H' \in \mathbb{P}^n_{\mathbb{C}}$ defined by the same equation $\ell_H = 0$. We denote by $M(\mathcal{A})$ and $M(\mathcal{A}')$ the complements of $\mathcal{A}$ and $\mathcal{A}'$.

Consider the Orlik–Solomon algebra $A^*_R(\mathcal{A})$ of $\mathcal{A}$ with coefficients in a unitary commutative ring R and the corresponding Aomoto complex $\left(A^*_R(\mathcal{A}), \omega_1 \wedge\right)$, where $\omega_1 = \sum_{H \in \mathcal{A}} a_H \in A^1_R(\mathcal{A})$. Here $a_H \in A^1_R(\mathcal{A})$ denotes the element of $A^1_R(\mathcal{A})$ corresponding to the hyperplane H, see [16].

Let $\lambda = \exp(2\sqrt{-1}\pi/d)$. For $q \geq 0$, we denote by $H^q(F_{\mathcal{A}})_{\lambda^k}$ the λ^k-eigenspace of the monodromy operator $h^q \colon H^q(F_{\mathcal{A}}, \mathbb{C}) \to H^q(F_{\mathcal{A}}, \mathbb{C})$, for $0 \leq k \leq d-1$. There is a well-known relation between these eigenspaces and the cohomology of $M(\mathcal{A}')$ with coefficients in a rank one local system [2, 5]:

$$H^q(F_{\mathcal{A}})_{\lambda^k} = H^q\left(M(\mathcal{A}'), \mathcal{L}_{\lambda^k}\right), \tag{1.1}$$

where $\mathcal{L}_{\lambda^k}$ is the rank one local system on $M(\mathcal{A}')$ whose monodromy around any hyperplane of $\mathcal{A}'$ is λ^k. The main result of this note, Theorem 1.1 below, is a vanishing result describing many situations where $H^q(F_{\mathcal{A}})_{\lambda^k} = 0$ for $k \neq 0$.

Let us begin by introducing a new combinatorial object associated with a hyperplane arrangement $\mathcal{A}$, namely a graph $G(\mathcal{A})$, as follows:

- The vertices of $G(\mathcal{A})$ correspond to the hyperplanes of $\mathcal{A}$.
- Two different vertices H_1 and H_2 are linked by an edge (we will write $H_1 - H_2$) if and only if $\mathcal{A}_X = \{H_1, H_2\}$, where $X = H_1 \cap H_2$.

We say that such a graph is connected if for any two vertices H_1 and H_2, we can find an edge sequence linking H_1 and H_2.

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Note that graphs have been considered whose vertices are intersection points (rather than hyperplanes), see for instance [10], [11], and [21].

With the previous notations we have the following main result.

Theorem 1.1 *Suppose the following assumptions are verified.*

(i) *The graph $G(\mathcal{A})$ is connected.*
(ii) *For every codimension 2 intersection X of hyperplanes in $\mathcal{A}$, we have $|\mathcal{A}_X| \leq 9$.*
(iii) *We have either $6 \nmid d$, or there exists a hyperplane $H \in \mathcal{A}$ such that if X is an intersection of hyperplanes of $\mathcal{A}$ of codimension 2, $X \subset H$, then $|\mathcal{A}_X| \neq 6$.*

Then $H^1(F_{\mathcal{A}}, \mathbb{C}) = H^1(F_{\mathcal{A}})_1$, i.e., *$H^1(F_{\mathcal{A}})_{\lambda^k} = 0$ for $k \neq 0$.*

Remark 1.2 (i) We have $H^1(F_{\mathcal{A}})_1 = H^1\big(M(\mathcal{A}'), \mathcal{L}_{\lambda^0}\big) = H^1\big(M(\mathcal{A}'), \mathbb{C}\big) = \mathbb{C}^{d-1}$. Thus, Theorem 1.1 gives a large number of situations where $H^1(F_{\mathcal{A}}, \mathbb{C})$ is determined by the intersection lattice $L(\mathcal{A})$. Indeed, the graph $G(\mathcal{A})$ is constructed with the information given by $L(\mathcal{A})$. In general, the question whether the cohomology of $F_{\mathcal{A}}$ is determined by $L(\mathcal{A})$ is still open, even if many advances have been made; see for instance the results of A. Măcinic and S. Papadima [14] for the first Betti number of graphic arrangements and the results of M. Yoshinaga [22, 23] on real line arrangements. More recently, S. Papadima and A. Suciu [18] gave a positive answer to this question and an explicit formula for h^1 in terms of the cohomology of the Aomoto complex $\big(A^*_R(\mathcal{A}), \omega_1 \wedge\big)$ with coefficients in $R = \mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z}$ when every codimension 2 intersection X of hyperplanes in $\mathcal{A}$ satisfies $|\mathcal{A}_X| \leq 3$.

(ii) By taking a generic 3-dimensional subspace $E \subset \mathbb{C}^{n+1}$ and replacing $\mathcal{A}'$ by the corresponding line arrangement in $\mathbb{P}(E) = \mathbb{P}^2_{\mathbb{C}}$, we can consider from the beginning that $n = 2$. This follows from the Zariski Theorem of Lefschetz type due to Hamm, Hamm–Lê and Goresky–MacPherson, see for instance [4, p. 25] for the simplest version. Moreover, in the case of a line arrangement, the action of h^1 determines all the actions h^* in view of the usual formula for the zeta-function of the monodromy of the Milnor fiber of a homogeneous polynomial, see for instance [4, p. 107]. Alternatively, one may use the equality of Euler characteristics $E\big(M(\mathcal{A}'), \mathcal{L}_{\lambda^k}\big) = E\big(M(\mathcal{A}'), \mathbb{C}\big)$, see for instance [5, p. 49].

