Canad. Math. Bull. Vol. 58 (3), 2015 pp. 632–650
http://dx.doi.org/10.4153/CMB-2015-023-0

Quantum Unique Ergodicity on Locally Symmetric Spaces: the Degenerate Lift

Lior Silberman

Abstract. Given a measure $\bar{\mu}_\infty$ on a locally symmetric space $Y = \Gamma\backslash G/K$ obtained as a weak-* limit of probability measures associated with eigenfunctions of the ring of invariant differential operators, we construct a measure $\bar{\mu}_\infty$ on the homogeneous space $X = \Gamma\backslash G$ that lifts $\bar{\mu}_\infty$ and is invariant by a connected subgroup $A_1 \subset A$ of positive dimension, where $G = NAK$ is an Iwasawa decomposition. If the functions are, in addition, eigenfunctions of the Hecke operators, then $\bar{\mu}_\infty$ is also the limit of measures associated with Hecke eigenfunctions on X. This generalizes results of the author with A. Venkatesh in the case where the spectral parameters stay away from the walls of the Weyl chamber.

1 Introduction

Previous work of the author with A. Venkatesh [17] investigated the asymptotic behaviour of eigenfunctions on high-rank locally symmetric spaces, under the assumption that the spectral parameters (see below) were *non-degenerate*, in that their imaginary parts were located away from the walls of the Weyl chamber (in particular, this forced the spectral parameters to lie on the unitary axis). This paper removes this assumption at the cost of a weaker invariance statement for the limiting measures. In addition, it obtains a useful bound on the norm of vectors in a unitarizable principal series representation with non-unitary parameter.

1.1 The Problem of Quantum Unique Ergodicity: Statement of the Result

Initially, let Y be a (compact) Riemannian manifold. To a non-zero eigenfunction ψ_n of the Laplace–Beltrami operator $\triangle$ with eigenvalue $-\lambda_n$ we attach the probability measure

$$\bar{\mu}_n(\phi) = \frac{1}{\|\psi_n\|^2} \int_Y |\psi_n(y)|^2 \phi(y)\,\mathrm{d}y\,.$$

Classifying the possible limits (in the weak-* sense) of sequences $\{\bar{\mu}_n\}_{n=1}^{\infty}$, where $\lambda_n \to \infty$ is known as the problem of "Quantum Unique Ergodicity" (specifically, "QUE on Y"). Nearly all attacks on this problem begin by associating with each measure $\bar{\mu}_n$ a distribution ("Wigner measure" or "microlocal lift") μ_n on the unit cotangent bundle S^*Y that projects to $\bar{\mu}_n$ on Y, in such a way that any weak-* limit of the μ_n is a probability measure, invariant under the geodesic flow on S^*Y. This approach

Received by the editors November 4, 2013; revised January 15, 2015.
Published electronically April 21, 2015.
The author is currently supported by an NSERC Discovery Grant.
AMS subject classification: 22E50, 43A85.
Keywords: quantum unique ergodicity, microlocal lift, spherical dual.

(due to Schnirel'man, Zelditch, and Colin de Verdière [6, 18, 22]) leads to a reformulation of the problem ("QUE on S^*Y"), where one seeks to classify the weak-* limits of sequences such as $\{\mu_n\}_{n=1}^{\infty}$. Here results from dynamical systems concerning measures invariant under the geodesic flow can be brought to bear. In particular, under the very general assumption that the geodesic flow on S^*Y is ergodic, it was shown by these authors that the Liouville measure on S^*Y (the Riemannian volume associated with the natural extension of the metric from Y to S^*Y) is the limit measure for a sequence of eigenfunctions of density 1. Thus the Riemannian volume on Y, being the projection of Liouville measure, is always a limit of a sequence of measures $\bar{\mu}_n$, and other limit measure are an exception and must correspond to sparse subsequences of eigenfunctions. The most spectacular realization of this approach to QUE is in the work of Lindenstrauss [13] on congruence hyperbolic surfaces, showing, for eigenfunctions ψ_n which are also eigenfunctions of the so-called Hecke operators, that the Riemannian volume is the only limiting measure.[1] In fact, Rudnick and Sarnak [15] conjecture that this phenomenon (uniqueness of the limit) holds for all compact manifolds Y of (possibly variable) negative sectional curvature. Results in that level of generality have also appeared recently, starting with the breakthrough of [1].

In this paper we consider a technical aspect of the problem on locally symmetric spaces $Y = \Gamma\backslash G/K$ of non-compact type. Here G is a semisimple Lie group with finite centre, K is a maximal compact subgroup, and $\Gamma < G$ is a lattice (thus Y is of finite volume but not necessarily compact). On such spaces there is a natural commutative algebra of differential operators containing the Laplace–Beltrami operator, the algebra of G-invariant differential operators on G/K, and it is more natural to consider joint eigenfunctions $\psi_n \in L^2(Y)$ of this algebra.

In [17], generalizing the works of Zelditch [23] and Wolpert [20] for hyperbolic surfaces (see also [12]), a representation-theoretic approach to the microlocal lift on Y is given under a genericity assumption ("non-degeneracy"), that the sequence of spectral parameters $\nu_n \in \mathfrak{a}_{\mathbb{C}}^*$ (here $\mathfrak{a} = \mathrm{Lie}(A)$, where $G = NAK$ is an Iwasawa decomposition) associated with the ψ_n be contained in a proper subcone of the open Weyl chamber in $i\mathfrak{a}_{\mathbb{R}}^*$. Under that assumption, and any weak-* limit $\bar{\mu}_\infty$ of a sequence as above was seen to be the projection of an A-invariant positive measure μ_∞ on $X = \Gamma\backslash G$. In this paper the non-degeneracy assumption is removed.

Theorem 1.1 *Assume that*

$$\bar{\mu}_n \xrightarrow[n\to\infty]{wk\text{-}*} \bar{\mu}_\infty.$$

Then there exists a non-trivial connected subgroup $A_1 \subset A$ and an A_1-invariant positive measure σ_∞ on X projecting to $\bar{\mu}_\infty$.

In more detail, let $C_c^\infty(X)_K$ be the space of right K-finite smooth functions of compact support on $X = \Gamma\backslash G$. By a distribution *we shall mean an element of its algebraic dual. Then, after passing to a subsequence, we obtain distributions $\mu_n \in C_c^\infty(X)_K'$ and functions $\widetilde{\psi}_n \in L^2(X)$ such that the following hold.*

(i) (Lift) *The distributions μ_n project to the measures $\bar{\mu}_n$ on Y. In other words, for $\phi \in C_c^\infty(Y)$ we have $\mu_n(\phi) = \bar{\mu}_n(\phi)$.*

[1] For non-compact surfaces this statement requires the result of [19].

https://doi.org/10.4153/CMB-2015-023-0 Published online by Cambridge University Press

(ii) *Let σ_n be the measure on X such that* $\mathrm{d}\sigma_n(x) = |\widetilde{\psi}_n(x)|^2\,\mathrm{d}x$. *Then*

(a) (Positivity) $\{\sigma_n\}_{n=1}^{\infty}$ *converges weak-* to a measure σ_∞ on X, necessarily a positive measure of total mass* ≤ 1.

(b) (Consistency) *For any* $\phi \in C_c^\infty(X)_K$, $|\sigma_n(\phi) - \mu_n(\phi)| \to 0$ *as* $n \to \infty$.

(iii) (Invariance) *Let the normalized spectral parameters*[2] $\widetilde{\nu}_n$ *converge to a limiting parameter* $\widetilde{\nu}_\infty$ *in the closed positive Weyl chamber of* $i\mathfrak{a}_{\mathbb{R}}^*$. *Then* μ_∞ *is invariant by* $A_1 Z_K(A_1)$, *where* $A_1 \subset A$ *is the set of elements fixed by the stabilizer* $W_1 = \mathrm{Stab}_W(\widetilde{\nu}_\infty)$ *(here* $W = N_G(A)/Z_G(A)$ *is the Weyl group of G; it acts naturally on A and* $\mathfrak{a}_{\mathbb{R}}^*$*).*

(iv) (Equivariance) $\widetilde{\psi}_n$ *belongs to the irreducible subrepresentation of G in* $L^2(X)$ *generated by* ψ_n *(the latter thought of as a K-invariant function on X). In particular, if* $\mathcal{H}$ *is a commutative algebra of bounded operators on* $L^2(X)$ *that commute with the G-action and* ψ_n *is a joint eigenfunction of* $\mathcal{H}$, *then so is* $\widetilde{\psi}_n$, *with the same eigenvalues.*

Remark 1.2 It is important to note that (if Y is compact) the microlocal approach of Schnirel'man, Zelditch, and Colin de Verdière can also be generalized to this setting: there is an equivariant microlocal calculus on the globally symmetric space G/K using the Helgason Fourier Transform (replacing the usual Fourier transform used in the usual microlocal calculus on $\mathbb{R}^n$). This construction may be found in [2] and (somewhat differently) in the work of Hansen–Hilgert–Schröder [9]. This microlocal approach lifts the measures on Y to distributions on S^*Y, but in fact all limits of these measures are supported on singular subsets there, which are isomorphic to submanifolds of the form $\Gamma\backslash G$, and moreover are M_1-invariant for appropriate compact subgroups M_1. The lift here is directly supported on the "correct" target $\Gamma\backslash G/M_1$. The approach given here has the advantage of applying equally well to finite-volume quotients, and in the congruence setting it is desirable to have the lift be manifestly equivariant with respect to the action of the Hecke algebra, something that is not obvious in the microlocal approach. Further discussion of the connection between the representation-theoretic and microlocal approaches may be found in Section 5.

Remark 1.3 When Y is compact, (ii)(a) can be strengthened: the limit measure must be a probability measure. However, in the non-compact case the methods of the present paper can only prove the theorem as stated and cannot rule out the possibility of *escape of mass*: that the limiting measure has total mass less than 1 (perhaps it vanishes entirely). Conjecturally, mass does not escape; this was recently shown for Hecke eigenfunctions on the modular surface in [19].

