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ORDERING UNIFORM COMPLETIONS OF PARTIALLY ORDERED SETS

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Let (P, Υ) be a (nearly) uniform ordered space. Let $(\mathbf{P}, \boldsymbol{\Upsilon})$ be the uniform completion of (P, Υ) at Υ. Several partial orders for $\mathbf{P}$ are introduced and discussed. One of these orders provides an adjoint to the functor which embeds the category of uniformly complete uniform ordered spaces in the category of uniform ordered spaces, both categories with uniformly continuous order-preserving functions. When P is a join-semilattice with uniformly continuous join, two of these orders coalesce to the unique partial order with respect to which $\mathbf{P}$ is a join-semilattice, P is a join-subsemilattice of $\mathbf{P}$, and the join on $\mathbf{P}$ is continuous. Let B be an abelian l-group with locally convex group and lattice topology T, and let $\mathbf{B}$ be the completion of B at the right uniformity associated with T. Then the two orders mentioned above are equivalent to the minimal partial order with respect to which $\mathbf{B}$ is an abelian l-group, $\mathbf{B}^+ \supseteq B^+$, and the lattice operations on $\mathbf{B}$ are continuous.

1. Introduction. For topological terminology left undefined, see [**3**; **7**]. Other definitions with which we assume the reader to be familiar may be found in [**1**; **2**].

Notation. Our undefined notation is standard with possibly the following exceptions: We denote a net with domain D by $\{x_\delta \mid \delta \in D\}$ or merely $\{x_\delta\}$ if the domain has been previously indicated. Let X, Y be sets and let $h : Y \rightarrow X$ be a function. The function $h \times h : Y \times Y \rightarrow X \times X$ is defined by $(x, y)h \times h = (xh, yh)$. Let L be a lattice, and suppose $M, N \subseteq L$. Then

$$M \vee N = \{x \vee a \mid x \in M, \quad a \in N\},$$
$$M \wedge N = \{x \wedge a \mid x \in M, \quad a \in N\}.$$

If $A, B \subseteq L \times L$, then

$$A \vee B = \{(x \vee a, y \vee b) \mid (x, y) \in A, (a, b) \in B\},$$
$$A \wedge B = \{(x \wedge a, y \wedge b) \mid (x, y) \in A, (a, b) \in B\}.$$

Let B be an l-group and suppose that $H \subseteq B$ and $J \subseteq B \times B$. Then

$$-H = \{-x \mid x \in H\},$$
$$-J = \{(-x, -y) \mid (x, y) \in J\}.$$

We note the following result, which is straightforward to prove.

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PROPOSITION 1.1. *Let (Y, Υ) be a separated uniform space. Let $(\mathbf{Y}, \boldsymbol{\Upsilon})$ be the completion of (Y, Υ) at Υ. Let $\mathbf{y} \in \mathbf{Y}$ and let $\{y_\delta | \delta \in D\}$ be a Cauchy net in Y converging to $\mathbf{y}$. If Υ^s is the set of symmetric entourages of Υ directed downwards, then there exists a Cauchy net $\{x_U | U \in \Upsilon^s\}$, with domain Υ^s, such that $\{x_U\}$ converges to $\mathbf{y}$, and such that, as subsets of Y, $\{x_U\} \subseteq \{y_\delta\}$. In particular, if $J, H \in \Upsilon^s$ are such that $J \subseteq H$, then $(\mathbf{y}, x_J) \in \mathbf{H}$.*

Let P be a set with partial order $\leq$. Let Υ be a separated uniformity on P and let $(\mathbf{P}, \boldsymbol{\Upsilon})$ be the completion of (P, Υ) at Υ. We may define a binary relation R on $\mathbf{P}$ by: $\mathbf{x}R\mathbf{y}$ if and only if there exist Cauchy nets $\{x_U | U \in \Upsilon^s\} \subseteq P$ converging to $\mathbf{x}$, and $\{y_U | U \in \Upsilon^s\} \subseteq P$ converging to $\mathbf{y}$ such that $x_U \leq y_U$ for all $U \in \Upsilon^s$. If P is a lattice, and if the lattice operations on P are uniformly continuous with respect to Υ, then Kiseleva [**5**] proves that R is a lattice order for $\mathbf{P}$.

We wish to investigate here the more general situation of partially ordered sets, and to consider in what way the orders we define there are "best" when restricted to lattices. It seems clear that some sort of restriction must be placed on Υ so that the uniform structure and the order structure mesh properly. The appropriate restriction turns out to be (a weakening of) Nachbin's concept of "uniform ordered space".

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2. (Nearly) Uniform ordered spaces. As above, let P be a set with partial order $\leq$. Let

$$\Delta(P) = \{(x, x) | x \in P\}$$

be the *diagonal of* P, and let

$$G(\leq) = \{(x, y) | x \leq y\}$$

be the *graph of* $\leq$.

A *semi-uniform structure for* P [**6**] is a filter $\mathscr{F}$ on $P \times P$ satisfying

(i) for all $V \in \mathscr{F}$, $\Delta(P) \subseteq V$;

(ii) if $V \in \mathscr{F}$, there exists $W \in \mathscr{F}$ such that $W \circ W \subseteq V$.

The set

$$\mathscr{F}^* = \{U \cap V^{-1} | U, V \in \mathscr{F}\}$$

is then a uniform structure for P [**6**].

We say that a partially ordered set P with uniformity Υ is a *nearly uniform ordered space* in case Υ is separated and there exists a semi-uniform structure $\mathscr{F}$ for P such that $\bigcap \mathscr{F} \supseteq G(\leq)$ and $\mathscr{F}^* = \Upsilon$. A partially ordered set P with uniformity Υ is a *uniform ordered space* [**6**] in case (P, Υ) is a nearly uniform ordered space and additionally $\bigcap \mathscr{F} \subseteq G(\leq)$.

Let P be a partially ordered set with topology T. We say that T is *locally convex* if for every $x \in P$ and every neighbourhood N of x, there exists a neighbourhood M of x such that M is convex and $M \subseteq N$.

PROPOSITION 2.1. *If* (P, Υ) *is a nearly uniform ordered space, then the topology on* P *induced by* Υ *is locally convex.*

Proof. The proof of [**6**, Proposition 9] does not use the assumption that $\bigcap \mathscr{F} \subseteq G(\leq)$.

Example 2.2. Let $\dot{Q}$ be the rational numbers with usual order. Then for any prime number p, T_p, the p-adic topology on $\dot{Q}$ [**3**], is not locally convex. Since T_p is a ring topology for $\dot{Q}$, there is a uniformity Υ on $\dot{Q}$ whose induced topology is T_p. By Proposition 1.1, $(\dot{Q}, \Upsilon)$ is not a nearly uniform ordered space.

Example 2.3. This example shows that not every nearly uniform ordered space is a uniform ordered space.

Let $\dot{R}$ be the real numbers and let $\dot{R}^- = \{r \in \dot{R} | r \leq 0\}$. Let $P = \dot{R} \times \dot{R}^-$. Let Υ be the uniformity P inherits from the usual uniformity on $\dot{R} \times \dot{R}$. Then Υ is separated. Define a binary relation L on P by: $(x, y)L(r, s)$ if and only if (1) $y \neq 0$, $x \leq r$, and $y \leq s$, or (2) $y = 0$, $s = 0$, and $0 < x \leq r$, or (3) $y = 0$, $s = 0$, and $x \leq r \leq 0$. A straightforward case-by-case argument shows that L is a partial order on P.

Let $G(L)$ be the graph of L. Suppose that $\mathscr{F}$ is a semi-uniform structure for P such that $\bigcap \mathscr{F} \supseteq G(L)$ and $\mathscr{F}^* = \Upsilon$. Let $H \in \mathscr{F}$. Then there exists $V \in \mathscr{F}$ such that $V \circ V \subseteq H$. Since $\mathscr{F}^* = \Upsilon$, $V \in \Upsilon$. Therefore, there exists $\delta < 0$ such that $((-1, 0), (-1, \delta)) \in V$. Since $V \in \mathscr{F}$, $V \supseteq G(L)$ and hence

$$((-1, \delta), (1, 0)) \in V.$$

Therefore

$$((-1, 0), (1, 0)) \in V \circ V \subseteq H.$$

Thus

$$((-1, 0), (1, 0)) \in \bigcap \mathscr{F}.$$

But $-1 < 0 < 1$ and hence $((-1, 0), (1, 0)) \notin G(L)$. Therefore

$$\bigcap \mathscr{F} \not\subseteq G(L),$$

i.e., (P, Υ) is not a uniform ordered space. However, if we let $\mathscr{F}$ be the semi-uniformity on P inherited from the usual semi-uniformity on $\dot{R} \times \dot{R}$, then clearly $\mathscr{F}^* = \Upsilon$ and $\bigcap \mathscr{F} \supseteq G(L)$. Therefore (P, Υ) is a nearly uniform ordered space.

A set T is *totally ordered* if T is partially ordered and if

$$T \times T = G(\leq) \cup G(\leq)^{-1}.$$

In this case, the notions of "nearly uniform ordered space" and "uniform space" are equivalent:

PROPOSITION 2.4. *Let T be a totally ordered set with uniformity Υ. Then (T, Υ) is a uniform ordered space if and only if it is a nearly uniform ordered space.*

Proof. Let $\mathscr{F}$ be a semi-uniform structure for T such that $\bigcap \mathscr{F} \supseteq G(\leq)$ and $\mathscr{F}^* = \Upsilon$. Suppose $(x, y) \in \bigcap \mathscr{F} \backslash G(\leq)$. Then $(y, x) \in G(\leq) \subseteq \bigcap \mathscr{F}$. Let $H \in \Upsilon$. Then $H \supseteq V \cap V^{-1}$ for some $V \in \mathscr{F}$. Since $(y, x) \in V$, $(x, y) \in V^{-1}$; thus $(x, y) \in H$. Therefore $(x, y) \in \bigcap \Upsilon$. But $(x, y) \notin G(\leq)$ and hence $(x, y) \notin \Delta(T)$. Therefore, Υ is not separated and the result follows.

3. Partial orders on some uniform completions. Let (P, Υ) be a nearly uniform ordered space, and let $(\mathbf{P}, \boldsymbol{\Upsilon})$ be the uniform completion of P at Υ. Define a binary relation $\lessdot$ on $\mathbf{P}$ as follows: $\mathbf{x} \lessdot \mathbf{y}$ if and only if for each Cauchy net $\{x_U | U \in \Upsilon^s\} \subseteq P$ converging to $\mathbf{x}$, there exists a Cauchy net $\{y_U | U \in \Upsilon^s\} \subseteq P$ converging to $\mathbf{y}$ such that for all $U \in \Upsilon^s$, $x_U \leq y_U$. We call this relation the *strong order* on $\mathbf{P}$.