(iii) The case where every two distinct lines H, H' of $\mathcal{A}$ are linked by an edge corresponds to the case where $\mathcal{A}$ is generic, and then Theorem 1.1 follows from [2, Theorem 3.2].

The proof of Theorem 1.1 uses a deep result of Papadima and Suciu [17] on resonance varieties with coefficients in a finite field and a vanishing result of D. Cohen, A. Dimca and P. Orlik [3], obtained via perverse sheaves as explained in detail in [5]. As a corollary, we apply Theorem 1.1 to the braid arrangement to recover the results of S. Settepanella, A. Măcinic and S. Papadima in this particular case.

2 Demonstration of Theorem 1.1

The first result explains the role played by the graph $G(\mathcal{A})$ in this story.

Lemma 2.1 *Suppose the graph $G(\mathcal{A})$ is connected. Then $H^1\big(A_R^*(\mathcal{A}),\omega_1\wedge\big) = 0$ for any unitary commutative ring R.*

Proof We show that $\ker\{A_R^1 \xrightarrow{\omega_1\wedge} A_R^2\} = R\cdot\omega_1$. Let $b = \sum_{H\in\mathcal{A}} b_H a_H$ be an element of $A_R^1(\mathcal{A})$. For all $X \in L(\mathcal{A})$, we let $\omega_{1X} = \sum_{H|X\subset H} a_H$, and $b_X = \sum_{H|X\subset H} b_H a_H$. We have that $A_R^2(\mathcal{A}) = \bigoplus_{X\in L_2(\mathcal{A})} A_R^2(\mathcal{A}_X)$, with $L_2(\mathcal{A}) = \{X \in L(\mathcal{A}) \mid \operatorname{codim} X = 2\}$, by the Brieskorn decomposition theorem [16]. Hence we have

$$\omega_1 \wedge b = 0_R \Leftrightarrow \omega_{1X} \wedge b_X = 0_R, \quad \forall X \in L_2(\mathcal{A}).$$

So, suppose $\omega_1 \wedge b = 0_R$, and let us show that $b_H = b_{H'}$, for all $H \neq H' \in \mathcal{A}$.

Let H, H' be two distinct hyperplanes of $\mathcal{A}$. Then $X = H \cap H' \in L_2(\mathcal{A})$, and one of the following holds:

- H and H' are linked with an edge and $\mathcal{A}_X = \{H, H'\}$. In this case we have $\omega_{1X} \wedge b_X = 0_R \Rightarrow (a_H + a_{H'}) \wedge (b_H a_H + b_{H'} a_{H'}) = 0_R \Rightarrow b_H = b_{H'}$.
- H and H' are not directly linked by an edge but there exist hyperplanes $H_{i_1}, \dots, H_{i_m}$ of $\mathcal{A}$ such that H and H_{i_1}, H_{i_1} and $H_{i_2}, \dots, H_{i_m}$ and H' are linked. With the same considerations as in the first case, we have that $b_H = b_{H_{i_1}}, b_{H_{i_1}} = b_{H_{i_2}}, \dots, b_{H_{i_m}} = b_{H'}$, so $b_H = b_{H'}$.

Hence b and ω_1 are proportional and $H^1(A_R^*, \omega_1\wedge) = 0$. ∎

A shorter proof of Lemma 2.1 can be obtained using the connectivity of $G(\mathcal{A})$ and [13, Lemma 3.3], or [14, Lemma 4.9] if $R = \mathbb{Z}_p$ with p prime.

Remark 2.2 In fact, a more general version of Lemma 2.1 holds. If, for all $H \in \mathcal{A}$, $\omega = \sum_{H\in\mathcal{A}} \omega_H a_H$ satisfies $\omega_H \in R^\times$ and the graph $G(\mathcal{A})$ is connected, then we have $H^1\big(A_R^*(\mathcal{A}), \omega\wedge\big) = 0$ for any unitary commutative ring R. Indeed, suppose $\omega \wedge b = 0_R$. If H and H' are linked by an edge, then

$$\begin{aligned}(\omega_H a_H + \omega_{H'} a_{H'}) \wedge (b_H a_H + b_{H'} a_{H'}) = 0_R &\Rightarrow \omega_H b_{H'} - \omega_{H'} b_H = 0_R \\ &\Rightarrow \exists t \in R \quad \text{such that } \begin{cases} b_H = t\omega_H, \\ b_{H'} = t\omega_{H'}. \end{cases}\end{aligned}$$

If there exist hyperplanes $H_{i_1}, \dots, H_{i_m}$ of $\mathcal{A}$ such that H and H_{i_1}, H_{i_1} and $H_{i_2}, \dots, H_{i_m}$ and H' are linked by an edge, then with the same considerations we have that there exist scalars $t, t_1, \dots, t_m$ in R such that

$$\begin{cases} b_H = t\omega_H, \\ b_{H_{i_1}} = t\omega_{H_{i_1}}, \end{cases} \quad \begin{cases} b_{H_{i_1}} = t_1\omega_{H_{i_1}}, \\ b_{H_{i_2}} = t_1\omega_{H_{i_2}}, \end{cases} \quad \dots, \quad \begin{cases} b_{H_{i_m}} = t_m\omega_{H_{i_m}}, \\ b_{H'} = t_m\omega_{H'}. \end{cases}$$

By identification we find that $t = t_1 = \cdots = t_m$. Hence b and ω are proportional.