As an aside, when Y is non-compact, the spectral decomposition of $L^2(Y)$ with respect to the algebra of invariant operators contains both a discrete part and a continuous part. We will only consider eigenfunctions in $L^2(Y)$ (that is, eigenfunctions in the discrete spectrum), ignoring the continuous spectrum. There is also a further partition of the discrete spectrum into its so-called *cuspidal* and *residual* parts, but

[2]For G simple, these are $\nu_n/\|\nu_n\|$. For G semisimple see the discussion in [17, §5.1], which also discusses the generalization to groups with infinite centre.

this distinction is immaterial to our discussion; we only use the assumption that the eigenfunctions belong to $L^2(Y)$.

1.2 Sketch of the Proof

As can be expected, we shall trace a path similar to that of the previous work. Given $\psi_n \in L^2(X)$ let $\psi'_n, \psi''_n \in L^2(X)$ be two other elements in the irreducible subrepresentation of $L^2(X)$ generated by ψ_n. Then for test functions $g \in C_c^\infty(X)$, the integral

$$g \mapsto \int_X \psi'_n \overline{\psi''_n} g \, \mathrm{d}\,\mathrm{vol} \tag{1.1}$$

defines a measure on X ($\bar{\mu}_n$ is the case where $\psi'_n = \psi''_n = \psi_n$), and we will study limits of this larger family of measures. We construct an asymptotic calculus for these measures by uniformizing the representation via the compact picture of a principal series representation induced from a potentially non-unitary character (we denote these representations (I_ν, V_K), where V_K is the space of K-finite functions on $M\backslash K$ and $\nu \in \mathfrak{a}^*_{\mathbb{C}}$ is the parameter; for the precise definition see Subsection 2.3). A prerequisite for taking limits in this setting is the following *a-priori* bound on these measures with respect to the uniformization, a technical contribution of this paper.

Theorem 1.4 *Let $(\pi, V_\pi) \in \widehat{G}$ be spherical, and let $R: (I_\nu, V_K) \to (\pi, V_\pi)$ be an*[3] *intertwining operator with the real part of $\nu \in \mathfrak{a}^*_{\mathbb{C}}$ in the closed positive chamber $\overline{\mathscr{C}}$, normalized such that $\|R(\phi_0)\|_{V_\pi} = 1$, where $\phi_0 \in V_K$ is the constant function* 1*. We then have $\|R(f)\|_{V_\pi} \le \|f\|_{L^2(K)}$ for any $f \in V_K$.*

Surprisingly, this useful fact is missing from the literature and was unknown to several experts consulted. It is proved in Section 3 as a consequence of the rationality of K-finite matrix coefficients by bounding the analytical continuation of the normalized intertwining operators $\widetilde{A}(\nu; w): (I_\nu, V_K) \to (I_{w\nu}, V_K)$ associated with elements w of the Weyl group W.

With this bound in hand we extend the asymptotic calculus of [17] to our setting. We construct the distributions μ_n in Section 4.1 (see Definition 4.3) by letting $\psi'_n = \psi_n$ be the spherical vector and letting ψ''_n be a distributional vector of the representation, specifically the one corresponding to the singular measure δ_1 supported on $M\backslash M_1 \subset M\backslash K$, where $M_1 = Z_K(A_1)$.

Integration by parts on X shows that the measures (1.1) are roughly unchanged if ψ'_n, ψ''_n are changed in compatible ways (if they correspond to $\phi', \phi'' \in V_K$ and at least one of ϕ', ϕ'' is left-M_1-invariant), then (in the limit of large spectral parameters) the measure in (1.1) approximately only depends on the product $\phi' \overline{\phi''}$. We have made the choice $\phi' = \phi_0$, $\phi'' = \delta_1$, so that the product $\phi_0 \delta_1$ is a positive measure on $M_1\backslash K$. We can approximate this measure by the function $|\phi|^2$ for some $\phi \in V_K$, and going back to (1.1) we have approximated our distribution by the one where $\psi'_n = \psi''_n$ correspond to ϕ, which is a positive measure on X. It follows that any limit of the μ_n is a positive measure.

[3]Such R always exist; of course ν depends on π.

https://doi.org/10.4153/CMB-2015-023-0 Published online by Cambridge University Press

A_1 invariance is obtained by finding a family of explicit differential equations satisfied by the distributions. The ring of invariant differential operators has an explicit isomorphism to the ring of polynomials in $\dim A$ variables (the *Harish-Chandra isomorphism*). Each invariant operator acts as a scalar on the representation generated by ψ_n, which gives an explicit differential equation satisfied by μ_n. Examining the highest-order terms shows that each generator of the ring gives invariance of the limit in a single direction in A (depending on $\widetilde{\nu}_\infty$). When $\widetilde{\nu}_\infty$ is non-degenerate (in the interior of the Weyl chamber), these directions are linearly independent and the measure is A-invariant. In the degenerate case these directions become dependent; we call the subgroup generated by them A_1.

1.3 A Measure Rigidity Problem on Locally Symmetric Spaces

An introduction to the relations between the general problem of Quantum Unique Ergodicity and the cases of manifolds of negative curvature and locally symmetric spaces of non-positive curvature may be found in [17]. We consider here only the latter case, where again we have the following conjecture.

Conjecture 1.5 (Silberman–Venkatesh) *The sequence $\{\bar{\mu}_n\}_{n=1}^\infty$ converges weak-* to the normalized volume measure $\frac{\mathrm{d}\,\mathrm{vol}_Y}{\mathrm{vol}(Y)}$.*

We recall the strategy pioneered by Lindenstrauss toward the "arithmetic" case of such conjectures, that is when Γ is a congruence lattice and ψ_n are also eigenfunctions of the algebra $\mathcal{H}$ of Hecke operators on $L^2(X)$. Assume the sequence $\bar{\mu}_n$ converges to a measure $\bar{\mu}_\infty$.

(a) *Lift*: Passing to a subsequence, lift $\bar{\mu}_\infty$ to a positive measure μ_∞ on X that projects to $\bar{\mu}_\infty$ under averaging by K and is invariant under a subgroup $H < G$, in a way that respects the $\mathcal{H}$-action.
(b) *Extra smoothness*: Using the geometry of the action of $\mathcal{H}$, show that any measure μ_∞ thus obtained is not too singular (for example, that the dimension of its support must be strictly larger than that of H).
(c) *Measure rigidity*: Using classification results for H-invariant measures on X, show that the additional information of Step (b) forces μ_∞ to be a G-invariant measure on X.

The result of this paper extend Step (a) of the strategy to the degenerate case; the methods used for Step (b) in [5] and [16] rely on the Hecke algebra alone. Realizing Step (c) then requires higher-rank measure classification results for A_1-invariant measures. Such results are now becoming available (see [7], but note the counter-example [14]). The results of this paper provide one of the few "natural" sources of A_1-invariant measures on $\Gamma\backslash G$ and raise the following challenge.

Problem 1.6 Let $\Gamma < G$ be a congruence lattice associated with a $\mathbb{Q}$-structure on G and let $A_1 \subset A$ be a non-trivial one-parameter subgroup fixed by a subgroup of the Weyl group. Let $\widetilde{\psi}_n \in L^2(X)$ be eigenfunctions of the Hecke operators on $X = \Gamma\backslash G$ such that their associated probability measures σ_n converge weak-* to an A_1-invariant measure σ_∞. Is it true that σ_∞ is then a (continuous) linear combination of algebraic measures on X?

2 Notation and Preliminaries

2.1 Structure Theory: Real Groups

Let G be a connected almost simple Lie group,[4] and let be $\mathfrak{g} = \mathrm{Lie}(G)$ its Lie algebra. Let Θ be a Cartan involution for G, let θ be the differential of Θ at the identity and let $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ be the associated polar decomposition. We fix a maximal Abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$. Its dimension is the (real) rank $\mathrm{rk}\, G$.

The dual vector space to $\mathfrak{a}$ will be denoted $\mathfrak{a}_{\mathbb{R}}^*$, and will be distinguished from the complexification $\mathfrak{a}_{\mathbb{C}}^* \stackrel{\mathrm{def}}{=} \mathfrak{a}_{\mathbb{R}}^* \otimes_{\mathbb{R}} \mathbb{C}$. For $\alpha \in \mathfrak{a}_{\mathbb{R}}^*$, set

$$\mathfrak{g}_\alpha = \left\{ X \in \mathfrak{g} \mid \forall H \in \mathfrak{a} : [H, X] = \alpha(H) X \right\}.$$

Let $\Delta = \Delta(\mathfrak{g} : \mathfrak{a})$ denote the set of *roots* (the non-zero $\alpha \in \mathfrak{a}_{\mathbb{R}}^*$ such that $\mathfrak{g}_\alpha \neq 0$). Then $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$, and $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$, where $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$. For $\alpha \in \mathfrak{a}_{\mathbb{R}}^*$ we set $p_\alpha = \dim \mathfrak{g}_\alpha$, $q_\alpha = \dim \mathfrak{g}_{2\alpha}$.

The Killing form B induces a positive-definite pairing $\langle X, Y \rangle = -B(X, \theta Y)$ on $\mathfrak{g}$ that remains non-degenerate when restricted to $\mathfrak{a}$. We identify $\mathfrak{a}$ and $\mathfrak{a}_{\mathbb{R}}^*$ via this pairing, giving us a non-degenerate pairing $\langle \cdot, \cdot \rangle$ on $\mathfrak{a}_{\mathbb{R}}^*$ and letting $H_\alpha \in \mathfrak{a}$ denote the element corresponding to $\alpha \in \Delta$. With this Euclidean structure on $\mathfrak{a}_{\mathbb{R}}^*$, the subset Δ is a root system, and we denote its Weyl group by $W(\mathfrak{g} : \mathfrak{a})$. A root $\alpha \in \Delta$ is *reduced* if $\frac{1}{2}\alpha \notin \Delta$. The set of reduced roots $\Delta^{\mathrm{r}} \subset \Delta$ is a root system as well. With $w \in W$ we associate the subset $\Phi_w = \Delta^{\mathrm{r}} \cap \Delta^+ \cap w^{-1}\Delta^-$ of positive reduced roots β such that $w\beta$ is negative.