Let $\mathscr{F}$ be a semi-uniform structure for P such that $\bigcap \mathscr{F} \supseteq G(\leq)$ and $\mathscr{F}^* = \Upsilon$. Define a binary relation $\leqslant(\mathscr{F})$ on $\mathbf{P}$ by: $\mathbf{x} \leqslant(\mathscr{F})\, \mathbf{y}$ if and only if for each $V \in \mathscr{F}$, for each Cauchy net $\{x_U | U \in \Upsilon^s\} \subseteq P$ converging to $\mathbf{x}$, there exists a Cauchy net $\{y_U | U \in \Upsilon^s\} \subseteq P$ converging to $\mathbf{y}$ such that for all $U \in \Upsilon^s$, $(x_U, y_U) \in V$. We call the relation $\leqslant(\mathscr{F})$ on $\mathbf{P}$ the *$\mathscr{F}$-order* on $\mathbf{P}$. Clearly if $\mathbf{x} \lessdot \mathbf{y}$, then $\mathbf{x} \leqslant(\mathscr{F})\, \mathbf{y}$ for every semi-uniform structure $\mathscr{F}$ for P such that $\bigcap \mathscr{F} \supseteq G(\leq)$ and $\mathscr{F}^* = \Upsilon$.

PROPOSITION 3.1. *If (P, Υ) is a nearly uniform ordered space, then the strong order and the $\mathscr{F}$-orders are partial orders on $\mathbf{P}$.*

Proof. Let $\mathscr{F}$ be a semi-uniform structure for P such that $\bigcap \mathscr{F} \supseteq G(\leq)$ and $\mathscr{F}^* = \Upsilon$.

Clearly $\mathbf{x} \lessdot \mathbf{x}$ and $\mathbf{x} \leqslant(\mathscr{F})\, \mathbf{x}$ for all $\mathbf{x} \in \mathbf{P}$.

Suppose $\mathbf{x} \lessdot \mathbf{y}$ ($\mathbf{x} \leqslant(\mathscr{F})\, \mathbf{y}$) and $\mathbf{y} \lessdot \mathbf{x}$ ($\mathbf{y} \leqslant(\mathscr{F})\, \mathbf{x}$). By Proposition 1.1, there exists a Cauchy net $\{y_U | U \in \Upsilon^s\} \subseteq P$ converging to $\mathbf{y}$. We will show that $\{y_U\}$ converges to $\mathbf{x}$. Let $W \in \Upsilon^s$. Let $H \in \Upsilon^s$ be such that $\mathbf{H} \circ \mathbf{H} \subseteq \mathbf{W}$, and let $V \in \mathscr{F}$ be such that

$$(V \circ V) \cap (V \circ V)^{-1} \subseteq H.$$

By definition of $\lessdot$ ($\leqslant(\mathscr{F})$), there exist Cauchy nets $\{x_U | U \in \Upsilon^s\} \subseteq P$ converging to $\mathbf{x}$ and $\{z_U | U \in \Upsilon^s\} \subseteq P$ converging to $\mathbf{y}$ such that for all $U \in \Upsilon^s$, $y_U \leq x_U$ ($(y_U, x_U) \in V$) and $x_U \leq z_U$ ($(x_U, z_U) \in V$). Hence

$$(y_U, x_U) \in G(\leq) \subseteq V \; ((y_U, x_U) \in V),$$

and

$$(x_U, z_U) \in G(\leq) \subseteq V \; ((x_U, z_U) \in V).$$

Since $\{y_U\}$ and $\{z_U\}$ are Cauchy nets converging to $\mathbf{y}$, there exists $K \in \Upsilon^s$ such that $K \subseteq H$ and if $J \subseteq K$, then

$$(y_J, z_J) \in V \cap V^{-1}.$$

Hence $(z_J, y_J) \in V$, and thus $(x_J, y_J) \in V \circ V$. Hence

$$(y_J, x_J) \in (V \circ V)^{-1} \cap V \subseteq H \subseteq \mathbf{H}.$$

But by Proposition 1.1, $(\mathbf{x}, x_J) \in \mathbf{K} \subseteq \mathbf{H}$ and hence

$$(\mathbf{x}, y_J) \in \mathbf{H} \circ \mathbf{H} \subseteq \mathbf{W}.$$

Therefore, $\{y_U\}$ converges to $\mathbf{x}$ and since $\boldsymbol{\Upsilon}$ is separated and $\{y_U\}$ also converges to $\mathbf{y}$, $\mathbf{x} = \mathbf{y}$.

Suppose $\mathbf{x} \lessdot \mathbf{y}$ ($\mathbf{x} \leqslant (\mathscr{F})\, \mathbf{y}$) and $\mathbf{y} \lessdot \mathbf{z}$ ($\mathbf{y} \leqslant (\mathscr{F})\, \mathbf{z}$). (Let $V \in \mathscr{F}$, and let $W \in \mathscr{F}$ be such that $W \circ W \subseteq V$.) Let $\{x_U | U \in \Upsilon^s\} \subseteq P$ be a Cauchy net converging to $\mathbf{x}$. Then there exist Cauchy nets $\{y_U | U \in \Upsilon^s\} \subseteq P$ converging to $\mathbf{y}$ and $\{z_U | U \in \Upsilon^s\} \subseteq P$ converging to $\mathbf{z}$ such that for all $U \in \Upsilon^s$, $x_U \leq y_U$ ($(x_U, y_U) \in W$) and $y_U \leq z_U$ ($(y_U, z_U) \in W$). Then for all $U \in \Upsilon^s$, $x_U \leq z_U$ ($(x_U, z_U) \in W \circ W \subseteq V$) and hence $\mathbf{x} \lessdot \mathbf{z}$ ($\mathbf{x} \leqslant (\mathscr{F})\, \mathbf{z}$).

4. Properties of the strong order and the $\mathscr{F}$-orders. In this section we show that if (P, Υ) is a uniform ordered space, then the $\mathscr{F}$-orders extend the order on P, and that any $\mathscr{F}$-order (the strong order) on a nearly uniform ordered space makes $(\mathbf{P}, \boldsymbol{\Upsilon})$ a (nearly) uniform ordered space.

PROPOSITION 4.1. *Let (P, Υ) be a nearly uniform ordered space. Any $\mathscr{F}$-order on* $\mathbf{P}$ *satisfies*

$$G(\leq) \subseteq G(\leqslant(\mathscr{F})) \cap (P \times P).$$

Proof. Let $x, y \in P$ be such that $x \leq y$. Let $V, W \in \mathscr{F}$ be such that $W \circ W \subseteq V$, and suppose that $\{x_U | U \in \Upsilon^s\} \subseteq P$ is a Cauchy net converging to x. If $J \subseteq W \cap W^{-1}$, then $(x, x_J) \in W \cap W^{-1} \subseteq W^{-1}$. Since

$$(x, y) \in G(\leq) \subseteq W, \; (x_J, y) \in W \circ W \subseteq V.$$

Let

$$y_U = \begin{cases} x_U, & \text{if } U \not\subseteq W \cap W^{-1} \\ y, & \text{if } U \subseteq W \cap W^{-1}. \end{cases}$$

Then $\{y_U\} \subseteq P$ is a Cauchy net converging to y and for all $U \in \Upsilon^s$, $(x_U, y_U) \in V$. Therefore, $x \leqslant (\mathscr{F})\, y$.

PROPOSITION 4.2. *Let (P, Υ) be a uniform ordered space, and suppose that* $\mathscr{F}$

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is a semi-uniform structure for P *such that* $\bigcap \mathscr{F} = G(\leq)$ *and* $\mathscr{F}^* = \Upsilon$. *Then the* $\mathscr{F}$*-order on* $\mathbf{P}$ *extends the order on* P, *i.e.*, $G(\leq) = G(\leqslant(\mathscr{F})) \cap (P \times P)$.

Proof. Let $x, y \in P$. By Proposition 4.1, if $x \leq y, x \leqslant (\mathscr{F})\, y$. Suppose $x \leqslant (\mathscr{F})\, y$. The net defined by $x_U = x$ for all $U \in \Upsilon^s$ is a Cauchy net converging to x. Let $V, W \in \mathscr{F}$ be such that $W \circ W \subseteq V$. Then there is a net $\{y_U | U \in \Upsilon^s\} \subseteq P$ converging to y such that $(x, y_U) = (x_U, y_U) \in W$ for all $U \in \Upsilon^s$. If $J \subseteq W \cap W^{-1}$, $(y, y_J) \in W \cap W^{-1} \subseteq W^{-1}$. Thus $(x, y) \in W \circ W \subseteq V$. We have shown that $(x, y) \in \bigcap \mathscr{F}$ and hence $(x, y) \in G(\leq)$, i.e., $x \leq y$.

That Proposition 4.1 (and therefore 4.2) does not necessarily hold for the strong order follows from the next example.

Example 4.3. In this example, we construct a uniform ordered space whose strong order does not extend the original order.

Let $\dot{R}$ be the real numbers and let

$$P = \{(x, y) \in \dot{R} \times \dot{R} | x = 0 \text{ or } y = 0\}.$$

Let Υ be the uniformity on P inherited from the usual uniformity on $\dot{R} \times \dot{R}$. Partially order P by $(x, y) \leq (r, s)$ if and only if $x \leq r$ and $y \leq s$. It is straightforward to show that $(P, \Upsilon) = (\mathbf{P}, \mathbf{\Upsilon})$ is a uniform ordered space.

Now $(0, 0) \leq (0, 1)$, and the net $\{(0, 1/n) | n = 1, 2, \ldots\}$ is a Cauchy net converging to $(0, 0)$. Let $\{x_U | U \in \Upsilon^s\} \subseteq \{(0, 1/n)\}$ be a Cauchy net as in Proposition 1.1. If $\{y_U | U \in \Upsilon^s\}$ is a net satisfying $x_U \leq y_U$ for all $U \in \Upsilon^s$, then $y_U = (0, y_U')$ for some $y_U' \in \dot{R}$, for all $U \in \Upsilon^s$. Hence if $\{y_U\}$ converges, it must converge to $(0, r)$ for some $r \in \dot{R}$. Therefore, there is no Cauchy net $\{y_U\}$ converging to $(1, 0)$ and satisfying $x_U \leq y_U$, i.e., $(0, 0)$ is not $\lessdot$ $(1, 0)$.

Let (P, Υ) be a nearly uniform ordered space. Let $\mathscr{F}$ be a semi-uniform structure for P such that $\bigcap \mathscr{F} \supseteq G(\leq)$ and $\mathscr{F}^* = \Upsilon$. For $V \in \mathscr{F}$, let $|V|$ be the subset of $\mathbf{P} \times \mathbf{P}$ consisting of all those $(\mathbf{x}, \mathbf{y})$ such that for each Cauchy net $\{x_U | U \in \Upsilon^s\} \subseteq P$ converging to $\mathbf{x}$, there exists a Cauchy net $\{y_U | U \in \Upsilon^s\} \subseteq P$ converging to $\mathbf{y}$ such that $(x_U, y_U) \in V$ for all $U \in \Upsilon^s$. Let $|\mathscr{F}|$ be the filter on $\mathbf{P} \times \mathbf{P}$ generated by $\{|V| | V \in \mathscr{F}\}$.

LEMMA 4.4. $|\mathscr{F}|$ *is a semi-uniform structure for* $\mathbf{P}$.

Proof. (i) Since $\Delta(P) \subseteq V$ for all $V \in \mathscr{F}$, $(\mathbf{x}, \mathbf{x}) \in |V|$ for all $\mathbf{x} \in \mathbf{P}$.