Let $(k_H)_{H\in\mathcal{A}}$ be a collection of integers with g.c.d. equal to 1 and define the corresponding multi-arrangement $\tilde{\mathcal{A}}\colon \tilde{Q} = \prod_{H\in\mathcal{A}} \ell_H^{k_H} = 0$, see [20]. Let $\tilde{F}\colon \tilde{Q} = 1$ be the corresponding Milnor fiber, and let $\tilde{h}$ be the corresponding monodromy $x \mapsto ux$ with $u \in \mathbb{C}^*$ a primitive root of the unity of order $N = \sum k_H = \deg \tilde{Q}$. Let $\rho_k\colon \pi_1\big(M(\mathcal{A})\big) \to \mathbb{C}^*$ be the representation such that $\rho_k(\gamma_H) = u^{k\cdot k_H}$, where γ_H is the meridian around the hyperplane H, and let $\mathcal{L}_{u^k}$ be the associated rank one local system on $M(\mathcal{A}')$. Then the analog of the formula (1.1) holds, and the interested reader may try to reformulate the main results of our note for the case of multi-arrangements. One must however add an extra condition, namely that $k_H \neq 0_{\mathbb{Z}_p}$ for all $H \in \mathcal{A}$, in order to apply the above extension of Lemma 2.1 when $R = \mathbb{Z}_p$.

The second result is rather general, and we include it here for the reader's sake, as we were not able to find a proper reference.

Lemma 2.3 *Let $\omega_1 \in A_R^*(\mathcal{A})$ be as above, and assume that $\partial\omega_1 = d = 0$ in R. Then $H^1\big(A_R^*(\mathcal{A}), \omega_1\wedge\big) = H^1\big(A_R^*(\mathcal{A}'), \omega_1\wedge\big)$.*

Proof Because $A_R^*(\mathcal{A}') = \ker\{A_R^*(\mathcal{A}) \xrightarrow{\partial} A_R^*(\mathcal{A})\} \subset A_R^*(\mathcal{A})$, it is clear that

$$H^1\big(A_R^*(\mathcal{A}'), \omega_1\wedge\big) \subset H^1\big(A_R^*(\mathcal{A}), \omega_1\wedge\big).$$

Let $b \in \ker\{A_R^1(\mathcal{A}) \xrightarrow{\omega_1\wedge} A_R^2(\mathcal{A})\}$. We have that $\partial(\omega_1\wedge b) = d\cdot b - \omega_1\cdot\partial b = -\omega_1\cdot\partial b = 0 \Rightarrow \partial b = 0$ in R. Hence $b \in \ker\{A_R^1(\mathcal{A}) \xrightarrow{\partial} A_R^0(\mathcal{A})\} = A_R^1(\mathcal{A}')$, and we have that $\ker\{A_R^1(\mathcal{A}) \xrightarrow{\omega_1\wedge} A_R^2(\mathcal{A})\} \subset \ker\{A_R^1(\mathcal{A}') \xrightarrow{\omega_1\wedge} A_R^2(\mathcal{A}')\}$, and $H^1\big(A_R^*(\mathcal{A}), \omega_1\wedge\big) \subset H^1\big(A_R^*(\mathcal{A}'), \omega_1\wedge\big)$. ∎

Remark 2.4 If we take H_d' to be the hyperplane at infinity, we can define $A_R^*(\mathcal{A}')$ as the Orlik–Solomon algebra of the affine arrangement $\mathcal{A}' = \{H_1', \dots, H_{d-1}'\} \subset \mathbb{C}^n$. Let $\omega_1' = \sum_{i=1}^{d-1} a_{H_i}' \in A_R^1(\mathcal{A}')$, where $a_{H_i}' \in A_R^1(\mathcal{A}')$ denotes the element of $A_R^1(\mathcal{A}')$ corresponding to the hyperplane H_i'. Then $a_{H_i}' = a_{H_i} - a_{H_d}$ for $1 \leq i \leq d-1$. So if R is a finite field of characteristic p, with p a prime dividing d, then we have $\omega_1' = \sum_{i=1}^{d-1}(a_{H_i} - a_{H_d}) = \sum_{i=1}^{d-1} a_{H_i} - (d-1)a_{H_d} = \sum_{i=1}^{d} a_{H_i} = \omega_1$. Hence $H^1\big(A_R^*(\mathcal{A}'), \omega_1'\wedge\big) = H^1\big(A_R^*(\mathcal{A}'), \omega_1\wedge\big)$.

Now we give the proof of Theorem 1.1. For this we consider two cases.

(1) If $6|d$, there exists $H \in \mathcal{A}$ such that if $X \in L_2(\mathcal{A})$, $X \subset H$, then $|\mathcal{A}_X| \neq 6$.

So the associated projective hyperplane $H' \in \mathcal{A}'$ is such that if $X \in L_2(\mathcal{A}')$, $X \subset H'$, then $|\mathcal{A}_X'| \neq 6$. Let $\lambda^k \neq 1$ be a d-th root of the unity. The only edge of $\mathcal{A}'$ contained in H' of codimension 1 is H', and the corresponding monodromy operator of $\mathcal{L}_{\lambda^k}$ is $T_{H'} = \lambda^k \neq 1$. Let $X \in L(\mathcal{A}')$ be a dense edge of $\mathcal{A}'$ of codimension 2 contained in H'. Then the corresponding monodromy operator of $\mathcal{L}_{\lambda^k}$ about the divisor associated to X is

$$T_X = \lambda^{k\cdot|\mathcal{A}_X'|},$$

with $|\mathcal{A}_X'| \in \{3,4,5,7,8,9\}$, see [5]. By using the vanishing result [5, Remark 2.4.20] applied to $\mathcal{L}_{\lambda^k}$, we have that

$$\operatorname{ord}(\lambda^k) \notin \{2,3,4,5,7,8,9\} \Rightarrow H^1(F_{\mathcal{A}})_{\lambda^k} = 0.$$

On the other hand, for a finite field $R = \mathbb{Z}_p$, [17, Theorem C] with Lemmas 2.1 and 2.3 and Remark 2.4 show that if $\operatorname{ord}(\lambda^k) = p^s$, with p prime and $s \geq 1$, *i.e.*, if $\operatorname{ord}(\lambda^k) \in \{2,3,4,5,7,8,9\}$, then again $H^1(F_{\mathcal{A}})_{\lambda^k} = 0$. Hence we have $H^1(F_{\mathcal{A}}, \mathbb{C}) = H^1(F_{\mathcal{A}})_1$ in all the above subcases.