We fix a simple system $\Pi \subset \Delta$, giving us a notion of positivity, and let Δ^+ (Δ^-) denote the set of positive (negative) roots, $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} p_\alpha \alpha \in \mathfrak{a}_{\mathbb{R}}^*$. For $\beta \in \Delta^{\mathrm{r}}$ and $\nu \in \mathfrak{a}_{\mathbb{C}}^*$, we set $\nu_\beta = \frac{2\langle \nu, \beta \rangle}{\langle \beta, \beta \rangle}$. Then

$$\mathscr{C} = \left\{ \nu \in \mathfrak{a}_{\mathbb{R}}^* \mid \forall \beta \in \Pi : \nu_\beta > 0 \right\}$$

is the open positive Weyl chamber. Its closure will be denoted $\overline{\mathscr{C}}$. We will also consider the open domain

$$\Omega = \mathscr{C} + i\mathfrak{a}_{\mathbb{R}}^* = \left\{ \nu \in \mathfrak{a}_{\mathbb{C}}^* \middle| \Re(\nu) \in \mathscr{C} \right\}$$

and its closure $\bar{\Omega}$. More generally, for $w \in W$, we set

$$\mathscr{C}_w = \left\{ \nu \in \mathfrak{a}_{\mathbb{R}}^* \mid \forall \beta \in \Phi_w : \nu_\beta > 0 \right\}$$

leading in the same fashion to $\overline{\mathscr{C}}_w \subset \mathfrak{a}_{\mathbb{R}}^*$ and $\Omega_w \subset \bar{\Omega}_w \subset \mathfrak{a}_{\mathbb{C}}^*$.

[4] The results of this paper hold (with natural modifications) for reductive G. The details may be found in [17, §5.1].

https://doi.org/10.4153/CMB-2015-023-0 Published online by Cambridge University Press

Returning to the Lie algebra we set $\mathfrak{n} = \oplus_{\alpha\in\Delta^+}\mathfrak{g}_\alpha$, $\bar{\mathfrak{n}} = \theta\mathfrak{n} = \oplus_{\alpha\in\Delta^-}\mathfrak{g}_\alpha$ and obtain the Iwasawa decomposition $\mathfrak{g} = \mathfrak{n}\oplus\mathfrak{a}\oplus\mathfrak{k}$. On the group level we set

$$K = \{g\in G \mid \Theta(g) = g\}, \quad A = \exp\mathfrak{a},\ N = \exp\mathfrak{n},\ \bar{N} = \exp\bar{\mathfrak{n}}.$$

These are closed subgroups with Lie algebras $\mathfrak{k}, \mathfrak{a}, \mathfrak{n}, \bar{\mathfrak{n}}$ respectively; K is a maximal compact subgroup, A a maximal diagonalizable subgroup, and N a maximal unipotent subgroup. With these we have the Iwasawa decomposition $G = NAK$. Another important subgroup is $M = Z_K(\mathfrak{a})$ which normalizes $N, \bar{N}$. Although M is not necessarily connected, $\mathfrak{m} = \mathrm{Lie}(M)$ holds. The action of $W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ on $\mathfrak{a}_\mathbb{R}^*$ gives an isomorphism of W and the algebraic Weyl group $W(\mathfrak{g}:\mathfrak{a})$ defined above.

Let dk be a probability Haar measure on K, da, dn Haar measures on A and N. Then $dn\cdot a^{2\rho}da\cdot dk$ is a Haar measure on G. The linear functional $f\mapsto\int_K f(k)dk$ on the space $\mathcal{F}^\rho = \{f\in C(G): f(nag) = a^{2\rho}f(g)\}$ is right G-invariant.

2.2 Complexification

Let $\mathfrak{b}$ be a maximal torus in the compact Lie algebra $\mathfrak{m}$, $\mathfrak{h} = \mathfrak{a}\oplus\mathfrak{b}$. Then $\mathfrak{h}$ is a maximal Abelian semisimple subalgebra of $\mathfrak{g}$, that is, a Cartan subalgebra.

Now $\mathfrak{g}_\mathbb{C}$ is a complex semisimple Lie algebra of which $\mathfrak{h}_\mathbb{C}$ is a Cartan subalgebra. We let $\Delta(\mathfrak{g}_\mathbb{C}:\mathfrak{h}_\mathbb{C})$ denote the associated root system, and $W(\mathfrak{g}_\mathbb{C}:\mathfrak{h}_\mathbb{C})$ its Weyl group. The restriction of any $\alpha\in\Delta(\mathfrak{g}_\mathbb{C}:\mathfrak{h}_\mathbb{C})$ to $\mathfrak{a}$ is either a root of $\mathfrak{g}$ or zero. We fix a notion of positivity on $\Delta(\mathfrak{g}_\mathbb{C}:\mathfrak{h}_\mathbb{C})$ compatible with our choice for $\Delta(\mathfrak{g}:\mathfrak{a})$, and let $\rho_\mathfrak{h}\in\mathfrak{h}_\mathbb{C}^*$ denote half the sum of the positive roots in $\Delta(\mathfrak{g}_\mathbb{C}:\mathfrak{h}_\mathbb{C})$. Once $\rho_\mathfrak{h}$ makes its appearance we shall use $\rho_\mathfrak{a}$ for ρ defined before.

The image of $N_K(\mathfrak{h})$ in $W(\mathfrak{g}_\mathbb{C}:\mathfrak{h}_\mathbb{C})$ is $\widetilde{W} = N_K(\mathfrak{h})/Z_M(\mathfrak{b})$, since any $k\in N_K(\mathfrak{h})$ must normalize $\mathfrak{a}$, $\mathfrak{b}$ separately.

Lemma 2.1 *$W(\mathfrak{m}:\mathfrak{b})\simeq N_M(\mathfrak{b})/Z_M(\mathfrak{b})$ is normal in $\widetilde{W}$; the quotient is naturally isomorphic to $W(\mathfrak{g}:\mathfrak{a})$.*

Proof That $N_M(\mathfrak{b}) = N_K(\mathfrak{h})\cap Z_K(\mathfrak{a})$ gives the first assertion, and shows that the quotient embeds in $W(\mathfrak{g}:\mathfrak{a})$ since $N_K(\mathfrak{h})\subset N_K(\mathfrak{a})$. To show that the embedding is surjective let $w\in N_K(\mathfrak{a})$ and consider $\mathrm{Ad}(w)\mathfrak{b}$. This is the Lie algebra of a maximal torus of M ($\mathrm{Ad}(w)$ is an automorphism of M), hence conjugate to $\mathfrak{b}$ in M. In other words, there exists $m\in M$ such that $\mathrm{Ad}(w)\mathfrak{b} = \mathrm{Ad}(m)\mathfrak{b}$ and hence $m^{-1}w\in N_K(\mathfrak{b})$. This element also normalizes $\mathfrak{a}$, and hence $wM\in W$ has a representative in $N_K(\mathfrak{h})$. ∎

Corollary 2.2 *Under the identification $\mathfrak{a}_\mathbb{C}^*\simeq\{\nu\in\mathfrak{h}_\mathbb{C}^* \mid \nu\restriction_\mathfrak{b}\equiv 0\}$ (dual to the identification $\mathfrak{a}\simeq\mathfrak{h}/\mathfrak{b}$) the group $\widetilde{W}\subset W(\mathfrak{g}_\mathbb{C}:\mathfrak{h}_\mathbb{C})$ acts on $\mathfrak{a}_\mathbb{C}^*$ via its quotient map to W.*

We let $U(\mathfrak{g}_\mathbb{C})$ denote the universal enveloping algebra of the complexification of $\mathfrak{g}$ (and similarly $U(\mathfrak{a}_\mathbb{C})$, $U(\mathfrak{n}_\mathbb{C}),\dots$). In such an algebra we let $U(\mathfrak{g}_\mathbb{C})^{\leq d}$ denote the subspace generated by all (non-commutative) monomials in $\mathfrak{g}_\mathbb{C}$ of degree at most d.

https://doi.org/10.4153/CMB-2015-023-0 Published online by Cambridge University Press

2.3 Representation Theory

For any continuous representation of K on a Fréchet space W and $\tau \in \widehat{K}$, we let W_τ denote the τ-isotypical subspace, and we let $W_K = \oplus_\tau W_\tau$ denote the (dense) subspace of K-finite vectors. We let $\widehat{W}_K = \prod_\tau W_\tau$ denote the completion of W_K with respect to this decomposition. This is the space of formal sums $\sum_\tau w_\tau$, where $w_\tau \in W_\tau$. We endow $\widehat{W}_K$ with the product topology, which is also the topology of convergence component-wise.

We specifically set $V = C(M\backslash K)$ with the right regular action of K and let V_K denote the space of K-finite vectors there. We also have $V_K = L^2(M\backslash K)_K$; $\widehat{V}_K$ can be identified with the algebraic dual V_K' via the pairing $(f, \sum_\tau \phi_\tau) \mapsto \sum_\tau \int_{M\backslash K} f \cdot \phi_\tau$; the product topology is the weak-* topology. We let $\phi_0 \in V_K$ denote the function everywhere equal to 1.

Definition 2.3 For $\nu \in \mathfrak{a}_{\mathbb{C}}^*$ let G act by the right regular representation on

$$\mathcal{F}^\nu = \left\{ \phi \in C^\infty(G) \;\middle|\; \forall n \in N, a \in A, m \in M, g \in G : f(namg) = a^{\nu+\rho} f(g) \right\}.$$

This induces a $(\mathfrak{g}, K)$-module structure on the space of K-finite vectors $\mathcal{F}_K^\nu$. By the Iwasawa decomposition the restriction map $\mathcal{F}_K^\nu \to V_K$ is an isomorphism of algebraic representations of K, giving us a model (I_ν, V_K) for F_K^ν. Given $\Phi = \sum_\tau \phi_\tau \in \widehat{V}_K$ and $X \in \mathfrak{g}$, we set $I_\nu(X)\Phi = \sum_\tau I_\nu(X)\phi_\tau$. The sum is locally finite: for $\tau' \in \widehat{K}$, the only summands having τ'-components are those corresponding to finite set of K-types τ such that the contragradient $\widehat{\tau}$ occurs in the tensor product of $\widehat{\tau'}$ and the adjoint representation of K on $\mathfrak{g}$. Let $\bar{\mathbb{1}}$ denote the trivial representation of $(\mathfrak{g}, K)$, where the complex number z acts by multiplication by $\bar{z}$. Let $(\bar{I}_\nu, \widehat{V}_K) = (I_\nu, \widehat{V}_K) \otimes \bar{\mathbb{1}}$.