(ii) Let $V, W \in \mathscr{F}$ be such that $W \circ W \subset V$. Suppose that $(\mathbf{x}, \mathbf{y})$, $(\mathbf{y}, \mathbf{z}) \in |W|$, and let $\{x_U | U \in \Upsilon^s\} \subseteq P$ be a Cauchy net converging to $\mathbf{x}$. Since $(\mathbf{x}, \mathbf{y}) \in |W|$ and $(\mathbf{y}, \mathbf{z}) \in |W|$, there are Cauchy nets $\{y_U | U \in \Upsilon^s\} \subseteq P$ converging to $\mathbf{y}$ and $\{z_U | U \in \Upsilon^s\} \subseteq P$ converging to $\mathbf{z}$ such that for all $U \in \Upsilon^s$, $(x_U, y_U) \in W$ and $(y_U, z_U) \in W$. Thus $(x_U, z_U) \in W \circ W \subseteq V$ for all $U \in \Upsilon^s$. Therefore $(\mathbf{x}, \mathbf{z}) \in |V|$, i.e. $|W| \circ |W| \subseteq |V|$. Hence $|\mathscr{F}|$ is a semi-uniform structure for $\mathbf{P}$.

LEMMA 4.5. $|\mathscr{F}|^* = \mathbf{\Upsilon}$.

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Proof. (If $A \in \Upsilon^s$, then $\mathbf{A}$ is the set of $(\mathbf{x}, \mathbf{y}) \in \mathbf{P} \times \mathbf{P}$ such that if $\{x_U | U \in \Upsilon^s\} \subseteq P$ converges to $\mathbf{x}$ and $\{y_U | U \in \Upsilon^s\} \subseteq P$ converges to $\mathbf{y}$, then there exists $U \in \Upsilon^s$ such that for all $J, K \subseteq U$, $(x_J, y_K) \in A$. Also $\boldsymbol{\Upsilon}$ is the filter generated by $\{\mathbf{A} | A \in \Upsilon^s\}$, and $\{\mathbf{A} | A \in \Upsilon^s\} \subseteq \boldsymbol{\Upsilon}^s$.)

Let $V, W \in \mathscr{F}$ be such that $W \circ W \subseteq V$, and let $H = V \cap V^{-1}$. We will first show that $|W| \cap |W|^{-1} \subseteq \mathbf{H}$: Let $(\mathbf{x}, \mathbf{y}) \in |W| \cap |W|^{-1}$, and suppose that $\{x_U | U \in \Upsilon^s\} \subseteq P$ and $\{y_U | U \in \Upsilon^s\} \subseteq P$ are Cauchy nets converging to $\mathbf{x}$ and $\mathbf{y}$ respectively. Then there exists a Cauchy net $\{a_U | U \in \Upsilon^s\} \subseteq P$ converging to $\mathbf{y}$ such that $(x_U, a_U) \in W$ for all $U \in \Upsilon^s$, and there exists $K \in \Upsilon^s$ such that for all $X, Y \subseteq K$, $(a_X, y_Y) \in W$. Hence $(x_X, y_Y) \in W \circ W \subseteq V$. Since $(\mathbf{x}, \mathbf{y}) \in |W|^{-1}$, there exists a Cauchy net $\{b_U | U \in \Upsilon^s\} \subseteq P$ converging to $\mathbf{x}$ such that $(y_U, b_U) \in W$, i.e. $(b_U, y_U) \in W^{-1}$, for all $U \in \Upsilon^s$. Since $\{b_U\}$ converges to $\mathbf{x}$, there exists $L \in \Upsilon^s$ such that for all $X, Y \subseteq L$, $(x_X, b_Y) \in W^{-1}$. Thus $(x_X, y_Y) \in W^{-1} \circ W^{-1} \subseteq V^{-1}$. Therefore, if $X, Y \subseteq L \cap K$, then $(x_X, y_Y) \in V \cap V^{-1}$, i.e. $(\mathbf{x}, \mathbf{y}) \in \mathbf{H}$.

Let $V \in \mathscr{F}$, and let $H = V \cap V^{-1}$. We next show that $\mathbf{H} \subseteq |V|$: Let $(\mathbf{x}, \mathbf{y}) \in \mathbf{H}$, let $\{x_U | U \in \Upsilon^s\} \subseteq P$ be a Cauchy net converging to $\mathbf{x}$, and let $\{z_U | U \in \Upsilon^s\} \subseteq P$ be a Cauchy net converging to $\mathbf{y}$. Suppose that $W \in \Upsilon^s$ is such that for all $J, K \subseteq W$, $(x_J, z_K) \in H \subseteq V$. Define

$$y_U = \begin{cases} x_U, & \text{if } U \not\subseteq W \\ z_U, & \text{if } U \subseteq W. \end{cases}$$

Then clearly $\{y_U\} \subseteq P$ is a Cauchy net converging to $\mathbf{y}$ such that $(x_U, y_U) \in V$ for all $U \in \Upsilon^s$. Therefore $(\mathbf{x}, \mathbf{y}) \in |V|$ and hence $\mathbf{H} \subseteq |V|$.

Now if $\mathbf{A} \in \boldsymbol{\Upsilon}$, then there exists $V \in \mathscr{F}$ such that $\mathbf{A} \supseteq \mathbf{H}$, where $H = V \cap V^{-1}$, and hence by the above $\mathbf{A} \supseteq |W| \cap |W|^{-1}$ for some $|W| \in |\mathscr{F}|$. Conversely, if $\mathbf{A} \in |\mathscr{F}|^*$, then there exists $V \in \mathscr{F}$ such that $\mathbf{A} \supseteq |V| \cap |V|^{-1}$. Let $H = V \cap V^{-1}$. Then by the above, $|V| \supseteq \mathbf{H}$, and thus, since $\mathbf{H}$ is symmetric, $|V| \cap |V|^{-1} \supseteq \mathbf{H}$. Hence, $\mathbf{A} \supseteq \mathbf{H}$, and therefore, $|\mathscr{F}|^* = \boldsymbol{\Upsilon}$.

LEMMA 4.6. $G(\leqslant(\mathscr{F})) = \bigcap |\mathscr{F}|$.

Proof. The result follows immediately from the definitions.

PROPOSITION 4.7. *Let (P, Υ) be a nearly uniform ordered space. Then any $\mathscr{F}$-order on P makes $(\mathbf{P}, \boldsymbol{\Upsilon})$ a uniform ordered space.*

Proof. By Lemmas 4.4, 4.5 and 4.6, $|\mathscr{F}|$ is a semi-uniform structure for $\mathbf{P}$ such that $G(\leqslant(\mathscr{F})) = \bigcap |\mathscr{F}|$ and $|\mathscr{F}|^* = \boldsymbol{\Upsilon}$.

PROPOSITION 4.8. *Let (P, Υ) be a nearly uniform ordered space. Then the strong order on $\mathbf{P}$ makes $(\mathbf{P}, \boldsymbol{\Upsilon})$ a nearly uniform ordered space.*

Proof. Clearly there exists an $\mathscr{F}$-order on $\mathbf{P}$ and clearly $G(\leqslant(\mathscr{F})) \supseteq G(\lessdot)$. Thus, by Lemmas 4.4, 4.5 and 4.6, $|\mathscr{F}|$ is a semi-uniform structure for $\mathbf{P}$ such that $\bigcap |\mathscr{F}| \supseteq G(\lessdot)$ and $|\mathscr{F}|^* = \boldsymbol{\Upsilon}$.

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To ensure that $(\mathbf{P}, \boldsymbol{\Upsilon})$ with the strong order is a uniform ordered space, we need an extra condition on (P, Υ). The condition we suggest in Section 6 is sufficient for this result and also implies the strong order version of Proposition 4.2.

5. The maximal semi-uniform structure. The results of Section 6 depend on a particular semi-uniform structure, whose definition and basic properties we discuss here.

Let Υ be a separated uniformity on the partially ordered set P; let

$$\mathscr{F}(\Upsilon) = \{V \in \Upsilon \mid \text{there exist } V_1, V_2, \ldots \in \Upsilon \text{ such that } V_1 = V, \text{ and for all } n,\ V_n \supseteq G(\leq) \text{ and } V_{n+1} \circ V_{n+1} \subseteq V_n\}.$$

PROPOSITION 5.1. *$\mathscr{F}(\Upsilon)$ is a semi-uniform structure for P.*

Proof. Clearly $\Delta(P) \subseteq V$ for all $V \in \mathscr{F}(\Upsilon)$. If $V \in \mathscr{F}(\Upsilon)$, then V_2 in the definition of $\mathscr{F}(\Upsilon)$ is an element of $\mathscr{F}(\Upsilon)$ such that $V_2 \circ V_2 \subseteq V$. To see that $\mathscr{F}(\Upsilon)$ is a filter on $P \times P$, we first note that clearly if $U \supseteq V \in \mathscr{F}(\Upsilon)$, then $U \in \mathscr{F}(\Upsilon)$. Furthermore, if $U, V \in \mathscr{F}(\Upsilon)$, then consider $U_1 \cap V_1$, $U_2 \cap V_2, \ldots$. Clearly $U_1 \cap V_1 = U \cap V$. For any n, $U_n \cap V_n \in \Upsilon$, $U_n \cap V_n \supseteq G(\leq)$, and

$$(U_{n+1} \cap V_{n+1}) \circ (U_{n+1} \cap V_{n+1}) \subseteq U_{n+1} \circ U_{n+1} \subseteq U_n,$$
$$(U_{n+1} \cap V_{n+1}) \circ (U_{n+1} \cap V_{n+1}) \subseteq V_{n+1} \circ V_{n+1} \subseteq V_n.$$

Thus

$$(U_{n+1} \cap V_{n+1}) \circ (U_{n+1} \cap V_{n+1}) \subseteq U_n \cap V_n.$$

PROPOSITION 5.2. *(P, Υ) is a nearly uniform ordered space if and only if $\mathscr{F}(\Upsilon)^* \supseteq \Upsilon$.*

Proof. Since $\mathscr{F}(\Upsilon) \subseteq \Upsilon$, clearly $\mathscr{F}(\Upsilon)^* \subseteq \Upsilon$. Thus, if $\mathscr{F}(\Upsilon)^* \supseteq \Upsilon$, $\mathscr{F}(\Upsilon)^* = \Upsilon$. Therefore, $\mathscr{F}(\Upsilon)$ is a semi-uniform structure for P such that $\bigcap \mathscr{F}(\Upsilon) \supseteq G(\leq)$ and $\mathscr{F}(\Upsilon)^* = \Upsilon$.

Conversely, suppose that $\mathscr{G}$ is a semi-uniform structure for P such that $\bigcap \mathscr{G} \supseteq G(\leq)$ and $\mathscr{G}^* = \Upsilon$. Clearly $\mathscr{G} \subseteq \mathscr{F}(\Upsilon)$. Hence, if $H \in \Upsilon$, then $H = U \cap V^{-1}$ for $U, V \in \mathscr{F}(\Upsilon)$, i.e., $H \in \mathscr{F}(\Upsilon)^*$.

PROPOSITION 5.3. *(P, Υ) is a uniform ordered space if and only if $\bigcap \mathscr{F}(\Upsilon) \subseteq G(\leq)$ and $\mathscr{F}(\Upsilon)^* \supseteq \Upsilon$.*

Proof. Let $\mathscr{G}$ be a semi-uniform structure for P such that $\bigcap \mathscr{G} = G(\leq)$ and $\mathscr{G}^* = \Upsilon$. Clearly $\mathscr{G} \subseteq \mathscr{F}(\Upsilon)$. Thus $\bigcap \mathscr{G} \supseteq \bigcap \mathscr{F}(\Upsilon)$, i.e.

$$G(\leq) \supseteq \bigcap \mathscr{F}(\Upsilon).$$

Proposition 5.3 then follows from Proposition 5.2.