(2) Suppose $6 \nmid d$, and let $X \in L(\mathcal{A}')$ be a dense edge of $\mathcal{A}'$ of codimension 2. With the same considerations as above, *i.e.*, using [5, Remark 2.4.20], we have

$$\operatorname{ord}(\lambda^k) \notin \{2,3,4,5,6,7,8,9\} \Rightarrow H^1(F_{\mathcal{A}})_{\lambda^k} = 0.$$

Because $\operatorname{ord}(\lambda^k)|d$, we have $\operatorname{ord}(\lambda^k) \neq 6$, so $\operatorname{ord}(\lambda^k) \in \{2,3,4,5,7,8,9\}$, and we conclude as in the previous case, using [17, Theorem C] and Lemma 2.1.

3 Applications and Related Results

We now apply Theorem 1.1 to the braid arrangement $\mathcal{A}_n \subset \mathbb{C}^{n+1}$ with Milnor fiber F_n. Recall that $\mathcal{A}_n$ is the collection of the hyperplanes

$$H_{ij}\colon x_i - x_j = 0, \quad 1 \leq i < j \leq n+1.$$

Corollary 3.1 *We have that $H^1(F_n, \mathbb{C}) = H^1(F_n)_1$, for any $n \geq 4$.*

Proof Let us show that $G(\mathcal{A}_n)$ is connected for $n \geq 4$. There are two types of intersections $X \in L_2(\mathcal{A}_n)$.

Type 1: The intersections $X = \{x_i = x_j, x_k = x_l, 1 \leq i < j < k < l \leq n+1\}$, with corresponding subarrangement $\mathcal{A}_X = \{H_{ij}, H_{kl}\}$.

Type 2: The intersections $X = \{x_i = x_j = x_k, 1 \leq i < j < k \leq n+1\}$, with corresponding subarrangement $\mathcal{A}_X = \{H_{ij}, H_{ik}, H_{jk}\}$.

Let $H_{ij}, H_{kl}, i < j, k < l$, be two distinct hyperplanes, and $X = H_{ij} \cap H_{kl} \in L_2(\mathcal{A}_n)$. Suppose $i \leq k$.

- If $\{i,j\} \cap \{k,l\} = \varnothing$, then X is type 1 and $\mathcal{A}_X = \{H_{ij}, H_{kl}\}$. Hence H_{ij} and H_{kl} are linked by an edge.
- If $\{i,j\} \cap \{k,l\} \neq \varnothing$, then three cases are possible:
  - (a) If $j = k$, then the set $I = \{i,j,k,l\}$ has three elements. Because $n \geq 4$, the set $\{1,2,\dots,n+1\}$ has at least five elements and so contains two elements $p < q$ such that $I \cap \{p,q\} = \varnothing$. Hence $H_{ij} \cap H_{pq}$ and $H_{jl} \cap H_{pq}$ are two type 1 intersections and we have that H_{ij} and H_{pq}, and H_{pq} and H_{jl}, are linked by an edge.
  - (b) If $j = l$, with the same considerations we have that there exist two elements $p < q$ such that H_{ij} and H_{pq}, and H_{pq} and H_{kj}, are linked by edges.
  - (c) If $i = k$, with the same considerations we have that there exist two elements $p < q$ such that H_{ij} and H_{pq}, and H_{pq} and H_{il}, are linked by edges.

This shows that $G(\mathcal{A}_n)$ is connected for $n \geq 4$.

Moreover, it is clear that $\mathcal{A}_n$ verifies the assumptions of Theorem 1.1 because $|\mathcal{A}_{nX}| = 2$ or 3, for all $X \in L_2(\mathcal{A}_n)$. ∎

Remark 3.2 For $n = 3$, the graph $G(\mathcal{A}_3)$ has three connected components, so is not connected. It is known that $H^1(F_3, \mathbb{C})_{\neq 1} = H^1(F_3)_{\lambda^2} \oplus H^1(F_3)_{\lambda^4}$ is 2-dimensional, see [14, 19]. Similarly, for $n = 2$, the graph $G(\mathcal{A}_2)$ has three connected components and $H^1(F_2, \mathbb{C})_{\neq 1} = H^1(F_2)_{\lambda} \oplus H^1(F_2)_{\lambda^2}$ is again 2-dimensional, see [14, 19]. For the

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Ceva arrangement given by

$$(x^3 - y^3)(y^3 - z^3)(x^3 - z^3) = 0,$$

the graph $G(\mathcal{A})$ has 9 connected components (there are no edges in this case). It is known that $H^1(F, \mathbb{C})_{\neq 1} = H^1(F)_{\lambda^3} \oplus H^1(F)_{\lambda^6}$ is 4-dimensional, see for instance [1].

Moreover, note that if $\mathcal{A}'$ is a line arrangement in $\mathbb{P}^2_{\mathbb{C}}$ coming from a pencil having $k \geq 3$ completely reducible fibers (see [9]) then the corresponding graph $G(\mathcal{A})$ has at least k connected components.