Notation 2.4 Let $(\mathcal{I}_\nu, \mathcal{V}_K)$ denote the $(\mathfrak{g}, K)$ module $(I_\nu \otimes \bar{I}_\nu, V_K \otimes \widehat{V}_K)$.

Fact 2.5 (Induced Representations)

(i) *The pairing $(f, g) \mapsto \int_{M\backslash K} fg$ is a G-invariant pairing on $\mathcal{F}^\nu \otimes \mathcal{F}^{-\nu}$. Equivalently, $(f, g) \mapsto \int_{M\backslash K} f\bar{g}$ is an invariant Hermitian pairing between (I_ν, V_K) and $(I_{-\bar{\nu}}, V_K)$. For $\nu \in i\mathfrak{a}_{\mathbb{R}}^*$ (the* unitary axis*), it follows that (I_ν, V_K) is unitarizable, its invariant Hermitian form given by the standard pairing of $L^2(M\backslash K)$.*

(ii) *The induced representation is irreducible for ν lying in an open dense subset of $i\mathfrak{a}_{\mathbb{R}}^*$.*

(iii) *Every irreducible spherical $(\mathfrak{g}, K)$-module (π, V_π) can be realized as a quotient via an intertwining operator $R: (I_\nu, V_K) \to (\pi, V_\pi)$, for some $\nu \in \overline{\mathscr{C}}$.*

2.4 Intertwining Operators

Given $w \in W$ and $\nu \in \mathfrak{a}_{\mathbb{C}}^*$, we can uniquely extend any $\phi \in V_K$ to an element of $\mathcal{F}^\nu$ (also denoted ϕ). For $\nu \in \mathscr{C}_w$ we can then define an endomorphism $A(\nu; w)$ of V_K by

$$\big(A(\nu; w)\phi\big)(k) = \int_{\bar{N} \cap wNw^{-1}} \phi(\bar{n}wk)\, d\bar{n}$$

(the integral converges absolutely in this case). This operator is manifestly K-equivariant, and it is easy to check that it intertwines the representations (I_ν, V_K) and

https://doi.org/10.4153/CMB-2015-023-0 Published online by Cambridge University Press

(I_{wv}, V_K), and that it is holomorphic in the domain Ω_w, as a sum of maps $\Omega_w \to \mathrm{End}(V_K^\tau)$.

Fact 2.6 (Intertwining Operators)

(i) [10, Prop. 60(i)] *The operators $A(v;w)$ admit a meromorphic continuation to all of $\mathfrak{a}_\mathbb{C}^*$, intertwining the representations (I_v, V_K) and (I_{wv}, V_K). For $v \in i\mathfrak{a}_\mathbb{R}^*$, they are unitary operators.*

(ii) [11, §VII.5] *For $v \in \mathfrak{a}_\mathbb{C}^*$ and $\beta \in \Delta$, set $v_\beta = \frac{2\langle v,\beta\rangle}{\langle\beta,\beta\rangle}$. For $w \in W$ set*

$$\Phi_w = \left\{ \beta \in \Delta \setminus 2\Delta | \beta \in \Delta^+ \cap w^{-1}\Delta^- \right\}.$$

Then $A(v;w)\phi_0 = r(v;w)\phi_0$, where

$$r(v;w) = \prod_{\beta\in\Phi_w}\Big[\frac{\Gamma(p_\beta+q_\beta)}{\Gamma(\frac{1}{2}(p_\beta+q_\beta))}\frac{\Gamma(\frac{1}{2}v_\beta)}{\Gamma(\frac{1}{2}(v_\beta+p_\beta))}\frac{\Gamma(\frac{1}{4}(v_\beta+p_\beta))}{\Gamma(\frac{1}{4}(v_\beta+p_\beta)+\frac{1}{2}q_\beta)}\Big].$$

We set $\widetilde{A}(v;w) = r^{-1}(v;w)A(v;w)$.

(iii) [11, Ch. XVI] *If the spherical representation (π, V_π) is unitarizable and realized as a quotient of (I_v, V_K) as before, these exists $w \in W$ with $w^2 = 1$ such that $wv = -\bar{v}$; furthermore, $\Re(v)$ belongs to a fixed compact set.*

(iv) *Conversely, let $w \in W$ satisfy $w^2 = 1$, and let $v \in \mathfrak{a}_\mathbb{C}^*$ such that $wv = -\bar{v}$. Then*

$$(f,g) \mapsto \left\langle A(v;w)f, g\right\rangle_{L^2(K)}$$

defines a non-zero $(\mathfrak{g}, K)$-equivariant Hermitian pairing on (I_v, V_K); the subspace where the pairing vanishes is the kernel of $A(v;w)$ and the quotient is irreducible. The quotient is unitarizable if and only if the pairing is semidefinite, and every unitary spherical representation arises this way.

(v) [4, Thm. 2.1(6)] *For fixed $\phi, \psi \in V_K$ the matrix coefficient*

$$v \mapsto \left\langle \widetilde{A}(v;w)\phi, \psi\right\rangle_{L^2(K)}$$

is a rational function of v where we identify $\mathfrak{a}_\mathbb{C}^*$ with $\mathbb{C}^{\dim\mathfrak{a}}$ via the map $v \mapsto (v(H_\alpha))_{\alpha\in\Pi}$.

Remark 2.7 Since V_K contains a unique copy of the trivial representation of K, we must have $A(v;w)\phi_0 = r(v;w)\phi_0$ for some meromorphic function $r(v;w)$. Showing that the integral defining $r(v;w)$ converges absolutely for $v \in \mathscr{C}_w$ proves the absolute convergence claim above.

Since $r(v;w)$ does not vanish in the open domain Ω_w, $\widetilde{A}(v;w)$ cannot have zeroes or poles there.

3 Interpolation Bounds on Intertwining Operators

Lemma 3.1 *Let $f \in \mathbb{C}(z)$ be a rational function of one variable. Suppose that f is bounded on the line $\Re(z) = 0$ and has no poles to the right of the line. Then*

$$\sup\left\{ f(z) \mid \Re(z) \geq 0\right\} = \sup\left\{ f(z) \mid \Re(z) = 0\right\}.$$

https://doi.org/10.4153/CMB-2015-023-0 Published online by Cambridge University Press

Proof Composing with a Möbius transformation we can instead consider the case of a rational function f holomorphic in the interior of the unit disk $\mathbb{D}$ and bounded on $\partial\mathbb{D} \setminus \{1\}$. The singularity of f at $z = 1$ is at most a pole since f is rational. The boundedness on the rest of the boundary then shows the singularity is removable so that f is continuous on the closed disk. Finally, apply the usual maximum principle. ∎

Proposition 3.2 *Let $w \in W$, and let $\widetilde{A}(\nu; w): (I_\nu, V_K) \to (I_{w\nu}, V_K)$ be the intertwining operator, normalized such that $\widetilde{A}(\nu; w)\phi_0 = \phi_0$. Then the operator norm $\|\widetilde{A}(\nu; w)\|_{L^2(K)\to L^2(K)}$ is at most 1 for $\nu \in \overline{\Omega}_w$.*

An immediate corollary is that the normalized operator has no poles on the boundary of Ω_w (the un-normalized integral $A(\nu; w)$ does have such poles).

Proof By duality, it suffices to show that

$$\left\langle \widetilde{A}(\nu; w)\phi, \psi \right\rangle_{L^2(K)} \le \|\phi\|_{L^2(K)} \|\psi\|_{L^2(K)}$$

holds for all non-zero $\phi, \psi \in V_K$ and all ν as above. As the left-hand side is a meromorphic function of ν, it suffices to establish the inequality for $\mathfrak{R}(\nu) \in \mathscr{C}_w$, which we assume henceforth.

We restrict the left-hand side to a one-parameter family of spectral parameters by considering the meromorphic one-variable function

$$f(z) = \frac{1}{\|\phi\|_{L^2(K)} \|\psi\|_{L^2(K)}} \left\langle \widetilde{A}(i\mathfrak{I}(\nu) + z\mathfrak{R}(\nu); w)\phi, \psi \right\rangle_{L^2(K)}.$$

It will be convenient to write $\nu_z = i\mathfrak{I}(\nu) + z\mathfrak{R}(\nu)$ so that $\nu_1 = \nu$, and note that the parameters in our family satisfy $\mathfrak{R}(\nu_z) = \mathfrak{R}(z)\mathfrak{R}(\nu)$ and in particular $\mathfrak{R}(\nu_z) \in \mathscr{C}_w$ when $\mathfrak{R}(z) > 0$. Arthur's result quoted above (Fact 2.6(iv)) says that $f(z)$ is a rational function of z. It has no poles in the domain $\mathfrak{R}(z) > 0$, since the intertwining operator has no poles in Ω_w. When $z = it \in i\mathbb{R}$, the parameter $\nu_z \in i\mathfrak{a}^*_{\mathbb{R}}$ is unitary and hence $\widetilde{A}(\nu_z; w)$ is a unitary operator, which implies $|f(z)| \le 1$ by Cauchy–Schwartz. In particular, f has no poles on this line, and the claim now follows from the lemma. ∎

Proof of Theorem 1.4 Let $(\pi, V_\pi) \in \widehat{G}$ be spherical, and let $R: (I_\nu, V_K) \to (\pi, V_\pi)$ be a non-zero intertwining operator with the real part of $\nu \in \mathfrak{a}^*_{\mathbb{C}}$ in the closed positive chamber $\overline{\mathscr{C}}$, normalized such that $\|R(\phi_0)\|_{V_\pi} = 1$.