PROPOSITION 5.4. *For every (nearly) uniform ordered space* (P, Υ), $\mathscr{F}(\Upsilon)$ *is the unique maximal semi-uniform structure for* P *satisfying*

$$(\bigcap \mathscr{F}(\Upsilon) \supseteq G(\leq)) \cap \mathscr{F}(\Upsilon) = G(\leq)$$

and $\mathscr{F}(\Upsilon)^* = \Upsilon$.

Proof. If $\mathscr{G}$ is such a structure, then clearly $\mathscr{G} \subseteq \mathscr{F}(\Upsilon)$. Proposition 5.4 then follows from Propositions 5.1, 5.2 and 5.3 and their proofs.

6. The strong order and condition (M). Let P be a partially ordered set with uniformity Υ. We say that Υ (or (P, Υ)) *satisfies condition* (M) in case

(M) for all $V \in \Upsilon$, there exists $W \in \Upsilon$ such that $W \circ G(\leq) \subseteq G(\leq) \circ V$.

Since $G(\leq) \circ (U \cap V) \subseteq (G(\leq) \circ U) \cap (G(\leq) \circ V)$ for all $U, V \in \Upsilon$, the set $\{G(\leq) \circ U | U \in \Upsilon\}$ is a filter-base on $P \times P$. Let $\mathscr{C}(\Upsilon)$ be the filter on $P \times P$ generated by $\{G(\leq) \circ U | U \in \Upsilon\}$.

PROPOSITION 6.1. *Let* P *be a partially ordered set with uniformity* Υ. *Then* Υ *satisfies* (M) *if and only if* $\mathscr{F}(\Upsilon) = \mathscr{C}(\Upsilon)$.

Proof. Suppose that Υ satisfies condition (M). If $H \in \mathscr{F}(\Upsilon)$, then there exists $J \in \mathscr{F}(\Upsilon)$ such that $J \circ J \subseteq H$. Since $G(\leq) \subseteq J$,

$$G(\leq) \circ J \subseteq J \circ J \subseteq H.$$

Since $J \in \Upsilon$, this implies $H \in \mathscr{C}(\Upsilon)$. Thus $\mathscr{F}(\Upsilon) \subseteq \mathscr{C}(\Upsilon)$.

Conversely, suppose $U \in \Upsilon$. By condition (M), there exist $V_1, V_2, \ldots \in \Upsilon$ such that $V_1 = U$, and for all n, $V_{n+1} \circ V_{n+1} \subseteq V_n$ and

$$V_{n+1} \circ G(\leq) \subseteq G(\leq) \circ V_n.$$

Consider $W_n = G(\leq) \circ V_{2n-1}$ for $n = 1, 2, \ldots$. Then $W_1 = G(\leq) \circ V_1 = G(\leq) \circ U$. Furthermore, for all n, $G(\leq) \subseteq W_n$ and

$$\begin{aligned} W_{n+1} \circ W_{n+1} &= G(\leq) \circ V_{2n+1} \circ G(\leq) \circ V_{2n+1} \\ &\subseteq G(\leq) \circ G(\leq) \circ V_{2n} \circ V_{2n} \\ &\subseteq G(\leq) \circ V_{2n-1} \\ &= W_n. \end{aligned}$$

Therefore, $G(\leq) \circ U \in \mathscr{F}(\Upsilon)$ and hence $\mathscr{C}(\Upsilon) \subseteq \mathscr{F}(\Upsilon)$.

Suppose $\mathscr{F}(\Upsilon) = \mathscr{C}(\Upsilon)$. Let $U \in \Upsilon$. Then $G(\leq) \circ U \in \mathscr{F}(\Upsilon)$ and hence there exist $V_1, V_2 \in \mathscr{F}(\Upsilon)$ such that $V_1 \circ V_2 \subseteq G(\leq) \circ U$. Since $\mathscr{C}(\Upsilon)$ is a filter-base, there exist $J_1, J_2 \in \Upsilon$ such that $G(\leq) \circ J_1 \subseteq V_1$ and $G(\leq) \circ J_2 \subseteq V_2$.

Thus

$$J_1 \circ G(\leq) \subseteq G(\leq) \circ J_1 \circ G(\leq) \circ J_2 \subseteq G(\leq) \circ U.$$

Therefore, Υ satisfies condition (M).

PROPOSITION 6.2. *Let* (P, Υ) *be a nearly uniform ordered space. If* Υ *satisfies condition* (M), *then the strong order is equivalent to the* $\mathscr{F}(\Upsilon)$*-order.*

Proof. Clearly, $G(\lessdot) \subseteq G(\leqslant(\mathscr{F}(\Upsilon))$.

Conversely, suppose $\mathbf{x} \leqslant(\mathscr{F}(\Upsilon))\, \mathbf{y}$ and let $\{x_U | U \in \Upsilon^s\} \subseteq P$ be a Cauchy net converging to $\mathbf{x}$. Let $V \in \Upsilon^s$. Then there exists a Cauchy net $\{y_U | U \in \Upsilon^s\} \subseteq P$ converging to $\mathbf{y}$ such that $(x_U, y_U) \in G(\leq) \circ V$ for all $U \in \Upsilon^s$. Hence there exists $a_V \in P$ such that $(x_V, a_V) \in G(\leq)$ and $(a_V, y_V) \in V$. Consider the net $\{a_V\} \subseteq P$. For all $V \in \Upsilon^s$, $x_V \leq a_V$, and $(a_V, y_V) \in V$. Let $V, V' \in \Upsilon^s$ be such that $V' \circ V' \subseteq V$. There exists $U \in \Upsilon^s$ such that for $J, K \subseteq U$, $(y_J, y_K) \in V'$. Thus, if $J, K \subseteq U \cap V'$, $(a_J, y_J) \in J \subseteq V'$ and $(y_J, y_K) \in V'$. Hence $(a_J, y_K) \in V$. Therefore $\{a_V\}$ is a Cauchy net converging to the same point to which $\{y_U\}$ converges; i.e., $\{a_V\} \subseteq P$ is a Cauchy net converging to $\mathbf{y}$. We conclude that $\mathbf{x} \lessdot \mathbf{y}$.

COROLLARY 6.3. *Let* (P, Υ) *be a nearly uniform ordered space. If* Υ *satisfies condition* (M), *then* $(\mathbf{P}, \mathbf{\Upsilon})$ *with strong order is a uniform ordered space.*

Proof. The result follows from Propositions 4.7 and 6.2.

COROLLARY 6.4. *Let* (P, Υ) *be a uniform ordered space. If* Υ *satisfies condition* (M), *then the strong order on* $\mathbf{P}$ *extends the order on* P, *i.e.,*

$$G(\leq) = G(\lessdot) \cap (P \times P).$$

Proof. The result follows from Propositions 4.2 and 6.2.

Example 6.5. Not every uniform ordered space satisfies condition (M): (P, Υ) of Example 4.3 is a uniform ordered space whose strong order does not extend its original order. Therefore, by Corollary 6.4, (P, Υ) does not satisfy condition (M).

7. Ordering uniform completions of semilattices. The first result of this section says that there is only one candidate for a topological extended semilattice operation on the uniform completion of a semilattice with separated uniformity.

Let $(S, \circ)$ be a semilattice with separated uniformity Υ. Let $(\mathbf{S}, \mathbf{\Upsilon})$ be its completion at Υ. Let $E(\circ) \subseteq \mathbf{S} \times \mathbf{S}$ be the set of all $(\mathbf{x}, \mathbf{y})$ such that there exist Cauchy nets $\{x_U | U \in \Upsilon^s\} \subseteq S$ converging to $\mathbf{x}$ and $\{y_U | U \in \Upsilon^s\} \subseteq S$ converging to $\mathbf{y}$ such that $x_U \circ y_U = y_U$ for all $U \in \Upsilon^s$. For the semilattice $(S, \circ)$, let

$$D(\circ) = \{(x, y) \in S \times S | x \circ y = y\}.$$

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Example 7.1. This example shows that $E(\circ)$ is not necessarily the graph of a partial order. As in Example 2.2, let $\dot{Q}$ be the rational numbers with semilattice operation $\vee$, and let T_p be the p-adic topology on $\dot{Q}$ for some prime p. Let Υ be the usual uniformity associated with T_p (see [**3**]), and let $\dot{N}$ be the natural numbers. For $x \in \dot{Q}$, let $v_p(x)$ be the exponent of p in the decomposition of x into prime factors. Then for $U \in \Upsilon^s$, there exists a minimal $n(U) \in \dot{N}$ such that

$$\{(x, y)|v_p(x - y) \geq n(U)\} \subseteq U.$$

Let

$$x_U = 1 + p^{n(U)-1}, \qquad y_U = p^{n(U)}, \qquad z_U = 1 + p^{n(U)}.$$

Then $\{x_U|U \in \Upsilon^s\}$ and $\{z_U|U \in \Upsilon^s\}$ are Cauchy nets in $\dot{Q}$ converging to 1, and $\{y_U|U \in \Upsilon^s\}$ is a Cauchy net in $\dot{Q}$ converging to 0. Furthermore, for all $U \in \Upsilon^s$,

$$x_U \vee y_U = y_U \quad \text{and} \quad y_U \vee z_U = z_U.$$

Therefore, $(1, 0) \in E(\vee)$ and $(0, 1) \in E(\vee)$, and hence $E(\vee)$ is not the graph of a partial order on the uniform completion of $\dot{Q}$ at Υ.

THEOREM 7.2. *Let* $(S, \circ)$ *be a semilattice separated uniformity* Υ. *Suppose that* ρ *is a continuous semilattice operation on* $\mathbf{S}$.

(a) *If* $D(\circ) \subseteq D(\rho)$, *then* $E(\circ) \subseteq D(\rho)$.

(b) *If* $(S, \circ)$ *is a subsemilattice of* $(\mathbf{S}, \rho)$, *then* $E(\circ) = D(\rho)$.

Proof. (a) Let $(\mathbf{x}, \mathbf{y}) \in E(\circ)$, and suppose that $\{x_U|U \in \Upsilon^s\} \subseteq S$, $\{y_U|U \in \Upsilon^s\} \subseteq S$ are Cauchy nets converging to $\mathbf{x}, \mathbf{y}$ respectively such that $x_U \circ y_U = y_U$ for all $U \in \Upsilon^s$. Since ρ is continuous, $\{x_U\rho y_U\}$ converges to $\mathbf{x}\rho\mathbf{y}$. Since $D(\circ) \subseteq D(\rho)$, $x_U\rho y_U = y_U$ for all $U \in \Upsilon^s$. Thus $\{x_U\rho y_U\}$ also converges to $\mathbf{y}$, and hence $\mathbf{x}\rho\mathbf{y} = \mathbf{y}$, i.e., $(\mathbf{x}, \mathbf{y}) \in D(\rho)$.