Corollary 3.3 *Assume that $\mathcal{A}'$ is a line arrangement in $\mathbb{P}^2_{\mathbb{C}}$ having only double and triple points. Assume that either*

(i) *the graph $G(\mathcal{A})$ is connected, or*

(ii) *there is one line containing exactly one triple point and d is even.*

Then $H^1(F_{\mathcal{A}}, \mathbb{C}) = H^1(F_{\mathcal{A}})_1$.

Proof The case (i) follows directly from Theorem 1.1.

(ii) Let $H \in \mathcal{A}$ be the line containing exactly one triple point p. Let $H_1, H_2 \in \mathcal{A}$ such that $\mathcal{A}_p = \{H, H_1, H_2\}$. Every $H_i \notin \mathcal{A}_p$ is linked by an edge with H, and we have that $G(\mathcal{A})$ is not connected if and only if H_1 or H_2 contains only triple points. For example, if H_1 would contain only triple points, we could count the hyperplanes of $\mathcal{A}$ in the following manner: $H_1, (H_{i_1}, H_{j_1}), (H_{i_2}, H_{j_2}), \ldots, (H_{i_{(d-1)/2}}, H_{j_{(d-1)/2}})$, where the pairs (H_i, H_j) correspond to the points of multiplicity 3 contained in H_1. Finally it would imply that $d = 2 \cdot (d-1)/2 + 1$ is odd, which contradicts our assumptions. Hence $G(\mathcal{A})$ is connected and we conclude directly with Theorem 1.1. ∎

Corollary 3.3 is a direct consequence of Theorem 1.1. The next result (with a proof similar to the proof of Theorem 1.1) is more general, and can be obtained also as a consequence of [12, Theorem 1.2] saying that if $H^1(F_{\mathcal{A}}, \mathbb{C}) \neq H^1(F_{\mathcal{A}})_1$, then $\mathcal{A}'$ comes from a pencil, so $G(\mathcal{A})$ is not connected. Indeed, for such a pencil, (ii) of Proposition 3.4 is not verified except for $d = 3$.

Proposition 3.4 *Assume that $\mathcal{A}'$ is a line arrangement in $\mathbb{P}^2_{\mathbb{C}}$ having only double and triple points. Assume that either*

(i) *the graph $G(\mathcal{A})$ is connected, or*

(ii) *$d = |\mathcal{A}'| > 3$ and there is one line containing exactly one triple point.*

Then $H^1(F_{\mathcal{A}}, \mathbb{C}) = H^1(F_{\mathcal{A}})_1$.

Proof The case (i) follows directly from Theorem 1.1.

(ii) Let $H' \in \mathcal{A}'$ be the line containing exactly one triple point p. Let $H_1', H_2' \in \mathcal{A}'$ be such that $\mathcal{A}_p' = \{H', H_1', H_2'\}$. Let $\lambda^k \neq 1$ be a d-th root of the unity. The only edge of $\mathcal{A}'$ of codimension 1 contained in H' is H', and the corresponding monodromy operator of $\mathcal{L}_{\lambda^k}$ is $T_{H'} = \lambda^k \neq 1$. The only dense edge of codimension 2 contained in H' is the triple point p, and the monodromy operator of $\mathcal{L}_{\lambda^k}$ about the divisor associated to p is $T_p = \lambda^{3k}$. By using the vanishing result [5, Remark 2.4.20] applied to $\mathcal{L}_{\lambda^k}$, we have that $\operatorname{ord}(\lambda^k) \neq 3 \Rightarrow H^1(F_{\mathcal{A}})_{\lambda^k} = 0$.

Now let us show that $H^1\left(A^*_{\mathbb{Z}_3}(\mathcal{A}), \omega_1 \wedge\right) = 0$.

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Let $b = \sum_{H \in \mathcal{A}} b_H a_H \in \ker\{ A^1_{\mathbb{Z}_3}(\mathcal{A}) \xrightarrow{\omega_1 \wedge} A^2_{\mathbb{Z}_3}(\mathcal{A}) \}$. If $H_k \in \mathcal{A} \backslash \mathcal{A}_p$, then $X = H \cap H_k$ is such that $\mathcal{A}_X = \{H, H_k\}$ and with the same considerations as in the proof of Lemma 2.1 we have $\omega_{1X} \wedge b_X = 0 \Leftrightarrow b_{H_k} = b_H$. We will show that $b_{H_1} = b_{H_2} = b_H$. Let $H_k \in \mathcal{A} \backslash \mathcal{A}_p$, and $X_1 = H_1 \cap H_k$, $X_2 = H_2 \cap H_k$, be the corresponding intersections with H_1 and H_2. We consider several cases.

- If $|\mathcal{A}_{X_1}| = 2$ and $|\mathcal{A}_{X_2}| = 2$, then $\omega_{1X_1} \wedge b_{X_1} = 0$ and $\omega_{1X_2} \wedge b_{X_2} = 0$ if and only if $b_{H_1} = b_{H_k}$ and $b_{H_2} = b_{H_k}$. As $b_{H_k} = b_H$, we obtain $b_{H_1} = b_{H_2} = b_H$.
- If $|\mathcal{A}_{X_1}| = 3$ and $|\mathcal{A}_{X_2}| = 2$, then $\mathcal{A}_{X_1} = \{H_1, H_k, H_j\}$, with $H_j \neq H, H_2$ (if $H_j = H$ or H_2, then $X_1 = p$ and $H_k \in \mathcal{A}_p$), and $\mathcal{A}_{X_2} = \{H_2, H_k\}$.