By Fact 2.6(iii) there exists an involution $w \in W$ such that $w\nu = -\bar{\nu}$ and such that $\langle \phi, A(\nu; w)\psi \rangle_{L^2(K)}$ is a G-equivariant Hermitian pairing on (I_ν, V_K). Also, the image of $A(w, \nu)$ is irreducible (in fact, isomorphic to π). By Schur's Lemma there is $c \ge 0$ such that for all K-finite ϕ we have $\|R(\phi)\|_{V_\pi} = c\langle \widetilde{A}(\nu; w)\phi, \phi \rangle_{L^2(K)}$. Our normalization implies that the constant of proportionality is 1, and the bound on the intertwining operator gives the claim $\|R(\phi)\|_{V_\pi} \le \|\phi\|_{L^2(K)}$. ∎

4 Degenerate Lift

In this section we establish Theorem 1.1.

https://doi.org/10.4153/CMB-2015-023-0 Published online by Cambridge University Press

4.1 The Basic Construction

One eigenfunction Let $\psi \in L^2(Y)$ be a normalized eigenfunction with the parameter $\nu \in \overline{\Omega}_w$; let $R: (I_\nu, V_K) \to (\mathcal{R}, L^2(X)_K)$ be an intertwining operator with $R(\phi_0) = \psi$. Note that the image of R is the subrepresentation generated by ψ, which is irreducible. Given $f_1, f_2 \in V_K$ and $g \in C_c^\infty(X)_K$, we set

$$\mu_R(f_1, f_2)(g) \stackrel{\text{def}}{=} \int_X R(f_1)\overline{R(f_2)} g \, d\,\mathrm{vol}_X .$$

By the Cauchy–Schwartz inequality and Theorem 1.4,

$$\left|\mu_R(f_1, f_2)(g)\right| \leq \|f_1\|_{L^2(K)} \|f_2\|_{L^2(K)} \|g\|_{L^\infty(X)} .$$

In particular, the $\mu_R(f_1, f_2)$ extend to finite Borel measures on X (positive measures when $f_1 = f_2$), but the key fact is that the bound on the total variation of these measures depends on f_1 and f_2 but not on ν or R.

This construction extends to the case where one of the two test vectors is not K-finite. Given $\Phi = \sum_{\tau \in \widehat{K}} \phi_\tau \in \widehat{V}_K$, we set

$$\mu_R(f, \Phi)(g) = \sum_{\tau \in \widehat{K}} \mu_R(f, \phi_\tau)(g),$$

noting that only finitely many τ can contribute. Letting $C_c^\infty(X)'_K$ denote the algebraic dual of $C_c^\infty(X)_K$, we have obtained the map

$$\mu_R: V_K \times \widehat{V}_K \longrightarrow C_c^\infty(X)'_K,$$

which is linear in the first variable and conjugate-linear in the second. Integration by parts on $\Gamma\backslash G$ shows that the extension $\mu_R: (\mathcal{I}_\nu, \mathcal{V}_K) \to C_c^\infty(X)'_K$ is an intertwining operator for the $(\mathfrak{g}, K)$-module structures.

Remark 4.1 By $C_c^\infty(X)'_K$ we denote the *algebraic* dual of our space $C_c^\infty(X)_K$ of test functions, whose elements we call *distributions* by abuse of terminology. Convergence of such "distributions" will be in the weak-* (pointwise) sense. Apart from limits of uniformly bounded sequences of measures, the limits we shall consider will be *positive* distributions (that is, take non-negative values at non-negative test functions), and such distributions are always Borel measures (finiteness will require an easy separate argument). For completeness we note, however, that when Φ defines a distribution on $M\backslash K$ in the ordinary sense (as is the case with δ), $\mu_R(f, \Phi)$ is bounded with respect to an appropriate Sobolev norm and hence $\mu_R(f, \Phi)$ is a distribution on X in the ordinary sense. Moreover, the bound depends on f and on the dual Sobolev norm of Φ but not ν or R.

Sequences of Eigenfunctions

Let $\{\nu_n\}_{n=1}^\infty \subset \bar{\Omega}$ such that $\|\nu_n\| \to \infty$, and let $R_n: (I_{\nu_n}, V_K) \to (\mathcal{R}, L^2(X)_K)$ be intertwining operators with $\|R_n(\phi_0)\|_{L^2(X)} = 1$. For $f_1, f_2 \in V_K$ the construction of the previous section gives a sequence of Borel measures $\mu_n(f_1, f_2) = \mu_{R_n}(f_1, f_2)$ of total variation at most $\|f_1\|_{L^2(K)} \|f_2\|_{L^2(K)}$. Assume that $\bar{\mu}_n = \mu_n(\phi_0, \phi_0)$ converge weak-* to a limiting measure $\bar{\mu}_\infty$, which we would like to study. By the Banach–Alaoglu theorem, for each pair f_1, f_2 there exists a subsequence $\{n_k\}_{k=1}^\infty \subset \mathbb{N}$ such that $\mu_{n_k}(f_1, f_2)$

https://doi.org/10.4153/CMB-2015-023-0 Published online by Cambridge University Press

converge weak-$\star$. Fixing a countable basis $\{\phi_i\}_{i=0}^{\infty} \subset V_K$, the standard diagonalization argument shows we may assume (after passing to a subsequence) that for any $f_1, f_2 \in V_K$ there exists a measure $\mu_\infty(f_1, f_2)$ such that for all $g \in C_c^\infty(X)_K$,

$$\lim_{n\to\infty} \mu_n(f_1, f_2)(g) = \mu_\infty(f_1, f_2)(g).$$

As before, given f_1 and g, the value of $\mu_\infty(f_1, f_2)(g)$ only depends on the projection of f_2 to a finite set of K-types. We can thus extend μ_∞ to all of $\mathcal{V}_K = V_K \otimes \widehat{V}_K$, and it is clear that μ_n converge weak-$\star$ to μ_∞ in the sense that for any fixed $F \in \mathcal{V}_{\mathcal{K}}$ and $g \in C_c^\infty(X)_K$, $\lim_{n\to\infty} \mu_n(F)(g) = \mu_\infty(F)(g)$.

The asymptotic properties of μ_n are governed by the normalized spectral parameters $\widetilde{\nu}_n = \frac{\nu_n}{\|\nu_n\|}$. Passing to a subsequence again we assume $\widetilde{\nu}_n \to \widetilde{\nu}_\infty$ as $n \to \infty$. Since the $\Re(\nu_n)$ are uniformly bounded (we are dealing with unitary representations), the limit parameter $\widetilde{\nu}_\infty$ is purely imaginary.

Definition 4.2 Call the sequence of intertwining operators $\{R_n\}_{n=1}^{\infty}$ *conveniently arranged* if $\widetilde{\nu}_n$ converge to some $\widetilde{\nu}_\infty \in i\mathfrak{a}_{\mathbb{R}}^*$ and if for any $f_1, f_2 \in V_K$, the sequence of measures $\{\mu_n(f_1, f_2)\}_{n=1}^{\infty}$ converges in the weak-$\star$ topology.

Given our limiting measure $\bar{\mu}_\infty$ we now fix once and for all a conveniently arranged sequence R_n such that $\mu_n(\phi_0, \phi_0)$ converges to $\bar{\mu}_\infty$, and set $M_1 = Z_K(\widetilde{\nu}_\infty)$. The motivation for the following choice will be come clear in the following section.

Definition 4.3 Let $\delta_1 \in V_K'$ be the distribution $\delta_1(f) = \int_{M\backslash M_1} f(m_1)dm_1$. Set

$$\mu_n = \mu_n(\phi_0 \otimes \delta_1),$$

which converge to the limit $\mu_\infty = \mu_\infty(\phi_0 \otimes \delta_1)$.

Note that for a K-invariant test function g, $\mu_n(g) = \bar{\mu}_n(g)$ since the spherical part of δ_1 is exactly ϕ_0. It follows that the μ_n indeed are lifts of the measures $\bar{\mu}_n$ to $X = \Gamma\backslash G$, which is Theorem 1.1(i).

Remark 4.4 Note that our definition of μ_n (and hence μ_∞) depends on the limit point $\widetilde{\nu}_\infty$, and not only on the limiting measure $\bar{\mu}_\infty$.

4.2 Integration by Parts: Positivity

Pointwise addition and multiplication give an algebra structure to V_K. Our asymptotic calculus for the measures $\mu_n(f_1, f_2)$ will depend on the following elements of this algebra.

For $X \in \mathfrak{g}$ and $k \in K$ we write the Iwasawa decomposition of $\mathrm{Ad}(k)X$ as $X_{\mathfrak{n}}(k) + X_{\mathfrak{a}}(k) + X_{\mathfrak{k}}(k)$. Now for $X \in \mathfrak{g}$ and $\widetilde{\nu}_\infty \in i\mathfrak{a}_{\mathbb{R}}^*$, set

$$p_X(k) = \frac{1}{i}\left\langle X_{\mathfrak{a}}(k), \widetilde{\nu}_\infty \right\rangle.$$

This is a left-M_1-invariant function on K, in particular a left-M-invariant function on K. It is K-finite, being a matrix element of the adjoint representation of K on $\mathfrak{g}$.

https://doi.org/10.4153/CMB-2015-023-0 Published online by Cambridge University Press

Lemma 4.5 *The subalgebra of V_K generated by $\{\phi_0\}\cup\{p_X\}_{X\in\mathfrak{g}}$ under pointwise addition and multiplication is precisely $\mathcal{F}_1 = C(M_1\backslash K)_K$, the algebra of left-$M_1$ invariant, right K-finite functions on K.*

Proof This follows from the Stone–Weierstrass Theorem, by which it suffices to check that the functions p_X separate the points of $M_1\backslash K$. Indeed, if $p_X(k) = p_X(k')$ for all X, then $M_1k' = M_1k$; recall that M_1 was defined as the centralizer of $\widetilde{\nu}_\infty$. ∎

Our calculation depends on the following basic formula, obtained by integration by parts.

Lemma 4.6 ([17, Lem. 3.10]) *There exists a seminorm $\|\cdot\|$ on $C_c^\infty(X)_K$ such that for any $f_1, f_2 \in V_K$ and $X \in \mathfrak{g}$,*

$$\left|\mu_n(p_X f_1, f_2)(g) - \mu_n(f_1, \overline{p_X} f_2)(g)\right| \ll_{f_1,f_2} \|g\|\left[\|\widetilde{\nu}_n - \widetilde{\nu}_\infty\| + \|\nu_n\|^{-1}\right].$$

Combining the two lemmata gives the following corollary.

Corollary 4.7 *Let $f \in \mathcal{F}_1$ and $f_1, f_2 \in V_K$. Then, for any $g \in C_c^\infty(X)_K$,*

$$\left|\mu_n(f\cdot f_1, f_2)(g) - \mu_n(f_1, \overline{f}\cdot f_2)(g)\right| \ll_{f,f_1,f_2} \|g\|\left[\|\widetilde{\nu}_n - \widetilde{\nu}_\infty\| + \|\nu_n\|^{-1}\right].$$

Claims (ii) and (iv) of the main theorem now follow from the following proposition.

