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A non-abelian Stickelberger theorem

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Abstract

Let L/k be a finite Galois extension of number fields with Galois group G. For every odd prime p satisfying certain mild technical hypotheses, we use values of Artin L-functions to construct an element in the centre of the group ring $\mathbb{Z}_{(p)}[G]$ that annihilates the p-part of the class group of L.

1. Introduction and statement of the main results

Let K/k be a finite Galois extension of number fields with Galois group G. Let $\mathcal{S}$ be a finite set of places of k containing the infinite places $\mathcal{S}_\infty$. For any (complex) character χ of G, we let $e_\chi = (\chi(1)/|G|)\sum_{g\in G}\chi(g^{-1})g$ denote the corresponding central idempotent of the group algebra $\mathbb{C}[G]$ and let $L_\mathcal{S}(s,\chi)$ denote the truncated Artin L-function attached to χ and $\mathcal{S}$. Summing over all irreducible characters of G gives a so-called 'Stickelberger element',

$$\Theta(K/k,\mathcal{S}) := \sum_{\chi\in\mathrm{Irr}(G)} L_\mathcal{S}(0,\bar{\chi})\cdot e_\chi.$$

Now suppose that k is totally real, K is a CM field, G is abelian and $\mathcal{S}$ contains the ramified places $\mathcal{S}_{\mathrm{ram}}(K/k)$. Let μ_K denote the roots of unity in K and let cl_K denote the class group of K. In [Cas79, DR80] it was shown independently that

$$\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)\Theta(K/k,\mathcal{S}) \subseteq \mathbb{Z}[G].$$

It is now easy to state Brumer's conjecture, which can be seen as a generalisation of Stickelberger's theorem.

Conjecture 1.1. In the above situation, $\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)\Theta(K/k,\mathcal{S})$ annihilates cl_K.

There is a large body of evidence in support of Brumer's conjecture; see the expository article [Gre04], for example. Furthermore, under the assumptions that the appropriate special case of the equivariant Tamagawa number conjecture (ETNC) holds (see § 6) and that the non-2-part of μ_K is a cohomologically trivial G-module, Greither has shown that Brumer's conjecture holds outside the 2-part (see [Gre07]).

By contrast, as far as we are aware, no Brumer-type annihilation result has yet been proved for any non-abelian extension. In the present article, we address this situation by proving an unconditional annihilation result for arbitrary (not necessarily abelian) extensions, from which a weak form of Brumer's conjecture can also be deduced.

Before stating the main result, we introduce some additional notation. For any natural number n, we let ζ_n denote a primitive nth root of unity. For any number field F, we write F^{cl} for the normal closure and F^+ for the maximal totally real subfield of F. For a complex character χ

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of a finite group G, we let $E = E_\chi$ denote a subfield of $\mathbb{C}$ over which χ can be realised that is both Galois and of finite degree over $\mathbb{Q}$, and we let $\mathcal{O} = \mathcal{O}_E$ denote the ring of algebraic integers of E. Furthermore, we write pr_χ for the associated 'projector' $\sum_{g\in G}\chi(g^{-1})g$ in the group algebra $E[G]$ and $\mathcal{D}_{E/\mathbb{Q}}$ for the different of the extension $E/\mathbb{Q}$.

We shall postpone two important definitions until § 2; for the moment we shall only give brief descriptions. In the theorem below, U_χ is an explicit fractional ideal of $\mathcal{O}$ that depends on $\mathcal{S}_{\mathrm{ram}}(K/k)$ (it is often the case that U_χ is trivial; see Remark 1.3(ii) and § 2.3). The fractional $\mathcal{O}$-ideal $h(\mu_K,\chi)$ is a natural truncated Euler characteristic of the χ-twist of μ_K. We abbreviate $L_{\mathcal{S}_\infty}(s,\chi)$ to $L(s,\chi)$.

Theorem 1.2. *Let L/k be a finite Galois extension of number fields with Galois group G. Fix a non-trivial irreducible character χ of G. Let $K := L^{\ker(\chi)}$ be the subfield of L cut out by χ. Let p be any odd prime satisfying the following condition.*

($*$) *Suppose that:*
  - (a) *k is totally real;*
  - (b) *K is a CM field; and*
  - (c) *$K^{\mathrm{cl}} \subset (K^{\mathrm{cl}})^+(\zeta_p)$.*

  Then no prime of K^+ above p is split in K/K^+.