  By taking $\{a_{H_1} a_{H_k}, a_{H_1} a_{H_j}\}$ as a base of $A^2_{\mathbb{Z}_3}(\mathcal{A})$, we have that

$$\begin{aligned}\omega_{1X_1} \wedge b_{X_1} = 0 &\Leftrightarrow a_{H_1} a_{H_k}(2b_{H_k} - b_{H_1} - b_{H_j}) + a_{H_1} a_{H_j}(2b_{H_j} - b_{H_1} - b_{H_k}) = 0\\ &\Leftrightarrow b_{H_1} + b_{H_k} + b_{H_j} = 0 \quad (*),\end{aligned}$$

  in $\mathbb{Z}_3$. With $b_{H_k} = b_H$ and $b_{H_j} = b_H$, we obtain $(*) \Leftrightarrow b_{H_1} = b_H$. We also have $\omega_{1X_2} \wedge b_{X_2} = 0 \Leftrightarrow b_{H_2} = b_{H_k}$. Finally $b_{H_1} = b_{H_2} = b_H$.
- If $|\mathcal{A}_{X_1}| = 3$ and $|\mathcal{A}_{X_2}| = 3$, then $\mathcal{A}_{X_1} = \{H_1, H_k, H_j\}$, with $H_j \neq H, H_2$, and $\mathcal{A}_{X_2} = \{H_2, H_k, H_l\}$, with $H_l \neq H, H_1$. With the same considerations as in the previous case we have

$$\omega_{1X_1} \wedge b_{X_1} = 0 \Leftrightarrow b_{H_1} + b_{H_k} + b_{H_j} = 0 \quad (*)$$

  and

$$\omega_{1X_2} \wedge b_{X_2} = 0 \Leftrightarrow b_{H_2} + b_{H_k} + b_{H_l} = 0. \quad (**)$$

  With $b_{H_k} = b_{H_j} = b_{H_l} = b_H$, we obtain $(*) \Leftrightarrow b_{H_1} = b_H$, and $(**) \Leftrightarrow b_{H_2} = b_H$. Finally $b_{H_1} = b_{H_2} = b_H$.

Hence b and ω_1 are proportional and $H^1\big(A^*_{\mathbb{Z}_3}(\mathcal{A}), \omega_1 \wedge\big) = 0$.

On the other hand, [17, Theorem C] with our Lemmas 2.1 and 2.3 and Remark 2.4 for $R = \mathbb{Z}_3$ give $\operatorname{ord}(\lambda^k) = 3 \Rightarrow H^1(F_{\mathcal{A}})_{\lambda^k} = 0$. ∎

The following examples show the difficulty of the problem in the general case. First we give an example where $G(\mathcal{A})$ is not connected, showing that the conditions (ii) and (iii) in Theorem 1.1 are not sufficient.

Example 3.5 Let $\mathcal{A}' \subset \mathbb{P}^2_{\mathbb{C}}$ be the arrangement defined by the homogeneous polynomial $Q(x{:}y{:}z) = xyz(x^4 - y^4)(y^4 - z^4)(x^4 - z^4)$. Lines of $\mathcal{A}'$ are

$$\{x = 0\}, \{y = 0\}, \{z = 0\}, d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8, d_9, d_{10}, d_{11}, d_{12},$$

where d_1, d_2, d_3, d_4 are of the form $x = \alpha y$, d_5, d_6, d_7, d_8 are of the form $y = \alpha z$, and $d_9, d_{10}, d_{11}, d_{12}$ are of the form $x = \alpha z$, with $\alpha^4 = 1$. The intersections between d_1, d_2, d_3, d_4 and $\{z = 0\}$ are double points, and the same is true for d_5, d_6, d_7, d_8 with $\{x = 0\}$, and for $d_9, d_{10}, d_{11}, d_{12}$ with $\{y = 0\}$. The other intersections of lines in $\mathcal{A}'$ are points of multiplicity 3 or 6. Indeed, if we take $i \in \{1, 2, 3, 4\}$ and $j \in \{5, 6, 7, 8\}$, we have $d_i \cap d_j := \{x = \alpha_i y\} \cap \{y = \alpha_j z\}$, with $(\alpha_i \alpha_j)^4 = 1$. So, $d_i \cap d_j = d_i \cap d_j \cap d_k$, where $d_k := \{x = \alpha_i \alpha_j z\}$, $k \in \{9, 10, 11, 12\}$. Similarly we have a point of multiplicity 3 if we take $i \in \{1, 2, 3, 4\}$ and $j \in \{9, 10, 11, 12\}$, or if we take $i \in \{5, 6, 7, 8\}$ and $j \in \{9, 10, 11, 12\}$. Then we have three points of multiplicity 6: $d_1 \cap d_2 \cap d_3 \cap d_4 \cap \{x = 0\} \cap \{y = 0\}$, $d_5 \cap d_6 \cap d_7 \cap d_8 \cap \{y = 0\} \cap \{z = 0\}$, and

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$d_9 \cap d_{10} \cap d_{11} \cap d_{12} \cap \{x = 0\} \cap \{z = 0\}$. Hence $G(\mathcal{A})$ has three connected components and is not connected. It's clear that $\mathcal{A}$ verifies (ii) and (iii) of Theorem 1.1, and we have that $H^1(F_{\mathcal{A}}, \mathbb{C}) \neq H^1(F_{\mathcal{A}})_1$, see [7, Remark 3.4 (iii)].

When the assumptions of Theorem 1.1 are not verified, it is very complicated to proceed, and we have to use other results.