Proposition 4.8 *We can choose $f_n \in V_K$ so that the measures $\sigma_n = \mu_n(f_n, f_n)$ converge weak-* to μ_∞ (in the notation of the main theorem, we set $\widetilde{\psi}_n = R_n(f_n)$).*

Proof Let $\{h_k\}_{k=1}^\infty \in \mathcal{F}_1$ be real-valued functions such that h_k^2 converge weak-* to δ_1, and let $h_0 = \phi_0$ (it is easy to see that such a sequence exists). By Corollary 4.7 there exists constants C_k depending only on the choice of f_k such that for any $g \in C_c^\infty(X)_K$ and n,

$$\left|\mu_n(\phi_0, h_k^2)(g) - \mu_n(h_k, h_k)(g)\right| \le C_k\|g\|\left[\|\widetilde{\nu}_n - \widetilde{\nu}_\infty\| + \|\nu_n\|^{-1}\right].$$

Noting that $C_0 = 0$, given $n \ge 1$ let $k(n)$ be the maximal $k \in \{0,\dots,n\}$ such that $C_k \le [\|\widetilde{\nu}_n - \widetilde{\nu}_\infty\| + \|\nu_n\|^{-1}]^{-1/2}$, and set $f_n = h_{k(n)}$, $\sigma_n = \mu_n(f_n, f_n)$. The sequence $k(n)$ is monotone and tends to infinity; it follows that f_n^2 converge weakly to δ_1.

Finally, we have

$$\left|\mu_n(g) - \sigma_n(g)\right| \le \left|\mu_n(\phi_0, \delta_1 - f_n^2)(g)\right| + \left[\|\widetilde{\nu}_n - \widetilde{\nu}_\infty\| + \|\nu_n\|^{-1}\right]^{1/2}\|g\|.$$

Let $T \subset \widehat{K}$ be a finite subset such that $g \in \sum_{\tau\in T} C_c^\infty(X)_\tau$. Let $d_n \in \sum_{\tau\in T} V_\tau$ be the projection of $\delta_1 - f_n^2$ to that space. Then $\overline{R(\delta_1 - f_n^2 - d_n)}$ has trivial pairing with $R(\phi_0)g$, since they do not transform under the same K-types. We may thus bound the first term in the inequality above by $|\mu_n(\phi_0, d_n)(g)| \le \|d_n\|_{L^2(K)}\|g\|_{L^\infty(X)}$. Since $\sum_{\tau\in T} V_\tau$ is finite-dimensional, that $d_n \to 0$ weakly implies that $d_n \to 0$ in norm. Since $\mu_n(g) \to \mu_\infty(g)$, we conclude that $\sigma_n(g) \to \mu_\infty(g)$ as well. ∎

Corollary 4.9 *We have that μ_∞ extends to a non-negative measure on X of total mass at most* 1*. When X is compact μ_∞ is a probability measure.*

Proof The σ_n extend to positive measures, hence μ_∞ extends to a non-negative measure. To bound the total mass it suffices to consider K-invariant test functions, for which μ_∞ agrees with $\bar{\mu}_\infty$, which is a weak-* limit of probability measures. ∎

Corollary 4.10 *When ψ_n are eigenfunctions of an algebra $\mathcal{H}$ of operators that commute with the G-action, then so are $\widetilde{\psi}_n$.*

Proof By Schur's Lemma each element of $\mathcal{H}$ acts as a scalar on the irreducible representation generated by ψ_n; $\widetilde{\psi}_n$ belongs to this representation. ∎

4.3 A_1-invariance.

Let $\delta \in V_K' \simeq \widehat{V}_K$ be the delta distribution, that is, $\delta(f) = f(1)$. Since $\mathfrak{a}$ is a quotient of $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ by a Lie ideal, we can consider any $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ as a Lie algebra homomorphism $\mathfrak{a}_{\mathbb{C}} \oplus \mathfrak{m}_{\mathbb{C}} \oplus \mathfrak{n}_{\mathbb{C}} \to \mathbb{C}$. It thus extends to an algebra homomorphism $U(\mathfrak{a}_{\mathbb{C}} \oplus \mathfrak{m}_{\mathbb{C}} \oplus \mathfrak{n}_{\mathbb{C}}) \to \mathbb{C}$, and there exists a unique algebra endomorphism $\tau_\lambda \colon U(\mathfrak{a}_{\mathbb{C}} \oplus \mathfrak{m}_{\mathbb{C}} \oplus \mathfrak{n}_{\mathbb{C}}) \to U(\mathfrak{a}_{\mathbb{C}} \oplus \mathfrak{m}_{\mathbb{C}} \oplus \mathfrak{n}_{\mathbb{C}})$ such that $\tau_\lambda(X) = X + \lambda(X)$ for $X \in \mathfrak{a}_{\mathbb{C}} \oplus \mathfrak{m}_{\mathbb{C}} \oplus \mathfrak{n}_{\mathbb{C}}$.

Lemma 4.11 *Let $\nu \in \mathfrak{a}_{\mathbb{C}}^*$, $u \in U(\mathfrak{a}_{\mathbb{C}} \oplus \mathfrak{m}_{\mathbb{C}} \oplus \mathfrak{n}_{\mathbb{C}})$.*

(i) $I_\nu(u)\delta = (-\rho + \nu)(u) \cdot \delta$.
(ii) $\mathcal{I}_\nu(\tau_{\rho+\nu-2\Re(\nu)}(u))(f \otimes \delta) = (I_\nu(u)f) \otimes \delta$.

Proof By induction, it suffices to prove both assertions for $u = X \in \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$. The first claim follows from the invariant pairing of $\mathcal{F}^\nu$ with $\mathcal{F}^{-\nu}$. Taking complex conjugates, this implies that

$$\mathcal{I}_\nu(X)(f \otimes \delta) = \big(I_\nu(X)f\big) \otimes \delta + \langle -\rho_{\mathfrak{a}} + \bar{\nu}, X\rangle (f \otimes \delta),$$

which is the second assertion. ∎

We next summarize the analysis of the centre of the universal enveloping algebra done in [17, §4].

Proposition 4.12 *Let $\mathcal{P} \in U(\mathfrak{h}_{\mathbb{C}})^{W(\mathfrak{g}_{\mathbb{C}}:\mathfrak{h}_{\mathbb{C}})}$ be homogeneous of degree d. Then there exist elements $b = b(\mathcal{P}) \in U(\mathfrak{n}_{\mathbb{C}})U(\mathfrak{a}_{\mathbb{C}})^{\leq d-2}$ and $c = c(\mathcal{P}) \in U(\mathfrak{g}_{\mathbb{C}})\mathfrak{k}_{\mathbb{C}}$ so that*

$$z = \tau_{-\rho_{\mathfrak{h}}}(\mathcal{P}) + b + c$$

belongs to the centre of the universal enveloping algebra. Furthermore, z acts on (I_ν, V_K) with the eigenvalue $\mathcal{P}(\nu + \rho_{\mathfrak{a}} - \rho_{\mathfrak{h}})$.

It follows that for such $\mathcal{P}$ we have

$$\mathcal{I}_\nu\big(\tau_{\rho_{\mathfrak{a}}-\rho_{\mathfrak{h}}+\nu-2\Re(\nu)}(\mathcal{P}) + \tau_{\rho_{\mathfrak{a}}+\nu-2\Re(\nu)}(b) - \mathcal{P}(\nu + \rho_{\mathfrak{a}} - \rho_{\mathfrak{h}})\big)(\phi_0 \otimes \delta) = 0$$

(to see this, unwind the definitions, using the fact that $I_\nu(c)\phi_0 = 0$).

Thinking of $\mathcal{P}$ as a function on $\mathfrak{h}_{\mathbb{C}}^*$, let $P'(\nu)$ denote its differential at $\nu \in \mathfrak{a}_{\mathbb{C}}^*$. This is an element of the cotangent space to $\mathfrak{h}_{\mathbb{C}}^*$, that is, an element of $\mathfrak{h}_{\mathbb{C}}$.

Proposition 4.13 *Let* $\mathcal{P} \in U(\mathfrak{h}_{\mathbb{C}})^{W(\mathfrak{g}_{\mathbb{C}}:\mathfrak{h}_{\mathbb{C}})}$. *Then there exists a polynomial map* $J: \mathfrak{a}_{\mathbb{C}}^* \to U(\mathfrak{g}_{\mathbb{C}})$ ($\mathfrak{a}_{\mathbb{C}}^*$ *thought of as a* real *vector space), of degree at most* $d-2$ *in the parameters* $\mathfrak{I}(\nu)$*, such that for any unitarizable parameter* $\nu \in \mathfrak{a}_{\mathbb{C}}^*$,

$$\mathcal{I}_\nu\Big(\mathcal{P}'(\widetilde{\nu}) + \frac{J(\nu)}{\|\nu\|^{d-1}}\Big)(\phi_0 \otimes \delta) = 0.$$

Proof Since $\mathcal{P}'$ is a homogeneous polynomial of degree $d-1$, it suffices to show that $\tau_{\rho_{\mathfrak{a}}-\rho_{\mathfrak{h}}+\nu-2\mathfrak{R}(\nu)}(\mathcal{P}) + \tau_{\rho_{\mathfrak{a}}+\nu-2\mathfrak{R}(\nu)}(b) - \mathcal{P}(\nu+\rho_{\mathfrak{a}}-\rho_{\mathfrak{h}}) - \mathcal{P}'(\nu)$ is a polynomial of degree at most $d-2$ in ν. It is clear that $J_1(\nu) = \tau_{\rho_{\mathfrak{a}}+\nu-2\mathfrak{R}(\nu)}(b)$ is such a polynomial, as is

$$J_2(\nu) = \tau_{\rho_{\mathfrak{a}}-\rho_{\mathfrak{h}}+\nu-2\mathfrak{R}(\nu)}\big(\mathcal{P}\big) - \mathcal{P}(\nu+\rho_{\mathfrak{a}}-\rho_{\mathfrak{h}}-2\mathfrak{R}(\nu)\big) - \mathcal{P}'\big(\nu+\rho_{\mathfrak{a}}-\rho_{\mathfrak{h}}-2\mathfrak{R}(\nu)\big).$$

Since$\mathcal{P}'$ is a polynomial of degree $d-1$ (valued in $\mathfrak{a}_{\mathbb{C}}$), $J_3(\nu) = \mathcal{P}'(\nu+\rho_{\mathfrak{a}}-\rho_{\mathfrak{h}}-2\mathfrak{R}(\nu)) - P'(\nu)$ is also of degree at most $d-2$. It remains to consider

$$J_4(\nu) = \mathcal{P}\big(\nu+\rho_{\mathfrak{a}}-\rho_{\mathfrak{h}}-2\mathfrak{R}(\nu)\big) - \mathcal{P}(\nu+\rho_{\mathfrak{a}}-\rho_{\mathfrak{h}}),$$

which is a polynomial map of degree $d-1$ in ν.