(b) Since $(S, \circ)$ is a subsemilattice of $(\mathbf{S}, \rho)$, $D(\circ) \subseteq D(\rho)$. Thus by (a), $E(\circ) \subseteq D(\rho)$. Suppose $\mathbf{x}\rho\mathbf{y} = \mathbf{y}$, and let

$$\{a_U|U \in \Upsilon^s\} \subseteq S, \qquad \{b_U|U \in \Upsilon^s\} \subseteq S$$

be Cauchy nets converging to $\mathbf{x}, \mathbf{y}$ respectively. Since $(S, \circ)$ is a subsemilattice of $(\mathbf{S}, \rho)$, $a_U\rho b_U \in S$ for all $U \in \Upsilon^s$. Since ρ is continuous, $\{a_U\rho b_U\} \subseteq S$ is a Cauchy net converging to $\mathbf{x}\rho\mathbf{y} = \mathbf{y}$. Since ρ is a semilattice operation,

$$(a_U) \circ (a_U\rho b_U) = (a_U\rho a_U)\rho b_U = a_U\rho b_U$$

for all $U \in \Upsilon^s$. Since $\{a_U\}$ converges to $\mathbf{x}$, this implies that $(\mathbf{x}, \mathbf{y}) \in E(\circ)$.

COROLLARY 7.3. *Let* $(S, \circ)$ *be a semilattice with separated uniformity* Υ. *If there is a continuous semilattice operation* ρ *on* $(\mathbf{S}, \mathbf{\Upsilon})$ *such that* $(S, \circ)$ *is a subsemilattice of* $(\mathbf{S}, \rho)$, *then* $E(\circ)$ *is the graph of a partial order.*

We would like to use the results of previous sections to produce such a

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ρ as in Theorem 7.2 (b). To this end, let $(L, \vee)$ $((L, \wedge), (L, \vee, \wedge))$ be a join-semilattice (meet-semilattice, lattice) with separated uniformity Υ. We call $(L, \Upsilon, \vee)$ $((L, \Upsilon, \wedge), (L, \Upsilon, \vee, \wedge))$ a *j-uniform semilattice* (*m-uniform semilattice*, *uniform lattice*) in case $\vee$ ($\wedge$, $\vee$ and $\wedge$) is (is, are) uniformly continuous.

PROPOSITION 7.4. *A j-uniform semilattice is a uniform ordered space which satisfies condition* (M)*; thus so is a uniform lattice.*

Proof. The result follows from [**6**, Proposition 11] and its proof.

Let $(L, \Upsilon, \vee)$ $((L, \Upsilon, \wedge))$ be a j-uniform (m-uniform) semilattice; let $(\mathbf{L}, \boldsymbol{\Upsilon})$ be its completion at Υ. Then $\vee$ ($\wedge$) may be extended to a uniformly continuous function $\blacktriangledown$ ($\blacktriangle$) from $\mathbf{L} \times \mathbf{L}$ to $\mathbf{L}$.

PROPOSITION 7.5. *If* $(L, \Upsilon, \vee)$ *is a j-uniform semilattice, then* $\blacktriangledown$ *is the least upper bound with respect to the strong order. If* $(L, \Upsilon, \vee, \wedge)$ *is a uniform lattice, then* $\blacktriangle$ *is the greatest lower bound with respect to the strong order.*

Proof. Let $\{x_U | U \in \Upsilon^s\} \subseteq L$ and $\{y_U | U \in \Upsilon^s\} \subseteq L$ be Cauchy nets converging to $\mathbf{x}$ and $\mathbf{y}$ respectively. Then $\{x_U \vee y_U | U \in \Upsilon^s\} \subseteq L$ is a Cauchy net converging to $\mathbf{x} \blacktriangledown \mathbf{y}$. Since $x_U \leq x_U \vee y_U$ and $y_U \leq x_U \vee y_U$ for all $U \in \Upsilon^s$, $\mathbf{x} \lessdot \mathbf{x} \blacktriangledown \mathbf{y}$ and $\mathbf{y} \lessdot \mathbf{x} \blacktriangledown \mathbf{y}$. Let $\{z_U | U \in \Upsilon^s\} \subseteq L$ be a Cauchy net converging to $\mathbf{x} \blacktriangle \mathbf{y}$. Then $z_U \leq x_U \vee z_U$ and $z_U \leq y_U \vee z_U$ for all $U \in \Upsilon^s$. We will show that $\{x_U \vee z_U\}$ converges to $\mathbf{x}$ and that $\{y_U \vee z_U\}$ converges to $\mathbf{y}$. Let $U \in \Upsilon^s$ and suppose $H \in \Upsilon^s$ is such that $H \vee H \subseteq U$. There exists $W \in \Upsilon^s$ such that if $J \subseteq W$,

$$(z_J, x_J \wedge y_J) \in H.$$

Thus

$$(z_J \vee x_J, (x_J \wedge y_J) \vee x_J) \in H \vee H,$$

i.e.,

$$(z_J \vee x_J, x_J) \in U.$$

Thus $\{z_U \vee x_U\}$ is a Cauchy net converging to the same point to which $\{x_U\}$ converges; i.e., $\{z_U \vee x_U\}$ converges to $\mathbf{x}$. Similarly, $\{z_U \vee y_U\}$ converges to $\mathbf{y}$ and hence $\mathbf{x} \blacktriangle \mathbf{y} \lessdot \mathbf{x}$ and $\mathbf{x} \blacktriangle \mathbf{y} \lessdot \mathbf{y}$.

Suppose $\mathbf{x} \lessdot \mathbf{z}$ and $\mathbf{y} \lessdot \mathbf{z}$. Let $\{p_U | U \in \Upsilon^s\} \subseteq L$ be a Cauchy net converging to $\mathbf{x} \blacktriangledown \mathbf{y}$, and let $\{x_U | U \in \Upsilon^s\} \subseteq L$, $\{y_U | U \in \Upsilon^s\} \subseteq L$ be Cauchy nets converging to $\mathbf{x}, \mathbf{y}$ respectively. There exist Cauchy nets $\{a_U | U \in \Upsilon^s\} \subseteq L$, $\{b_U | U \in \Upsilon^s\} \subseteq L$ converging to $\mathbf{z}$ such that $x_U \leq a_U$, $y_U \leq b_U$ for all $U \in \Upsilon^s$. Then $\{a_U \vee b_U\}$ is a Cauchy net converging to $\mathbf{z}$ and $x_U \vee y_U \leq a_U \vee b_U$ for all $U \in \Upsilon^s$. Let $z_U = (a_U \vee b_U) \vee p_U$ for all $U \in \Upsilon^s$. Then $z_U \geq p_U$ for all U. Let $V \in \Upsilon^s$ and let $H \in \Upsilon^s$ be such that $H \vee H \subseteq V$. There exists $W \in \Upsilon^s$ such that if $U \subseteq W$,

$$(p_U, x_U \vee y_U) \in H.$$

Thus

$$((a_U \vee b_U) \vee p_U, (a_U \vee b_U) \vee (x_U \vee y_U)) \in H \vee H,$$

i.e.,

$$(z_U, a_U \vee b_U) \in V.$$

Hence $\{z_U\}$ is a Cauchy net converging to $\mathbf{z}$ and thus $\mathbf{x} \blacktriangledown \mathbf{y} \leqslant \mathbf{z}$. We conclude that $\blacktriangledown$ is the least upper bound with respect to the strong order on $\mathbf{L}$.

Suppose that $\mathbf{z} \leqslant \mathbf{x}$ and $\mathbf{z} \leqslant \mathbf{y}$. Let $\{z_U | U \in \Upsilon^s\} \subseteq L$ be a Cauchy net converging to $\mathbf{z}$ and suppose $\{x_U | U \in \Upsilon^s\} \subseteq L$, $\{y_U | U \in \Upsilon^s\} \subseteq L$ are Cauchy nets converging to $\mathbf{x}, \mathbf{y}$ respectively such that $z_U \leq x_U$ and $z_U \leq y_U$ for all $U \in \Upsilon^s$. Then $z_U \leq x_U \wedge y_U$ for all $U \in \Upsilon^s$ and clearly $\{x_U \wedge y_U\} \subseteq L$ is a Cauchy net converging to $\mathbf{x} \blacktriangle \mathbf{y}$. Thus $\mathbf{z} \leqslant \mathbf{x} \blacktriangle \mathbf{y}$. We conclude that $\blacktriangle$ is the greatest lower bound with respect to the strong order on $\mathbf{L}$.

COROLLARY 7.6. *A j-uniform semilattice* $(L, \Upsilon, \vee)$ (*uniform lattice* $(L, \Upsilon, \vee, \wedge)$) *is a join-subsemilattice* (*sublattice*) *of* $(\mathbf{L}, \leqslant)$ *and* $(\mathbf{L}, \leqslant(\mathscr{F}(\Upsilon))$.

Proof. By Propositions 6.2, 7.4, and 7.5, the join (join and meet) on L agrees (agree) with the join (join and meet) on $\mathbf{L}$ restricted to L.

COROLLARY 7.7. *Let* $(L, \Upsilon, \vee)$ $((L, \Upsilon, \vee, \wedge))$ *be a j-uniform semilattice* (*uniform lattice*). *Then the strong order on* $\mathbf{L}$ *is the unique order on* $\mathbf{L}$ *such that* $\mathbf{L}$ *is a join-semilattice* (*lattice*), L *is a join-subsemilattice* (*sublattice*) *of* $\mathbf{L}$, *and the join* (*join and meet*) *on* $\mathbf{L}$ *is* (*are*) *continuous.*

Proof. By Proposition 7.5 and Corollary 7.6, the strong order satisfies the conditions. By Theorem 7.2 (b), such an order is unique.

COROLLARY 7.8. *Let* $(L, \Upsilon, \vee)$ $((L, \Upsilon, \vee, \wedge))$ *be a j-uniform semilaltice* (*uniform lattice*). *Suppose* $\mathbf{L}$ *is ordered such that* $\mathbf{L}$ *is a join-semilattice* (*lattice*) *and* L *is a join-subsemilattice* (*sublattice*) *of* $\mathbf{L}$. *Then the join* (*lattice operations*) *is* (*are*) *continuous if and only if it is* (*they are*) *uniformly continuous.*

Proof. If the join (lattice operations) is (are) continuous, then the order is the strong order by Corollary 7.7. Thus it is (they are) in fact uniformly continuous.

We also have another characterization of the strong order on a j-uniform semilattice.

COROLLARY 7.9. *For a j-uniform lattice* $(L, \Upsilon, \vee)$, $\mathbf{x} \leqslant \mathbf{y}$ *if and only if there exist Cauchy nets* $\{x_U | U \in \Upsilon^s\} \subseteq L$, $\{y_U | U \in \Upsilon^s\} \subseteq L$ *converging to* $\mathbf{x}, \mathbf{y}$ *respectively such that* $x_U \leq y_U$ *for all* $U \in \Upsilon^s$.

Proof. The statement of the condition merely re-phrases the definition of $E(\vee)$ in terms of the join-semilattice order.

We note that this version of the strong order is precisely the definition used in [**5**] to extend the order on a uniform lattice to its completion. (See Section 1.)

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Remark 7.10. A *dual strong order*, $\lessdot_d$, could be defined on the uniform completion of a nearly uniform ordered space $(P, \Upsilon, \leq)$ by: $\mathbf{x} \lessdot_d \mathbf{y}$ if and only if for each Cauchy net $\{y_U | U \in \Upsilon^s\} \subseteq P$ converging to $\mathbf{y}$, there exists a Cauchy net $\{x_U | U \in \Upsilon^s\} \subseteq P$ converging to $\mathbf{x}$ such that $x_U \leq y_U$ for all $U \in \Upsilon^s$. Similarly, *dual* $\mathscr{F}$*-orders* could be defined, but these would be equivalent to the corresponding $\mathscr{F}$-orders. The results of Sections 3 and 4 would then hold with the obvious (dual) modifications.