Example 3.6 Let $\mathcal{A}' \subset \mathbb{P}^2_{\mathbb{C}}$ be the arrangement defined by the homogeneous polynomial $Q(x:y:z) = xyz(x^2 - y^2)(y^2 - z^2)(x^2 - z^2)$. With the same arguments as in Example 3.5 we can show that $G(\mathcal{A})$ is not connected and that $\mathcal{A}$ verifies points (ii) and (iii) of Theorem 1.1. But here, $H^1(F_{\mathcal{A}}, \mathbb{C}) = H^1(F_{\mathcal{A}})_1$, see [2] or [7, Remark 3.4 (ii)].

Example 3.7 Let $\mathcal{A}' \subset \mathbb{P}^2_{\mathbb{C}}$ be the arrangement with defining polynomial

$$\begin{aligned} Q(x:y:z) = & xy(x+y)(x-y)(x+2y)(x-2y)(2x+y+z)(2x+y+2z) \\ & \times (2x+y+3z)(2x+y-z)(2x+y-2z)(2x+y-3z). \end{aligned}$$

Here $d = 12$ and we have two intersections of multiplicity 6: $\{x = y = 0\}$, and $\{y = -2x\} \cap \{z = 0\}$. One can easily verify that each hyperplane contains one of these two intersections and that $G(\mathcal{A})$ is connected. Indeed, any hyperplane in

$$\big\{ \{x=0\}, \{y=0\}, \{x+y=0\}, \{x-y=0\}, \{x+2y=0\}, \{x-2y=0\} \big\}$$

is linked by an edge with any hyperplane in

$$\begin{aligned} \big\{ & \{2x+y+z=0\}, \{2x+y+2z=0\}, \{2x+y+3z=0\}, \\ & \{2x+y-z=0\}, \{2x+y-2z=0\}, \{2x+y-3z=0\} \big\}. \end{aligned}$$

Hence (i) and (ii) of Theorem 1.1 are verified, but not (iii).

The minimal number of lines in $\mathcal{A}'$ containing all the points of multiplicity at least 3 is 2, so with [15, Theorem 1.1] we have that $\mathcal{A}'$ belongs to the class $\mathcal{C}_2$, and any rank one local system on $M(\mathcal{A}')$ is admissible. Hence if we take $\lambda^k \neq 1$, there exists

$$\omega = \sum_{H \in \mathcal{A}'} \omega_H \frac{d\ell_H}{\ell_H} \in H^1\big(M(\mathcal{A}'), \mathbb{C}\big)$$

such that

$$\dim H^1(F_{\mathcal{A}})_{\lambda^k} = \dim H^1\big(M(\mathcal{A}'), \mathcal{L}_{\lambda^k}\big) = \dim H^1\Big(H^*\big(M(\mathcal{A}'), \mathbb{C}\big), \omega\wedge\Big).$$

Furthermore, it is known that $\exp(2\pi\sqrt{-1}\omega_H) = \lambda^k$, for all H, so $\omega_H \neq 0$, $\forall H$.

Assume $\dim H^1\big(H^*(M(\mathcal{A}'), \mathbb{C}), \omega\wedge\big) \neq 0$. Then

$$\omega \in \mathcal{R}_1(\mathcal{A}') = \Big\{ \alpha \in H^1\big(M(\mathcal{A}'), \mathbb{C}\big) \ \Big|\ \dim H^1\Big(H^*\big(M(\mathcal{A}'), \mathbb{C}\big), \alpha\wedge\Big) \geq 1 \Big\}.$$

With the description of the irreductible components of the first resonance variety for a $\mathcal{C}_2$ arrangement, see [8, Theorem 4.3], we have a contradiction with the fact that $\omega_H \neq 0$, for all H.

Hence $H^1\big(H^*(M(\mathcal{A}'), \mathbb{C}), \omega\wedge\big) = 0$ and $H^1(F_{\mathcal{A}}, \mathbb{C}) = H^1(F_{\mathcal{A}})_1$.

Example 3.8 Let $\mathcal{A} \subset \mathbb{C}^4$ be the arrangement defined by the homogeneous polynomial $Q(x, y, z, t) = xy(x-y)(x+y)(x-2y)(x+2y)zt(z-t)(z+t)(z-2t)(z+2t)$. Here $d = 12$ and we have two intersections in $L_2(\mathcal{A})$ of multiplicity 6: $\{x = y = 0\}$, and $\{z = t = 0\}$. One can easily verify that each hyperplane contains one of these two intersections and that $G(\mathcal{A})$ is connected. Indeed, any hyperplane in

$$\big\{\{x = 0\}, \{y = 0\}, \{x - y = 0\}, \{x + y = 0\}, \{x - 2y = 0\}, \{x + 2y = 0\}\big\}$$

is linked by an edge with any hyperplane in

$$\big\{\{z = 0\}, \{t = 0\}, \{z - t = 0\}, \{z + t = 0\}, \{z - 2t = 0\}, \{z + 2t = 0\}\big\}.$$

Hence (i) and (ii) of Theorem 1.1 are verified, but not (iii).

One can decompose $\mathcal{A}$ into two arrangements with distinct variables: $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$, where $\mathcal{A}_1$ is defined by $Q_1(x, y) = xy(x - y)(x + y)(x - 2y)(x + 2y)$ and $\mathcal{A}_2$ is defined by $Q_2(z, t) = zt(z - t)(z + t)(z - 2t)(z + 2t)$. Let us take $\lambda^k \neq 1$ and denote by F_1 and F_2 the Milnor fibers of the subarrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ in $\mathbb{C}^2$. Then applying [6, Theorem 1.4 (i)] we have that

$$H^1(F_{\mathcal{A}})_{\lambda^k} = \bigoplus_{a+b+c=1} H^a(\mathbb{T}, \mathbb{C}) \otimes H^b(F_1)_{\lambda^k} \otimes H^c(F_2)_{\lambda^k} = 0,$$

so $H^1(F_{\mathcal{A}}, \mathbb{C}) = H^1(F_{\mathcal{A}})_1$.