The first two terms are the difference of the values of a polynomial at two points; we may write this in the form $\langle \mathcal{P}'(\nu+\rho_{\mathfrak{a}}-\rho_{\mathfrak{h}}), -2\mathfrak{R}(\nu)\rangle + J_5(\nu)$ where $J_5(\nu)$ includes the terms of degree $d-2$ or less and the pairing is the one between $\mathfrak{h}_{\mathbb{C}}$and $\mathfrak{a}_{\mathbb{C}}^*$. Finally, $\mathcal{P}'(\nu+\rho_{\mathfrak{a}}-\rho_{\mathfrak{h}}) - \mathcal{P}'(\mathfrak{I}(\nu))$ has degree $d-2$ in $\mathfrak{I}(\nu)$. Setting

$$J_6(\nu) = \big\langle \mathcal{P}'(\nu+\rho_{\mathfrak{a}}-\rho_{\mathfrak{h}}) - \mathcal{P}'(\mathfrak{I}(\nu)), -2\mathfrak{R}(\nu)\big\rangle,$$

we see that $\phi_0 \otimes \delta$ is annihilated by

$$P'(\nu) + \sum_{i=1}^{6} J_i(\nu) - 2\big\langle \mathcal{P}'(\mathfrak{I}(\nu)), \mathfrak{R}(\nu)\big\rangle.$$

We conclude by showing that the final scalar vanishes. By assumption, there exists $w \in W$ such that $w\nu = -\bar{\nu}$. By Corollary 2.2 there exist $\widetilde{w} \in W(\mathfrak{g}_{\mathbb{C}}:\mathfrak{h}_{\mathbb{C}})$ such that $\widetilde{w}\nu = -\bar{\nu}$. Applying the chain rule to $\mathcal{P} = \mathcal{P} \circ \widetilde{w}$ we see that $\mathcal{P}'(\mathfrak{I}(\nu))$ is fixed by $\widetilde{w}$, while $\widetilde{w}\mathfrak{R}(\nu) = -\widetilde{w}\mathfrak{R}(\nu)$. ∎

Corollary 4.14 *Let* $\{R_n\}_{n=1}^{\infty}$ *be a conveniently arranged sequence of intertwining operators from* (I_{ν_n}, V_K) *to* $L^2(X)$*. Then the limit distribution* $\mu_\infty = \mu_\infty(\phi_0 \otimes \delta_1)$ *is* H*-invariant for any* $H = \mathcal{P}'(\widetilde{\nu}_\infty)$*, where* $\mathcal{P} \in U(\mathfrak{h}_{\mathbb{C}})^{W(\mathfrak{g}_{\mathbb{C}}:\mathfrak{h}_{\mathbb{C}})}$.

Proof By additivity, it suffices to prove this when $\mathcal{P}$ is homogeneous. Next, since $\mu_n = \mu_n(\phi_0 \otimes \delta_1)$ are M_1-invariant distributions (in fact, we are lifting to $\Gamma\backslash G/M_1$, not to $\Gamma\backslash G$), it suffices to consider M_1-invariant test functions $g \in C_c^\infty(X)_K^{M_1}$. For these we have $\mu_n(\phi_0 \otimes \delta_1)(g) = \mu_n(\phi_0 \otimes \delta)(g)$. We conclude that it is enough to show that $\mu_\infty(\phi_0 \otimes \delta)$ are $\mathcal{P}'(\widetilde{\nu}_\infty)$-invariant for homogeneous $\mathcal{P}$, but this follows immediately by passing to the limit in the proposition. ∎

Since the $W(\mathfrak{g}_{\mathbb{C}}:\mathfrak{h}_{\mathbb{C}})$-invariant polynomials on $\mathfrak{h}_{\mathbb{C}}$ are dense in the space of smooth functions on the sphere there, it is clear that $\{\mathcal{P}'(\widetilde{\nu}_\infty)\}$ is precisely the set $\mathfrak{h}_{\mathbb{C}}^{W_1'}$ where $W_1' = \mathrm{Stab}_{W(\mathfrak{g}_{\mathbb{C}}:\mathfrak{h}_{\mathbb{C}})}(\widetilde{\nu}_\infty)$. Theorem 1.1(iii) is then contained in the following lemma.

https://doi.org/10.4153/CMB-2015-023-0 Published online by Cambridge University Press

Lemma 4.15 *Let* $W_1 = \mathrm{Stab}_W(\widetilde{\nu}_\infty)$. *Then* $\mathfrak{a}_{\mathbb{C}}^{W_1} = \mathfrak{h}_{\mathbb{C}}^{W_1'} \cap \mathfrak{a}_{\mathbb{C}}$.

Proof The subgroup of a Weyl group fixing a point in $\mathfrak{a}$ (or its dual) is generated by the root reflections it contains. It follows that W_1 is generated by the root reflections s_α where $\alpha \in \Delta(\mathfrak{g}:\mathfrak{a})$ satisfies $B(\alpha, \widetilde{\nu}_\infty) = 0$ (pairing given by the Killing form on $\mathfrak{g}$) while W_1' is generated by the root reflections $s_{\alpha'}$ where $\alpha' \in \Delta(\mathfrak{g}_{\mathbb{C}}:\mathfrak{h}_{\mathbb{C}})$ satisfies $B'(\alpha', \widetilde{\nu}_\infty) = 0$ (Killing form on $\mathfrak{g}_{\mathbb{C}}$). Now $B'(\alpha', \widetilde{\nu}_\infty) = B(\alpha' \restriction_{\mathfrak{a}}, \widetilde{\nu}_\infty)$, since $\mathfrak{b}$ is orthogonal to $\mathfrak{a}$, where the restrictions are either roots (or zero). Since s_α fixes H if and only if $\alpha(H) = 0$, while $s_{\alpha'}$ fixes H if and only if $\alpha'(H) = 0$, it follows that

$$\mathfrak{h}_{\mathbb{C}}^{W_1'} \cap \mathfrak{a}_{\mathbb{C}} = \bigcap_{B(\alpha, \widetilde{\nu}_\infty)=0} \mathrm{Ker}(\alpha) = \mathfrak{a}_{\mathbb{C}}^{W_1}. \qquad \blacksquare$$

5 Comparison of the Representation-Theoretic and Microlocal Approaches in the Degenerate Case

Zelditch's proof [22] of Schnirel'man's Theorem for hyperbolic surfaces uses the microlocal calculus developed in [21]. First, an the equivariant microlocal calculus on the hyperbolic plane $\mathbb{H}$ is given by using the Helgason Fourier Transform instead of the usual Fourier transform, and this is then projected using averaging with respect to Γ to a microlocal calculus on compact $Y = \Gamma\backslash\mathbb{H}$. A similar calculus can be developed on any compact locally symmetric space of non-compact type; this is described in [2, Sec. 3] and in a different form in [9, Sec. 6]. Both papers continue to construct a microlocal lift (Wigner measure) and prove its positivity, as well as the flow-invariance as in Corollary 4.14 (see [2, Thm. 1.3, Eq. (1.3)] and [9, Thm. 6.7]).

Following [17, Sec. 5.4] (which discusses the non-degenerate case) we sketch a proof that the microlocal and representation-theoretic lifts agree asymptotically in the degenerate case as well. We need to show that limits of the lifts obtained by microlocal methods will be M_1-invariant, supported on submanifolds of the form $\Gamma\backslash G$, and agree with limits of the lifts constructed in this paper.

5.1 On the Quantization Scheme

Is essentially shown in [22] (but not in this language) that in the case of a compact hyperbolic surface and for pseudodifferential calculus of [21], we have

$$\mu_R(\phi_0 \otimes \delta)(g) = \langle \mathrm{Op}(g)\psi, \psi \rangle, \tag{5.1}$$

where $\psi = R(\phi_0)$, and δ is the Dirac delta on the circle $K = \mathrm{SO}(2)$. More generally, Zelditch shows that if ψ_n, ψ_m are two eigenfunctions, then

$$\big\langle \mathrm{Op}(g)\psi_n, \psi_m \big\rangle = \int_X R_n(\delta)\overline{R_m(\phi_0)} g \,\mathrm{d\,vol}\,.$$

It is tempting to believe that the same is true in our case: that we can *define* a microlocal calculus on Y by

$$\big\langle \mathrm{Op}(g)\psi_n, \psi_m \big\rangle = \int_X R_n(\delta_1)\overline{R_m(\phi_0)} g \,\mathrm{d\,vol}\,. \tag{5.2}$$

https://doi.org/10.4153/CMB-2015-023-0 Published online by Cambridge University Press

Unfortunately, our choice of δ_1 depends on the limiting parameter $\widetilde{\nu}_\infty$, and not only on the parameter ν_n. That would not be an obstacle in the non-degenerate case (especially in rank 1) but it would an issue in general. It is probably possible to generalize the equivariant calculus of Zelditch to general locally symmetric spaces and then hope for an exact identity like (5.1) or even (5.2), but this has not been done.

We sidestep this issue in the rest of this section by considering only diagonal matrix coefficients, and by giving up on the exact identity. Instead we show that $\mu_R(\phi_0 \otimes \delta_1)(g)$ and $\langle \mathrm{Op}(g)\psi, \psi\rangle$ are approximately equal for any quantization scheme Op.