In Section 6, we could use condition

(J) for all $V \in \Upsilon$, there exists $W \in \Upsilon$ such that $G(\leq) \circ W \subseteq V \circ G(\leq)$

to replace condition (M), and the set $\mathscr{D}(\Upsilon)$ = the filter generated by $\{U \circ G(\leq) | U \in \Upsilon\}$ to replace $\mathscr{C}(\Upsilon)$. The essential part of [**6**, Proposition 11], viz. [**6**, Theorem 10], remains true if we replace condition (M) with condition (J). Then [**6**, Proposition 11] holds when we replace the hypothesis of a uniformly continuous join with that of a uniformly continuous meet. This would permit a "dualizing" of hypotheses (that is, replacing "j-uniform" with "m-uniform") in such results as Propositions 7.4 and 7.5. To prove the dual versions of Corollaries 7.6 through 7.9 would then require the following meet-semilattice interpretation of Theorem 7.2:

THEOREM 7.2d. *Under the hypotheses of Theorem* 7.2, *we have the following:*
(a) *If* $D(\circ)^{-1} \subseteq D(\rho)^{-1}$, *then* $E(\circ)^{-1} \subseteq D(\rho)^{-1}$.
(b) *If* $(S, \circ)$ *is a subsemilattice of* $(\mathbf{S}, \rho)$, *then* $E(\circ)^{-1} = D(\rho)^{-1}$.

8. Ordering uniform completions of l-groups. In this section, let B be an l-group with Hausdorff topology T such that (B, T) is both a topological group and a topological lattice. Let Υ_r, Υ_l, Υ be, respectively, the right, left, and two-sided uniformities associated with T. Then Υ_r is generated by sets of the form

$$\bar{W} = \{(x, y) | x - y \in W\}$$

where W is a neighbourhood of 0; for the same W's, the sets

$$\underline{W} = \{(x, y) | -x + y \in W\}$$

generate Υ_l; and Υ is the filter generated by $\Upsilon_r \cup \Upsilon_l$. Let $\mathbf{B}_r$, $\mathbf{B}_l$, $\mathbf{B}$ be the completions of B with respect to Υ_r, Υ_l, Υ, respectively.

Corollary 7.7 (and its proof) indicates part of the proof of a conjecture of Conrad [**4**]. If B is abelian, then $\mathbf{B}_r = \mathbf{B}_l = \mathbf{B}$ may be considered as a supergroup of B. Conrad conjectured – in different terminology – that the strong order is the minimal lattice order on $\mathbf{B}$ such that $\mathbf{B}$ is an abelian l-group, $\mathbf{B}^+ \supseteq B^+$, and the lattice operations on $\mathbf{B}$ are continuous. We will show that this conjecture is true in a more general situation, provided that at least one of the lattice operations on B is uniformly continuous. This hypothesis is not as restrictive as it may appear; we will show that in fact it is equivalent to the assumption of local convexity for T.

LEMMA 8.1. *The following are equivalent:*
(a) $\vee$ *is uniformly continuous with respect to* Υ_r (Υ_l, Υ)*;*
(b) $\wedge$ *is uniformly continuous with respect to* Υ_l (Υ_r, Υ).

Proof. We note that

$$a \wedge b = -[(-a) \vee (-b)] \quad \text{and} \quad a \vee b = -[(-a) \wedge (-b)]$$

for all $a, b \in B$. Since $- : B \to B$ is uniformly continuous with respect to Υ [**7**], the equivalence of (a) and (b) in the case of Υ is clear.

Suppose $\vee$ is uniformly continuous with respect to Υ_r, and let $V \in \Upsilon_l$. Then by [**3**, III, 3, Proposition 2] $-V \in \Upsilon_r$. Thus there exists $H \in \Upsilon_r$ such that $H \vee H \subseteq -V$. Then $-H \in \Upsilon_l$. Let $(x, y), (a, b) \in -H$; thus $(-x, -y)$, $(-a, -b) \in H$ and hence $((-x) \vee (-a), (-y) \vee (-b)) \in -V$; i.e., $(-[(-x) \vee (-a)], -[(-y) \vee (-b)]) \in V$. Therefore $(x \wedge a, y \wedge b) \in V$, i.e. $(-H) \wedge (-H) \subseteq V$. Thus $\wedge$ is uniformly continuous with respect to Υ_l, i.e. (a) implies (b). Similarly (b) implies (a). The equivalence of (a) in the case of Υ_l and (b) in the case of Υ_r may be proven similarly.

LEMMA 8.2. *If* T *is locally convex, then the lattice operations on* B *are uniformly continuous with respect to* Υ_r, Υ_l *and* Υ.

Proof. Let W be a neighbourhood of 0 and let $\mathscr{N}$ be the set of convex neighbourhoods of 0. Since T is locally convex, there exists $M \in \mathscr{N}$ such that $M \subseteq W$. Since the lattice operations on B are continuous, and again since T is locally convex, there exists $N \in \mathscr{N}$ such that $N \vee N \subseteq M$ and $N \wedge N \subseteq M$ (M is a neighbourhood of 0 and $0 \vee 0 = 0 = 0 \wedge 0$). We first consider the uniformity U_r. Suppose that $(x, y), (a, b) \in \bar{N}$. We will show that $(x \vee a, y \vee b)$, $(x \wedge a, y \wedge b) \in \bar{W}$, and hence that $\bar{N} \vee \bar{N} \subseteq \bar{W}$ and $\bar{N} \wedge \bar{N} \subseteq \bar{W}$.

We note that $x - y, a - b \in N$ and thus since M is convex, that any t such that

$$(x - y) \wedge (a - b) \leq t \leq (x - y) \vee (a - b)$$

is an element of $M \subseteq W$. Furthermore,

$$\begin{aligned} (1) \quad [x \vee a] - [y \vee b] &= [x \vee a] + [(-y) \wedge (-b)] \\ &= [(x \vee a) - y] \wedge [(x \vee a) - b] \\ &= [(x - y) \vee (a - y)] \wedge [(x - b) \vee (a - b)], \end{aligned}$$

$$\begin{aligned} (2) \quad [x \vee a] - [y \vee b] &= [x \vee a] + [(-y) \wedge (-b)] \\ &= [x + ((-y) \wedge (-b))] \vee [a + ((-y) \wedge (-b))] \\ &= [(x - y) \wedge (x - b)] \vee [(a - y) \wedge (a - b)], \end{aligned}$$

$$(3) \quad [x \wedge a] - [y \wedge b] = [(x - y) \vee (x - b)] \wedge [(a - y) \vee (a - b)],$$

$$(4) \quad [x \wedge a] - [y \wedge b] = [(x - y) \wedge (a - y)] \vee [(x - b) \wedge (a - b)].$$

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Thus

$$(x - y) \wedge (a - b) \leq [x \vee a] - [y \vee b] \text{ by (1)},$$
$$(x - y) \vee (a - b) \geq [x \vee a] - [y \vee b] \text{ by (2)},$$
$$(x - y) \wedge (a - b) \leq [x \wedge a] - [y \wedge b] \text{ by (3)},$$
$$(x - y) \vee (a - b) \geq [x \wedge a] - [y \wedge b] \text{ by (4)}.$$

Therefore,

$$[x \vee a] - [y \vee b], [x \wedge a] - [y \wedge b] \in W,$$

i.e., $(x \vee a, y \vee b), (x \wedge a, y \wedge b) \in \bar{W}$.

Therefore $\wedge$ and $\vee$ are uniformly continuous with respect to Υ_r.

Similarly, $\wedge$ and $\vee$ are uniformly continuous with respect to Υ_l and hence with respect to Υ.

PROPOSITION 8.3. *The following are equivalent:*

(a) $\vee$ *is uniformly continuous with respect to* Υ_r, Υ_l, *or* Υ;

(b) $\wedge$ *is uniformly continuous with respect to* Υ_r, Υ_l, *or* Υ;

(c) T *is locally convex.*

Proof. By Propositions 2.1 and 7.4, (a) implies (c). By Lemma 8.2, (c) implies (a) and (b). It remains to show that (b) implies (a). Suppose $\wedge$ is uniformly continuous with respect to Υ_r (Υ_l, Υ). By Lemma 8.1, $\vee$ is uniformly continuous with respect to Υ_l (Υ_r, Υ). Then, as above, T is locally convex, and hence $\vee$ is uniformly continuous with respect to Υ_r, Υ_l and Υ. Thus (b) implies (a).

We call an l-group B with Hausdorff group and lattice topology T a *ul-group* in case the equivalent conditions (a), (b), (c) of Proposition 8.3 hold.

PROPOSITION 8.4. *Let* (B, T) *be an abelian ul-group. Then* $(\mathbf{B}, \mathbf{T}, \lesssim)$ *is an abelian ul-group.*

Proof. By [**3**, III, 3, Theorem 2], $(\mathbf{B}, \mathbf{T})$ is a topological abelian group. By Propositions 7.5 and 8.3, the lattice operations on $(\mathbf{B}, \lesssim)$ are uniformly continuous. It remains to show that $(\mathbf{B}, \lesssim)$ is a partially ordered group. Let $\mathbf{x} \lesssim \mathbf{y}$ and suppose $\mathbf{b} \in \mathbf{B}$. Let $\{x_U | U \in \Upsilon^s\} \subseteq B$, $\{y_U | U \in \Upsilon^s\} \subseteq B$, $\{b_U | U \in \Upsilon^s\} \subseteq B$ be Cauchy nets converging to $\mathbf{x}$, $\mathbf{y}$, $\mathbf{b}$ respectively such that $x_U \leq y_U$ for all $U \in \Upsilon^s$. Then $x_U + b_U \leq y_U + b_U$ for all $U \in \Upsilon^s$. Since $(\mathbf{B}, \mathbf{T})$ is a topological group, $\{x_U + b_U\} \subseteq B$, $\{y_U + b_U\} \subseteq B$, converge to $\mathbf{x} + \mathbf{b}$, $\mathbf{y} + \mathbf{b}$ respectively. Thus $\mathbf{x} + \mathbf{b} \lesssim \mathbf{y} + \mathbf{b}$, and hence $(\mathbf{B}, \lesssim)$ is a partially ordered group.

PROPOSITION 8.5. *Let* (B, T) *be a ul-group. Then the strong order on* $\mathbf{B}_r$ $(\mathbf{B}_l, \mathbf{B})$ *is the minimal lattice order* $\lesssim$ *on* $\mathbf{B}_r$ $(\mathbf{B}_l, \mathbf{B})$ *such that* $G(\leq) \subseteq G(\lesssim)$ *and the join on* $\mathbf{B}_r$ $(\mathbf{B}_l, \mathbf{B})$ *is continuous.*

Proof. By Propositions 3.1, 7.4, 7.5, 8.3, and Corollary 6.4, the strong order on $\mathbf{B}_r$ ($\mathbf{B}_l$, $\mathbf{B}$) satisfies the conditions. By Corollary 7.9 and Theorem 7.2 (a),

$$G(\lessgtr) = E(\vee) \subseteq G(\lesssim)$$

for any lattice order $\lesssim$ on $\mathbf{B}_r$ ($\mathbf{B}_l$, $\mathbf{B}$) satisfying the conditions.