In fact, a more general version of Corollary 3.1 holds. Let Γ be a simple graph (that is to say it contains no loop and no double edge) and let $\mathcal{A}_\Gamma$ be the corresponding graphic arrangement, see for instance [14]. In other words, $\mathcal{A}_\Gamma$ is a subarrangement of the braid arrangement. Such a graph Γ is composed of edges (ij), $i < j$, and the corresponding arrangement $\mathcal{A}_\Gamma$ is composed of the hyperplanes $H_{ij} : x_i - x_j = 0$, $(ij) \in \Gamma$. We will denote by $|\Gamma|$ the number of vertices of Γ. We say that Γ is *connected* if we can link any two different vertices with an edge sequence. Let $\omega_1 = \sum_{(ij)\in\Gamma} a_{ij} \in A^1_R(\mathcal{A}_\Gamma)$.

We have the following result.

Lemma 3.9 *Suppose Γ is connected and $|\Gamma| \geq 5$, then $H^1\big(A^*_R(\mathcal{A}_\Gamma), \omega_1\wedge\big) = 0$ for any unitary commutative ring R.*

Proof Let $b = \sum_{(ij)\in\Gamma} b_{ij} a_{ij} \in A^1_R(\mathcal{A}_\Gamma)$ be such that $\omega_1 \wedge b = 0$. Let us show that b and ω_1 are proportional. With [13, Lemma 3.3] or [14, Lemma 4.9] we have the following:

(i) For all intersections of type 1: $H_{ij} \cap H_{kl}$, $i < j < k < l$, we have $b_{ij} = b_{kl}$.
(ii) For all intersections of type 2: $H_{ij} \cap H_{ik} \cap H_{jk}$, $i < j < k$, we have either $b_{ij} = b_{ik} = b_{jk}$ if $3 \neq 0_R$, or $b_{ij} + b_{ik} + b_{jk} = 0$ if $3 = 0_R$.

So let us suppose that $3 = 0_R$, and let us take an intersection of type 2: $H_{ij} \cap H_{ik} \cap H_{jk}$, $i < j < k$. We will show that $b_{ij} = b_{ik} = b_{jk}$.

Because $|\mathcal{A}_\Gamma| \geq 5$, there exist two additional vertices s and m, so there exist two additional edges. Because Γ is connected, these two edges are linked either

(1) with one of the vertices of the triangle ijk,
(2) or with two different vertices of the triangle ijk,

(3) or one of these edges is linked with a vertex of the triangle ijk, and the other is linked with the new vertex of the first one.

By symmetry, we can assume that we are in one of the following cases (here we have chosen $i < j < k < s < m$, but of course the order does not matter).

(1) If (is), $(im) \in \Gamma$, then one has the following.
   (a) If (js), $(km) \notin \Gamma$, then $H_{ij} - H_{is} - H_{jk} - H_{im} - H_{ik}$, and $b_{ij} = b_{ik} = b_{jk}$.
   (b) If (js), $(km) \in \Gamma$, then $H_{ij} - H_{km} - H_{js} - H_{ik} - H_{js} - H_{im} - H_{jk}$, and $b_{ij} = b_{ik} = b_{jk}$.
   (c) If $(js) \in \Gamma$, $(km) \notin \Gamma$, then $H_{ik} - H_{im} - H_{jk}$, and $b_{ik} = b_{jk}$. With $b_{ij} + b_{ik} + b_{jk} = 0$, we have $b_{ij} = -2b_{ik} = b_{ik}$ in R.
   (d) If $(js) \notin \Gamma$, $(km) \in \Gamma$, it is the symmetric case of the previous one.
(2) If (is), $(jm) \in \Gamma$, then $H_{ik} - H_{jm} - H_{is} - H_{jk}$, and $b_{ik} = b_{jk}$. With $b_{ij} + b_{ik} + b_{jk} = 0$, we have $b_{ij} = b_{ik}$.
(3) If (is), $(sm) \in \Gamma$, then H_{ij}, H_{ik} and H_{jk} are linked by an edge with H_{sm} in $G(\mathcal{A}_\Gamma)$ and we can conclude directly. ∎

Remark 3.10 (i) We have that Γ is connected and $|\Gamma| \geq 5$ does not imply $G(\mathcal{A}_\Gamma)$ is connected; the graphic arrangement $\mathcal{A}_\Gamma = \{H_{12}, H_{13}, H_{14}, H_{15}, H_{23}, H_{34}, H_{35}\}$ verifies Γ is connected and $|\Gamma| \geq 5$, but $G(\mathcal{A}_\Gamma)$ is not connected (H_{13} is linked with no hyperplane of $\mathcal{A}_\Gamma$) and we cannot apply Lemma 2.1.

(ii) If Γ is connected and $|\Gamma| \geq 5$, then we recover Măcinic and Papadima's results for graphic arrangements [14]. Indeed, with similar considerations as in the proof of Theorem 1.1 and by using Lemma 3.9 instead of Lemma 2.1, we have that $H^1(F_{\mathcal{A}_\Gamma}, \mathbb{C}) = H^1(F_{\mathcal{A}_\Gamma})_1$.

If the graph Γ is not connected, but each of its connected components Γ_i satisfies $|\Gamma_i| \geq 5$, then $\mathcal{A}_\Gamma$ is a product of arrangements $\mathcal{A}_{\Gamma_i}$ and we can conclude using [6, Theorem 1.4(i)].

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