Then, for any element x of $\mathcal{D}_{E/\mathbb{Q}}^{-1}\cdot h(\mu_K,\chi)\cdot U_\chi$, the sum

$$\sum_{\omega\in\mathrm{Gal}(E_\chi/\mathbb{Q})} x^\omega L(0,\bar{\chi}^\omega)\cdot \mathrm{pr}_{\chi^\omega}$$

belongs to the centre of $\mathbb{Z}_{(p)}[G]$ and annihilates $\mathbb{Z}_{(p)}\otimes_{\mathbb{Z}}\mathrm{cl}_L$.

Remark 1.3. The statement of Theorem 1.2 can be simplified in several cases.

(i) If an odd prime p is unramified in $K/\mathbb{Q}$, then (b) forces $K^{\mathrm{cl}} \not\subset (K^{\mathrm{cl}})^+(\zeta_p)$ and so condition ($*$) holds trivially. Furthermore, if k is normal over $\mathbb{Q}$ and $[(K^{\mathrm{cl}})^+ : \mathbb{Q}]$ is odd, then condition ($*$) holds for all odd primes p (these hypotheses together with (a), (b) and (c) imply that all the primes of K^+ above p are in fact ramified in K/K^+).

(ii) If every inertia subgroup of $\mathrm{Gal}(K/k)$ is normal (for example, if χ is linear or every $\mathfrak{p}\in\mathcal{S}_{\mathrm{ram}}(K/k)$ is non-split in K/k), then under the assumption that χ is non-trivial and irreducible, it is straightforward to show that U_χ is trivial (see § 2.3).

(iii) If an odd prime p does not divide $|\mu_K|$, then $h(\mu_K,\chi)$ is relatively prime to p and so this term can be ignored. In particular, this is the case when p is unramified in $K/\mathbb{Q}$.

Remark 1.4. The purpose of assuming condition ($*$) is to ensure that when (a) and (b) hold, the strong Stark conjecture at p as formulated by Chinburg in [Chi83, Conjecture 2.2] will hold for the (odd) character χ. Hence condition ($*$) can be ignored completely in each of the following cases, where the strong Stark conjecture is already known to be valid.

(i) The character χ is rational-valued: this was proved by Tate in [Tat84, ch. II, Theorem 8.6].

(ii) The field k is $\mathbb{Q}$ and the character χ is linear: this was proved by Ritter and Weiss in [RW97] (in fact, they showed that the conjecture holds if 2 is unramified in $K/\mathbb{Q}$ and that otherwise it holds outside the 2-part; but this is all we need since p is odd).

(iii) The field k is imaginary quadratic of class number one and χ is a linear character whose order is divisible only by primes which split completely in $k/\mathbb{Q}$: this follows from [BF03, § 3] and the result of Bley in [Ble06, Theorem 4.2].

Note that, in particular, we are in case (i) if G is isomorphic to the symmetric group on any number of elements, the quaternion group of order eight, or any direct product of such groups.

We give the proof of the following corollary in § 12, after the proof of Theorem 1.2.

Corollary 1.5. *Let L/k be a finite Galois extension of number fields with Galois group G. Suppose that every inertia subgroup is normal in G (which is the case, for example, when every $\mathfrak{p} \in \mathcal{S}_{\mathrm{ram}}(L/k)$ is non-split in L/k). Let $\mathcal{S}$ be any finite set of places of k containing the infinite places $\mathcal{S}_\infty$. For any irreducible character χ of G, let $\mathbb{Q}(\chi)$ denote the character field of χ and let d_χ be the minimum of $[E_\chi : \mathbb{Q}(\chi)]$ over all possible choices of E_χ. Let p be any odd prime that is unramified in $L/\mathbb{Q}$. Then the element*

$$\sum_{\chi \in \mathrm{Irr}(G), \chi \neq 1} L_{\mathcal{S}}(0, \bar{\chi}) \cdot d_\chi \mathrm{pr}_\chi$$

belongs to the centre of $\mathbb{Z}_{(p)}[G]$ and annihilates $\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} \mathrm{cl}_L$.

Remark 1.6. Note that E_χ can always be taken to be $\mathbb{Q}(\zeta_n)$ where n is the exponent of G (see [CR81, (15.18)]), and so $[\mathbb{Q}(\zeta_n) : \mathbb{Q}(\chi)]$ is an upper bound for d_χ. In fact, $d_\chi = 1$ whenever G is abelian or of odd prime power order, or when it is isomorphic to the symmetric group on any number of letters, the dihedral group of any order, or any direct product of such groups.

