CORRESPONDENCE.

THE GENERAL THEORY OF SUMMATION FORMULAS.

[We have received from Mr. Julius Altenburger a letter giving a resumé of the second paper referred to in the review on p. 449. Owing to the pressure on our space we are unable to print the letter in full, but the following slightly modified extracts will indicate the general course of Mr. Altenburger's investigation.—ED. *J.I.A.*]

Mechanical operations for graduation can be represented by the general formula $A(u_x)=\Sigma_{-n}^{n}a_\kappa u_{x+\kappa}$, where A is the symbol of operation, u the series of values to be graduated, and a. the coefficients of the formula of operation. If $a_\kappa=a_{-\kappa}$ (as is usual in practice), the operation is a symmetrical one. The investigation, however, is general, and applicable to asymmetrical as well as symmetrical operations.

Let $\eta_x=y_x+\epsilon_x$, where η_x is the observed value, y the (unknown) true value, and ϵ the (unknown) error of observation.

Then $A(\eta_x)=A(y_x)+A(\epsilon_x)$, and A will be a graduating operation if $A(y_x)=y_x$, and $\Sigma[A(\epsilon)_x]^2$ is $<\Sigma[\epsilon_x]^2$.

The second condition will be fulfilled if Σa^2_κ is <1, and the less the sum of the squares of a the greater will be the smoothing effect of the formula. Since Σa^2 cannot be zero, no operation can give the true value y, the result of graduation being merely to restrict the limits of the errors of observation.

The first condition leads to the general problem of the reproduction of functions by mechanical operations. On the assumption that $y_{x+\kappa}$ can be expanded in the series—

$$y_x+\kappa y'_x+\frac{\kappa^2}{2!}y''_x+\ldots$$

where $y^{(\nu)}$ denotes $\frac{d^\nu y}{dx^\nu}$.

$$A(y_x)=y_x\Sigma a_\kappa+y'_x\Sigma\kappa a_\kappa+\ldots+\frac{y^\nu}{\nu!}\Sigma\kappa^\nu a_\kappa+\ldots$$

and the conditions of reproduction are

$$\Sigma a_\kappa=1\text{ ; and}$$

$$\Sigma\kappa^\nu a_\kappa=0\text{ for }\nu=1,\ 2,\ 3,\ \ldots$$

The number of coefficients a being limited, these conditions can be satisfied for only a limited value of ν, so that only rational integral functions are exactly reproducible.

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Let A_r denote an operation of degree r, for which the condition $\Sigma\kappa^\nu a_\kappa = 0$ is satisfied for $\nu = 1, 2 \ldots r$. Then, if y be of the sth degree,

$$A_r(y_x) = y_x + \sum_{\nu=r+1}^{\nu=s} \left[\frac{y_x^{(\nu)}}{\nu\,!} \Sigma\kappa^\nu a\kappa \right].$$

$$= y_x + \phi_r, \text{ say.}$$

The expression ϕ_r is of the $(s-r-1)$th degree; and, consequently,

$$A_r(\phi_r) = \phi_r + \phi_{2r+1} \text{ (say)},$$

where ϕ_{2r+1} is of the $\{s-2(r+1)\}$th degree.

Hence, $A_r^2(y_x) = y_x + 2\phi_r + \phi_{2r+1}$,

and generally

$$A_r^m(y_x) = y_x + m\phi_r + \frac{m(m-1)}{2\,!}\phi_{2r+1} + \ldots + \frac{m\,!}{(m-p)\,!\,p\,!}\phi_{p(r+1)-1}$$

The elimination of the expressions ϕ leads to the general formula of approximation—

$$y = A^m - m\Delta_m + \frac{(m+1)\,!}{2\,!\,(m-1)\,!}\Delta^2{}_m - \ldots + (-1)^i\frac{(m+i-1)\,!}{i\,!\,(m-1)\,!}\Delta^i{}_m + \;\ldots$$

where A^m is written for $A_r{}^m(y_x)$ and $\Delta_m = A^{m+1} - A^m$.

[This result may be readily obtained by the method of operators. Since $\phi_r = A(y) - y = \Delta y$; $\phi_{2r+1} = A(\phi_r) - \phi_r = \Delta\phi_r = \Delta^2 y$, &c.; where the differences have reference to the series y, $A(y)$, $A^2(y)$, &c., the expression for $A^m(y)$ may be written in the form $(1+\Delta)^m y$. Hence, $y = (1+\Delta)^{-m}A^m = A^m - m\Delta_m + \frac{(m+1)m}{2}\Delta_m^2 - \ldots$—Ed. *J.I.A.*]

Let $A(u_x) = \frac{1}{2n+1}\Sigma_{-n}^n u_{x+\kappa}$.

Then A is an operation of the first degree, because $\Sigma a_\kappa = 1$, $\Sigma\kappa a_\kappa = 0$,

and $\Sigma\kappa^2 a_\kappa = \frac{2}{2n+1}\Sigma_1^n\kappa^2$. Since $\Sigma\left(\frac{1}{2n+1}\right)^2 = \frac{1}{2n+1}$, which is <1,

A is a smoothing operation. The substitution of this operation in the general formula gives a very simple process of graduating to any requisite degree of reproduction and smoothing. The expression A^m then gives an approximate representation of the function on the assumption that *first* differences (of η) are constant, $A^m - m\Delta_m$, on the assumption that *third* differences are constant, $A^m - m\Delta_m + \frac{(m+1)m}{2}\Delta^2{}_m$, on the assumption that *fifth* differences are constant, and so on.

The resulting errors may be represented generally by $\eta - y^{(N)}$. The degree of approximation (N) may be considered sufficient if these errors are distributed about zero. This may be tested by the application to the errors of a strong graduating operation of the first degree. If the $A(\epsilon^N)$ form a series of positive or negative values, the reproduction is not sufficiently accurate, and an approximation of higher degree must be obtained.

[Mr. Altenburger proceeds to show that formulas of reproduction can be obtained by combining different operations A, B, instead of by repetition of the single operation A, and deduces Woolhouse's, Higham's, and Karup's formulas. He also investigates the effect of the operator A^m on a series following Makeham's law.]

The general process admits of the graduation of θ_x and E_x separately. The ratio of the graduated values then gives a graduated value of q, where account is taken of the weights of the observations.

The method may be applied to functions of two or more variables (*e.g.*, Select Tables). The process then consists of two or more graduations, one of the variables at a time being regarded as varying while the others remain constant.