5.2 Equivalence of the Microlocal and Representation-Theoretic Constructions

Fix a "quantization scheme" $a \mapsto \mathrm{Op}(a)$ coming from a microlocal calculus on Y. "Step one" of the argument in [17, Sec. 5.4] shows, without using the non-degeneracy assumption, that under a natural identification $T^*Y \simeq X \times_K \mathfrak{p}^*$, any limit of the distributions

$$a \longmapsto \big\langle \mathrm{Op}(a)\psi_n, \psi_n \big\rangle$$

must be supported on the submanifold $X \times \{\widetilde{\nu}_\infty\}$. This is a special case of a much more general fact (see [8, Thm. 5.4], as noted by [9, Sec. 6]),

We stop here to see how Proposition 4.12 above connects the microlocal and representation-theoretic proofs of flow invariance (see also [2, Lem. 3.3]). Suppose for simplicity that G is $\mathbb{R}$-split, so that $\mathfrak{a} = \mathfrak{h}$ and $\mathfrak{a}/W = \mathfrak{p}/K$. Each element of $\mathcal{P}(\mathfrak{a}_{\mathbb{C}})^W$ can also be thought of as an element of $\mathcal{P}(\mathfrak{p}_{\mathbb{C}})^K$, that is, a K-invariant polynomial function on $\mathfrak{p}^*$ or equivalently, a symbol on $T^*Y = X \times_K \mathfrak{p}^*$ depending only on the second coordinate. Now both approaches show that, the limiting measure is invariant by $P'(\widetilde{\nu}_\infty)$, where P' is the gradient of P, an element of $\mathfrak{a}$.

That gives A_1-invariance of the limit measure by Lemma 4.15.

"Step two" of the argument there considers a symbol of the special form $a(x,\nu) = 1/\|\nu\|^d \int_K \sigma(xk) u_d(k^{-1}\nu k)$ on $X \times \mathfrak{p}$, defined using a K-finite function $\sigma \in C^\infty(X)_K$ and a degree-d element $u_d \in U(\mathfrak{g})$ (thought as a polynomial function on $\mathfrak{g}^*$, and as a differential operator, as appropriate). Combining [17, Eq. (5.4),(5.5),(5.11)] gives

$$\begin{aligned}\big\langle \mathrm{Op}(a)\psi_n, \psi_n \big\rangle &= \int_X \overline{\psi_n(x)} R_n(p_u)(x)\sigma(x)\,\mathrm{d}x + o(1)\\ &= \mu_n(p_u, \phi_0)(\sigma) + o(1),\end{aligned}$$

where $p_u(k) = u_d(k^{-1}\widetilde{\nu}_\infty k)$, notably a function on $M_1\backslash K$. Setting $g(x) = a(x, \widetilde{\nu}_\infty)$, we then have

$$\begin{aligned}\mu_n(\phi_0, \delta_1)\big(a(x,\nu_n)\big) &= \mu_n(\phi_0,\delta_1)(g)\\ &= \mu_n(\phi_0,\delta)(g) && (5.3)\\ &= \big\langle \psi_n(x)\sigma(x), R_n(\overline{p_u})\big\rangle_{L^2(X)} && (5.4)\\ &= \mu_n(\phi_0, \overline{p_u})(\sigma)\\ &= \mu_n(p_u, \phi_0)(\sigma) + o(1), && (5.5)\end{aligned}$$

where (5.3) follows from the M_1-invariance of g by averaging, (5.4) is [17, Eq. (5.6)], and (5.5) is an application of Corollary 4.7. Comparing the last two calculations, we conclude that

$$\langle \mathrm{Op}(a)\psi_n, \psi_n \rangle = \mu_n(\phi_0, \delta_1)(g) + o(1).$$

Finally, to prove equality of the distributions we need to show that the functions g considered in step two span the space $C^\infty(X/M_1)_K$ of K-finite M_1-invariant functions on X. However, a being defined by convolution, this amounts to checking that the functions $p_u(k)$ span $C^\infty(M_1\backslash K)_K$, which is the content of Lemma 4.5.

5.3 Possible Connection to Patterson–Sullivan Distributions

Papers [3, 9] use the boundary values of (non-degenerate) eigenfunctions ψ_n to construct "Patterson–Sullivan" distributions PS_n on $\Gamma\backslash G/M$ ($G = \mathrm{SL}_2(\mathbb{R})$ in the first paper, general in the second). These distributions are A-invariant by construction, and are shown to be asymptotically the equal to the Wigner distributions $a \mapsto \langle \mathrm{Op}(a)\psi_n, \psi_n \rangle$ (which are invariant by the "Schrödinger flow" of conjugation by the propagator for the appropriate Schrödinger equations), giving a different proof of the A-invariance of quantum limits. The distributions $\mu_n(\phi_0, \delta_1)$ considered here are an intermediate family between the two. They are not explicitly invariant by either the classical or the quantum flow. However, in the case of $G = \mathrm{SL}_2(\mathbb{R})$, Anantharaman–Zelditch discovered a surprising connection between the Wigner and Patterson–Sullivan distributions via an explicit integral transform, and it is natural to ask whether the applying the integral transform to the Patterson–Sullivan distribution gives a distribution related to $\mu_n(\phi_0, \delta)$. Finding such a relation would perhaps permit a generalization of this integral transform beyond hyperbolic surfaces and would assist in constructing Patterson–Sullivan distributions in the degenerate case. We leave this question for future work.

Acknowledgments This work was done while the author was a member at the Institute for Advanced Study in Princeton; his stay there was partly supported by the Institute's NSF grant. He would like to thank Werner Müller for a useful discussion and encouragement, and the anonymous referees for their careful reading and for many useful comments and suggestions.

References

[1] N. Anantharaman, *Entropy and the localization of eigenfunctions.* Ann. of Math. (2) **168**(2008), no. 2, 435–475. http://dx.doi.org/10.4007/annals.2008.168.435

[2] N. Anantharaman and L. Silberman, *A Haar component for quantum limits on locally symmetric spaces.* Israel J. Math. **195**(2013), no. 1, 393–447. http://dx.doi.org/10.1007/s11856-012-0133-x

[3] N. Anantharaman and S. Zelditch, *Patterson-Sullivan distributions and quantum ergodicity.* Ann. Henri Poincaré **8**(2007), no. 2, 361–426. http://dx.doi.org/10.1007/s00023-006-0311-7

[4] J. Arthur, *Intertwining operators and residues. I. Weighted characters.* J. Funct. Anal. **84**(1989), no. 1, 19–84. http://dx.doi.org/10.1016/0022-1236(89)90110-9

[5] J. Bourgain and E. Lindenstrauss, *Entropy of quantum limits.* Comm. Math. Phys. **233**(2003), no. 1, 153–171. http://dx.doi.org/10.1007/s00220-002-0770-8

[6] Y. Colin de Verdière, *Ergodicité et fonctions propres du laplacien.* Comm. Math. Phys. **102**(1985), no. 3, 497–502. http://dx.doi.org/10.1007/BF01209296

[7] M. Einsiedler and E. Lindenstrauss, *On measures invariant under diagonalizable actions: the rank-one case and the general low-entropy method.* J. Mod. Dyn. **2**(2008), no. 1, 83–128.
[8] L. C. Evans and M. Zworski, *Lectures on semiclassical analysis.* (version 0.2) http://math.berkeley.edu/~evans/semiclassical.pdf
[9] S. Hansen, J. Hilgert, and M. Schröder, *Patterson-Sullivan distributions in higher rank.* Math. Z. **272**(2012), no. 1–2, :607–643. http://dx.doi.org/10.1007/s00209-011-0952-1
[10] A. W. Knapp and E. M. Stein, *Intertwining operators for semisimple groups.* Ann. of Math. (2) **93**(1971), 489–578. http://dx.doi.org/10.2307/1970887
[11] A. W. Knapp, *Representation theory of semisimple groups. An overview based on examples.* Princeton Mathematical Series, 36, Princeton University Press, Princeton, NJ, 1986.
[12] E. Lindenstrauss, *On quantum unique ergodicity for* $\Gamma\backslash\mathbb{H}\times\mathbb{H}$. Internat. Math. Res. Notices **2001**, no. 17, 913–933.
[13] ———, *Invariant measures and arithmetic quantum unique ergodicity.* Ann. of Math. (2) **163**(2006), 165–219. http://dx.doi.org/10.4007/annals.2006.163.165
[14] F. Maucourant, *A nonhomogeneous orbit closure of a diagonal subgroup.* Ann. of Math. (2) **171**(2010), no. 1, 557–570. http://dx.doi.org/10.4007/annals.2010.171.557
[15] Z. Rudnick and P. Sarnak, *The behaviour of eigenstates of arithmetic hyperbolic manifolds.* Comm. Math. Phys. **161**(1994), no. 1, 195–213. http://dx.doi.org/10.1007/BF02099418
[16] L. Silberman and A. Venkatesh, *Entropy bounds for Hecke eigenfunctions on division algebras.* In revision.
[17] ———, *Quantum unique ergodicity for locally symmetric spaces.* Geom. Funct. Anal. **17**(2007), no. 3, 960–998. http://dx.doi.org/10.1007/s00039-007-0611-1
[18] A. I. Šhnirel'man, *Ergodic properties of eigenfunctions.* Uspekhi Mat. Nauk. **29**(1974), no. 6(180), 181–182.
[19] K. Soundararajan, *Quantum unique ergodicity for* $SL_2(\mathbb{Z})\backslash\mathbb{H}$. Ann. of Math. (2) **172**(2010), no. 2, 1529–1538.
[20] S. A. Wolpert, *Semiclassical limits for the hyperbolic plane.* Duke Math. J. **108**(2001), no. 3, 449–509. http://dx.doi.org/10.1215/S0012-7094-01-10833-8
[21] S. Zelditch, *Pseudodifferential analysis on hyperbolic surfaces.* J. Funct. Anal. **68**(1986), no. 1, 72–105. http://dx.doi.org/10.1016/0022-1236(86)90058-3
[22] ———, *Uniform distribution of eigenfunctions on compact hyperbolic surfaces.* Duke Math. J. **55**(1987), no. 4, 919–941. http://dx.doi.org/10.1215/S0012-7094-87-05546-3
[23] ———, *The averaging method and ergodic theory for pseudo-differential operators on compact hyperbolic surfaces.* J. Funct. Anal. **82**(1989), 38–68. http://dx.doi.org/10.1016/0022-1236(89)90091-8

Department of Mathematics, University of British Columbia, Vancouver, BC, V6T 1Z2
e-mail: lior@math.ubc.ca