COROLLARY 8.6. *Let* (B, T) *be an abelian ul-group. Then the strong order on* $\mathbf{B} = \mathbf{B}_r = \mathbf{B}_l$ *is the minimal partial order on* $\mathbf{B}$ *such that* $\mathbf{B}$ *is an abelian l-group,* $\mathbf{B}^+ \supseteq B^+$, *and the lattice operations on* $\mathbf{B}$ *are continuous.*

Proof. Let $\lesssim$ be a partial order on $\mathbf{B}$ satisfying the conditions. Since $\mathbf{B}$ is an abelian l-group and $\mathbf{B}^+ \supseteq B^+$, $G(\leq) \subseteq G(\lesssim)$. The result follows from Propositions 8.4 and 8.5.

COROLLARY 8.7. *Let* (B, T) *be a ul-group. Then the strong order on* $\mathbf{B}_r$ ($\mathbf{B}_l$, $\mathbf{B}$) *is the minimal lattice order* $\lesssim$ *on* $\mathbf{B}_r$ ($\mathbf{B}_l$, $\mathbf{B}$) *such that* $G(\leq) \subseteq G(\lesssim)$ *and the lattice operations on* $\mathbf{B}_r$ ($\mathbf{B}_l$, $\mathbf{B}$) *are uniformly continuous.*

Proof. By Proposition 7.5, the strong order satisfies the additional condition. The result follows from Proposition 8.5.

9. Categorical considerations. Definitions of terms which are left undefined in this section and which do not appear in a preceeding section, may be found in [**2**] or any book on category theory. (NB: We use "functor" for "covariant functor".)

Let $\mathfrak{A}$, $\mathfrak{B}$ be categories and let $\alpha : \mathfrak{B} \to \mathfrak{A}$ be a functor. A functor $\beta : \mathfrak{A} \to \mathfrak{B}$ is *adjoint* to α in case for each object A of $\mathfrak{A}$ there is a morphism $t_A : A \to A\beta\alpha$ such that if X is an object of $\mathfrak{B}$ and if $g : A \to X\alpha$ is a morphism of $\mathfrak{A}$, then there exists a unique morphism $k : A\beta \to X$ of $\mathfrak{B}$ such that $(t_A)(k\alpha) = g$. The functor which takes a separated uniform space to its completion (and a uniformly continuous function to its extension) is adjoint to the functor which embeds the category of complete separated uniform spaces and uniformly continuous functions in the category of separated uniform spaces and uniformly continuous functions.

Let $\mathfrak{nuD}$ be the category of nearly uniform ordered spaces and uniformly continuous order-preserving functions. Let $\mathfrak{uD}$ be the category of uniform ordered spaces and uniformly continuous order-preserving functions. Let $\mathfrak{CnuD}$ and $\mathfrak{CuD}$ be the categories of uniformly complete nearly uniform ordered spaces and uniformly complete uniform ordered spaces, respectively. Let $\eta : \mathfrak{CuD} \to \mathfrak{uD}$, $\zeta : \mathfrak{CnuD} \to \mathfrak{nuD}$ be the natural embedding functors. We wish to investigate possible adjoint functors to η and ζ.

Specifically, we will investigate the following problem: Let $(P, \mathrm{T}, \leq)$ be a (nearly) uniform ordered space. Let $(\mathbf{P}, \boldsymbol{\Upsilon})$ be its uniform completion, and let $t_p : P \to \mathbf{P}$ be the usual uniform embedding (see [**3**]). Can we specify a partial order $\precsim$ on $\mathbf{P}$ with the following properties:

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(i) $(\mathbf{P}, \mathbf{\Upsilon}, \lesssim)$ is a (nearly) uniform ordered space;

(ii) $t_p : (P, \leq) \to (\mathbf{P}, \lesssim)$ preserves order;

(iii) if $(\mathbf{X}, \mathbf{\Xi}, \lesssim)$ is a uniformly complete (nearly) uniform ordered space, if $f : P \to \mathbf{X}$ is a uniformly continuous order-preserving function, and if $\mathbf{f} : (\mathbf{P}, \mathbf{\Upsilon}) \to (\mathbf{X}, \mathbf{\Xi})$ is the unique uniformly continuous function such that $t_p\mathbf{f} = f$, then $\mathbf{f} : (\mathbf{P}, \lesssim) \to (\mathbf{X}, \lesssim)$ preserves order?

Let $(P, \Upsilon, \leq)$ be a nearly uniform ordered space, and let $(\mathbf{P}, \mathbf{\Upsilon})$ and $t_p : P \to \mathbf{P}$ be as above. For simplicity, we sometimes ignore t_p and identify P with its t_p-image in $\mathbf{P}$. Let

$$\mathscr{E}(\mathbf{\Upsilon}) = \{V \in \mathbf{\Upsilon} \mid \text{there exist } V_1, V_2, \ldots \in \mathbf{\Upsilon} \text{ such that } V_1 = V, \text{ and for all } n,\ V_n \supseteq G(\leq) \text{ and } V_{n+1} \circ V_{n+1} \subseteq V_n\}.$$

LEMMA 9.1. *$\mathscr{E}(\mathbf{\Upsilon})$ is a semi-uniform structure for* $\mathbf{P}$.

Proof. The proof is similar to that of Proposition 5.1.

Let $\leq_1$ and $\leq_2$ be the binary relations defined on $\mathbf{P}$ as follows:

$\mathbf{x} \leq_1 \mathbf{y}$ if and only if $\mathbf{x} = \mathbf{y}$ or $\mathbf{x}, \mathbf{y} \in P$ and $\mathbf{x} \leq \mathbf{y}$;
$\mathbf{x} \leq_2 \mathbf{y}$ if and only if $(\mathbf{x}, \mathbf{y}) \in \bigcap \mathscr{E}(\mathbf{\Upsilon})$.

PROPOSITION 9.2. *Both $\leq_1$ and $\leq_2$ are partial orders on* $\mathbf{P}$.

Proof. Clearly $\leq_1$ is a partial order. Since $\mathbf{\Upsilon}$ is separated, Lemma 9.1 and the argument of [**6**, pp. 58–59] show that $\bigcap \mathscr{E}(\mathbf{\Upsilon})$ is the graph of a partial order.

We will show that $\leq_1$ and $\leq_2$ can be used to define adjoint functors to ζ and η respectively.

Let $\mathscr{F}(\Upsilon)$ be the maximal semi-uniform structure for P defined in Section 5. Let $|\mathscr{F}(\Upsilon)|$ be the collection of subsets of $\mathbf{P} \times \mathbf{P}$ defined following Example 4.3.

LEMMA 9.3. $|\mathscr{F}(\Upsilon)| \subseteq \mathscr{E}(\mathbf{\Upsilon})$.

Proof. If $V \in |\mathscr{F}(\Upsilon)|$, then by Proposition 4.1 and Lemma 4.6, $G(\leq) \subseteq V$. Thus by Lemma 4.4, for every $V \in |\mathscr{F}(\Upsilon)|$, there exist $V_1, V_2, \ldots \in \mathbf{\Upsilon}$ such that $V = V_1$, and for all n, $V_{n+1} \circ V_{n+1} \subseteq V_n$ and $G(\leq) \subseteq V_n$. Therefore, $|\mathscr{F}(\Upsilon)| \subseteq \mathscr{E}(\mathbf{\Upsilon})$.

PROPOSITION 9.4. *$(\mathbf{P}, \mathbf{\Upsilon}, \leq_1)$ is a nearly uniform ordered space, and $(\mathbf{P}, \mathbf{\Upsilon}, \leq_2)$ is a uniform ordered space.*

Proof. By Proposition 9.2, $(\mathbf{P}, \leq_1)$ and $(\mathbf{P}, \leq_2)$ are partially ordered sets; by Lemma 9.1, $\mathscr{E}(\mathbf{\Upsilon})$ is a semi-uniform structure for $\mathbf{P}$; and from the definitions, $G(\leq_1) \subseteq \bigcap \mathscr{E}(\mathbf{\Upsilon})$ and $G(\leq_2) = \bigcap \mathscr{E}(\mathbf{\Upsilon})$. Thus it suffices to show that $\mathscr{E}(\mathbf{\Upsilon})^* = \mathbf{\Upsilon}$. By Lemma 4.5, $|\mathscr{F}(\Upsilon)|^* = \mathbf{\Upsilon}$, and hence by Lemma 9.3, $\mathscr{E}(\mathbf{\Upsilon})^* \supseteq |\mathscr{F}(\Upsilon)|^* = \mathbf{\Upsilon}$. Clearly $\mathscr{E}(\mathbf{\Upsilon})^* \subseteq \mathbf{\Upsilon}$, and hence $\mathscr{E}(\mathbf{\Upsilon})^* = \mathbf{\Upsilon}$.

PROPOSITION 9.5. *Let $\bar{\zeta} : \mathfrak{nu}\mathfrak{D} \to \mathfrak{Cnu}\mathfrak{D}$ take an object $(P, \Upsilon, \leq)$ to $(\mathbf{P}, \mathbf{\Upsilon}, \leq_1)$*

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and a morphism to its uniformly continuous extension. Then $\bar{\zeta}$ is a functor which is adjoint to ζ.

Proof. Let $(P, \Upsilon, \leq)$ be a nearly uniform ordered space. By Proposition 9.4, $(\mathbf{P}, \boldsymbol{\Upsilon}, \leq_1)$ be a uniformly complete nearly uniform ordered space. Clearly, t_p preserves order. Let $(\mathbf{X}, \boldsymbol{\Xi}, \lesssim)$ be a uniformly complete nearly uniform ordered space, let $f : P \to \mathbf{X}$ be a uniformly continuous order-preserving function, and let $\mathbf{f} : (\mathbf{P}, \boldsymbol{\Upsilon}) \to (\mathbf{X}, \boldsymbol{\Xi})$ be the unique uniformly continuous function such that $t_p\mathbf{f} = f$. It suffices to show that $\mathbf{f} : (\mathbf{P}, \leq_1) \to (\mathbf{X}, \lesssim)$ preserves order. To this end, suppose that $\mathbf{x} \leq_1 \mathbf{y}$. If $\mathbf{x} = \mathbf{y}$, then $\mathbf{xf} = \mathbf{yf}$, and thus $\mathbf{xf} \lesssim \mathbf{yf}$. If $\mathbf{x} \neq \mathbf{y}$, then there exist $a, b \in P$ such that $a \leq b$ and $at_p = \mathbf{x}$, $bt_p = \mathbf{y}$. Thus

$$\mathbf{xf} = at_p\mathbf{f} = af \lesssim bf = bt_p\mathbf{f} = \mathbf{yf}.$$

The proof that an adjoint for η may be defined similarly to $\bar{\zeta}$ for ζ is somewhat more complicated than the proof of Proposition 9.5. We will need the following two lemmas, the first of which is straightforward to prove.

LEMMA 9.6. *Let X, Y be sets and let $h : Y \to X$ be a function. Let $L, J \subseteq X \times X$ be such that $L \circ L \subseteq J$. Then*

$$L(h \times h)^{-1} \circ L(h \times h)^{-1} \subseteq J(h \times h)^{-1}.$$

LEMMA 9.7. *Let $(P, \Upsilon, \leq)$, $(X, \Xi, \lesssim)$, and $(Y, \Psi, \precsim)$ be nearly uniform ordered spaces. Let $g : P \to X$, $t : P \to Y$, and $k : Y \to X$ be uniformly continuous functions such that $tk = g$, and suppose that g and t preserve order. If $V \in \mathscr{F}(\Xi)$, then there exist $E_1, E_2, \ldots \in \Psi$ such that $V(k \times k)^{-1} = E_1$, and for all n, $G(\leq)(t \times t) \subseteq E_n$ and $E_{n+1} \circ E_{n+1} \subseteq E_n$.*

Proof. Let $V_1, V_2, \ldots \in \Xi$ be such that $V_1 = V$, and for all n, $G(\lesssim) \subseteq V_n$ and $V_{n+1} \circ V_{n+1} \subseteq V_n$. For each n, let $E_n = V_n(k \times k)^{-1} \in \Psi$. If $a, b \in P$ are such that $a \leq b$, then $ag \lesssim bg$, i.e., $(a)tk \lesssim (b)tk$. Hence for all n, $(at, bt)(k \times k) \in G(\lesssim) \subseteq V_n$, i.e., $(at, bt) \in V_n(k \times k)^{-1} = E_n$. Thus, $G(\leq)(t \times t) \subseteq E_n$ for all n. By Lemma 9.6, since $V_{n+1} \circ V_{n+1} \subseteq V_n$, $E_{n+1} \circ E_{n+1} \subseteq E_n$. By definition, $V(k \times k)^{-1} = E_1$.

PROPOSITION 9.8. *Let $\bar{\eta} : \mathfrak{uO} \to \mathfrak{CuO}$ take an object $(P, \Upsilon, \leq)$ to $(\mathbf{P}, \boldsymbol{\Upsilon}, \leq_2)$ and a morphism to its uniformly continuous extension. Then $\bar{\eta}$ is a functor which is adjoint to η.*

Proof. Let $(P, \Upsilon, \leq)$ be a uniform ordered space. By Proposition 9.4, $(\mathbf{P}, \boldsymbol{\Upsilon}, \leq_2)$ is a uniform ordered space. Clearly, t_p preserves order. Let $(\mathbf{X}, \boldsymbol{\Xi}, \lesssim)$ be a uniformly complete uniform ordered space, let $f : (P, \Upsilon, \leq) \to (\mathbf{X}, \boldsymbol{\Xi}, \lesssim)$ be a uniformly continuous order-preserving function, and let $\mathbf{f} : (\mathbf{P}, \boldsymbol{\Upsilon}) \to (\mathbf{X}, \boldsymbol{\Xi})$ be the unique uniformly continuous function such that $t_p\mathbf{f} = f$. It suffices to show that $\mathbf{f} : (\mathbf{P}, \leq_1) \to (\mathbf{X}, \lesssim)$ preserves order. To this end, suppose that $\mathbf{x} \leq_2 \mathbf{y}$. Then $(\mathbf{x}, \mathbf{y}) \in \bigcap \mathscr{E}(\boldsymbol{\Upsilon})$. Thus by Lemma 9.7, $(\mathbf{x}, \mathbf{y}) \in V(\mathbf{f} \times \mathbf{f})^{-1}$

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for all $V \in \mathscr{F}(\boldsymbol{\Xi})$, i.e., $(\mathbf{xf}, \mathbf{yf}) \in V$ for all $V \in \mathscr{F}(\boldsymbol{\Xi})$. Thus

$$(\mathbf{xf}, \mathbf{yf}) \in \bigcap \mathscr{F}(\boldsymbol{\Xi}) = G(\lesssim),$$

i.e., $\mathbf{xf} \lesssim \mathbf{yf}$.

In fact, $\leq_2$ is the partial ordering of $\mathbf{P}$ we would expect in view of the preceding sections, namely $\leqslant (\mathscr{F}(\Upsilon))$.

PROPOSITION 9.9. $G(\leq_2) = G(\leqslant (\mathscr{F}(\Upsilon)))$.

Proof. By Lemma 9.3, $|\mathscr{F}(\Upsilon)| \subseteq \mathscr{E}(\boldsymbol{\Upsilon})$, and hence by Lemma 4.6,

$$G(\leqslant (\mathscr{F}(\Upsilon))) = \bigcap |\mathscr{F}(\Upsilon)| \supseteq \bigcap \mathscr{E}(\boldsymbol{\Upsilon}) = G(\leq_2).$$

Conversely, suppose that $\mathbf{x} \leqslant (\mathscr{F}(\Upsilon))\, \mathbf{y}$. Let $T(\boldsymbol{\Upsilon})$ be the topology on $\mathbf{P}$ associated with $\boldsymbol{\Upsilon}$, and for all $V \in \boldsymbol{\Upsilon}$, let V^c be the closure of V with respect to $T(\boldsymbol{\Upsilon}) \times T(\boldsymbol{\Upsilon})$.

We first show that $(\mathbf{x}, \mathbf{y}) \in V^c$ for all $V \in \mathscr{E}(\boldsymbol{\Upsilon})$. Let $V \in \mathscr{E}(\boldsymbol{\Upsilon})$. Clearly, for all $H \in \mathscr{E}(\boldsymbol{\Upsilon})$, $H(t_p \times t_p)^{-1} \supseteq G(\leq)$ and $H(t_p \times t_p)^{-1} \in \Upsilon$. Thus, by Lemma 9.6 as applied in Lemma 9.7, $V(t_p \times t_p)^{-1} \in \mathscr{F}(\Upsilon)$. Thus there exist Cauchy nets $\{x_U | U \in \Upsilon^s\} \subseteq P$, $\{y_U | U \in \Upsilon^s\} \subseteq P$ such that $\{x_U t_p\}$ converges to $\mathbf{x}$, $\{y_U t_p\}$ converges to $\mathbf{y}$, and $(x_U, y_U) \in V(t_p \times t_p)^{-1}$ for all $U \in \Upsilon^s$. Thus $(x_U t_p, y_U t_p) \in V$ for all $U \in \Upsilon^s$, and hence $(\mathbf{x}, \mathbf{y}) \in V^c$.

If $H \in \mathscr{E}(\boldsymbol{\Upsilon})$, then there exists $V \in \mathscr{E}(\boldsymbol{\Upsilon})$ such that $V \circ V \circ V \subseteq H$. Thus by [**3**, II, 1, Proposition 2],

$$V^c = \bigcap_{U \in \Upsilon^s} (U \circ V \circ U) \subseteq V \circ V \circ V \subseteq H.$$

Hence $(\mathbf{x}, \mathbf{y}) \in H$ for all $H \in \mathscr{E}(\boldsymbol{\Upsilon})$, i.e., $(\mathbf{x}, \mathbf{y}) \in \bigcap \mathscr{E}(\boldsymbol{\Upsilon}) = G(\leq_2)$.

The following examples show that $\leq_1$ may or may not be equivalent to $\leq_2$.

Example 9.10. Let $(P, \Upsilon, \leq)$ be a uniformly complete uniform ordered space. Then clearly $G(\leq_1) = G(\leq_2)$.

Example 9.11. Let $\dot{Q}$ be the rational numbers with usual order $\leq$ and usual uniformity Υ. Then clearly, $(\dot{Q}, \Upsilon, \leq)$ is a uniform ordered space. Clearly, $\dot{R}$ is the completion of $\dot{Q}$ at Υ and $0 \leqslant (\mathscr{F}(\Upsilon))\, \pi$, i.e., $0 \leq_2 \pi$. However, since $\pi \in \dot{R} \backslash \dot{Q}$, $0 \nleq_1 \pi$, and thus $G(\leq_1) \neq G(\leq_2)$.

Let $\nu : \mathfrak{u}\mathfrak{O} \rightarrow \mathfrak{n}\mathfrak{u}\mathfrak{O}$ be the natural embedding functor. Let $\bar{\nu} : \mathfrak{n}\mathfrak{u}\mathfrak{O} \rightarrow \mathfrak{u}\mathfrak{O}$ be defined by $f\bar{\nu} = f$ for all morphisms f and $(P, \Upsilon, \leq)\bar{\nu} = (P, \Upsilon, \precsim)$ where $G(\precsim) = \bigcap \mathscr{F}(\Upsilon)$.

PROPOSITION 9.12. *$\bar{\nu}$ is a functor which is adjoint to ν.*

Proof. As in [**6**, pp. 58–59], $\precsim$ is a partial order on P. Then clearly $(P, \Upsilon, \precsim)$ is a uniform ordered space. Since $G(\leq) \subseteq \bigcap \mathscr{F}(\Upsilon) = G(\precsim)$, the identity map from $(P, \Upsilon, \leq)$ to $(P, \Upsilon, \precsim)$ preserves order. Let $(X, \Xi, \lesssim)$ be a uniform ordered space, and suppose that $f : P \rightarrow X$ is a uniformly continuous order-preserving (with respect to $\leq$) function. It suffices to show that if $x \bar{<} y$, then $xf \lesssim yf$. By Lemma 9.7, if $V \in \mathscr{F}(\Xi)$, then $V(f \times f)^{-1} \in \mathscr{F}(\Upsilon)$. Thus

$$(x, y) \in \bigcap \mathscr{T}(\Upsilon) \subseteq \bigcap_{V \in \mathscr{F}(\Xi)} V(f \times f)^{-1},$$

i.e., $(xf, yf) \in \bigcap \mathscr{F}(\Xi)$. Since $(X, \Xi, \lesssim)$ is a uniform ordered space, $(xf, yf) \in G(\lesssim)$, i.e., $xf \lesssim yf$.

Remark. Let $\mathfrak{juS}(\mathfrak{muS}, \mathfrak{uL})$ and $\mathfrak{CjuS}(\mathfrak{CmuS}, \mathfrak{CuL})$ be, respectively, the categories of j-uniform semilattices (m-uniform semilattices, uniform lattices) and uniformly complete j-uniform semilattices (m-uniform semilattices, uniform lattices), both with uniformly continuous join-preserving (meet-preserving, join- and meet-preserving) functions. Using Corollary 7.7 (Remark 7.10, Corollary 7.7) as a guide, we see that the strong order is the only candidate for a solution to the problem of ordering the completion of a j-uniform semilattice (m-uniform semilattice, uniform lattice) so as to describe an adjoint to the natural embedding functor from $\mathfrak{CjuS}$ ($\mathfrak{CmuS}$, $\mathfrak{CuL}$) to $\mathfrak{juS}(\mathfrak{muS}, \mathfrak{uL})$. Routine manipulation shows that it is indeed the proper order, i.e. that the extension of a uniformly continuous join-preserving (meet-preserving, join- and meet-preserving) function to the completion does preserve the join (meet, join and meet).

Let $\mathfrak{A}$ and $\mathfrak{CA}$ be, respectively, the categories of abelian ul-groups and uniformly complete abelian ul-groups, both with uniformly continuous group and lattice homomorphisms. That the strong order on the completion is an "adjoint" ordering may be seen by the following argument (cf. Corollary 8.6): The completion with strong order is an abelian ul-group by Proposition 8.4; the unique extension of a morphism to the completion is a uniformly continuous group homomorphism by [**3**, III, 3, Propositions 3 and 8], and a lattice homomorphism by the routine manipulation mentioned above.

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