Remark 1.7. Suppose that k is totally real, L is a CM field, G is abelian and $\mathcal{S}$ contains the ramified primes $\mathcal{S}_{\mathrm{ram}}(L/k)$. Then Corollary 1.5 says that for p odd and unramified in $L/\mathbb{Q}$,

$$\sum_{\chi \in \mathrm{Irr}(G), \chi \neq 1} L_{\mathcal{S}}(0, \bar{\chi}) \cdot \mathrm{pr}_\chi = |G| \cdot \Theta(L/k, \mathcal{S}) \text{ annihilates } \mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} \mathrm{cl}_K.$$

(Note that there is a slight adjustment to be made in the case of $k = \mathbb{Q}$.) Under the hypotheses on p we have $\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_L) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} = \mathbb{Z}_{(p)}[G]$, so the above statement is the same as the 'p-part' of Brumer's conjecture (Conjecture 1.1) but with an extra factor of $|G|$ in the annihilator (of course, this makes no difference if p does not divide $|G|$). □

Remark 1.8. In [Bur09] a conjecture is given which, in the setting of the present article, uses values of derivatives of Artin L-functions to construct explicit annihilators of ideal class groups. Upon restriction to the abelian case and to consideration of values of Artin L-functions, the central conjecture of [Bur09] precisely recovers Brumer's conjecture. Explicit examples are also studied in [Bur09] which show that, in some cases, Theorem 1.2 is essentially the strongest possible annihilation result.

2. Definition of U_χ and $h(\mu_K, \chi)$

In this section we give the necessary background material to make precise the definitions of U_χ and $h(\mu_K, \chi)$ in Theorem 1.2.

2.1 χ-twists

We largely follow the exposition of [Bur08, § 1]. Fix a finite group G and an irreducible (complex) character χ of G. Let $E = E_\chi$ be a subfield of $\mathbb{C}$ over which χ can be realised that is both Galois

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and of finite degree over $\mathbb{Q}$. We write $\mathcal{O}$ for the ring of algebraic integers in E and set

$$e_\chi := \frac{\chi(1)}{|G|}\sum_{g\in G}\chi(g^{-1})g = \frac{\chi(1)}{|G|}\sum_{g\in G}\overline{\chi}(g)g, \quad \mathrm{pr}_\chi := \frac{|G|}{\chi(1)}e_\chi = \sum_{g\in G}\chi(g^{-1})g = \sum_{g\in G}\overline{\chi}(g)g.$$

Here e_χ is a primitive central idempotent of $E[G]$ and pr_χ is the associated 'projector'.

We choose a maximal $\mathcal{O}$-order $\mathfrak{M}$ in $E[G]$ containing $\mathcal{O}[G]$ and fix an indecomposable idempotent f_χ of $e_\chi\mathfrak{M}$. We define an $\mathcal{O}$-torsion-free right $\mathcal{O}[G]$-module by setting $T_\chi := f_\chi\mathfrak{M}$. (Note that this is slightly different from the definition given in [Bur08, § 1].) The associated right $E[G]$-module $E\otimes_{\mathcal{O}} T_\chi$ has character χ, and T_χ is locally free of rank $\chi(1)$ over $\mathcal{O}$.

For any (left) G-module M we set $M[\chi] := T_\chi\otimes_{\mathbb{Z}} M$, upon which G acts on the left by $t\otimes_{\mathbb{Z}} m\mapsto tg^{-1}\otimes_{\mathbb{Z}} g(m)$ for each $t\in T_\chi, m\in M$ and $g\in G$. For any G-module M and integer i, we write $\widehat{H}^i(G,M)$ for the Tate cohomology in degree i of M with respect to G. We also write M^G and M_G for, respectively, the maximal submodule and maximal quotient module of M upon which G acts trivially. Then we obtain a left exact functor $M\mapsto M^\chi$ and a right exact functor $M\mapsto M_\chi$ from the category of left G-modules to the category of $\mathcal{O}$-modules, by setting $M^\chi := M[\chi]^G$ and $M_\chi := M[\chi]_G = T_\chi\otimes_{\mathbb{Z}[G]} M$. The action of $\mathrm{Norm}_G := \sum_{g\in G} g$ on $M[\chi]$ induces a homomorphism of $\mathcal{O}$-modules $t(M,\chi): M_\chi\to M^\chi$ with kernel $\widehat{H}^{-1}(G,M[\chi])$ and cokernel $\widehat{H}^0(G,M[\chi])$. Thus $t(M,\chi)$ is bijective whenever M, and hence also $M[\chi]$, is a cohomologically trivial G-module.

We shall henceforth take 'module' to mean 'left module' unless explicitly stated otherwise.

2.2 Reducing to the $L=K$ case

Assume the setting and notation of Theorem 1.2 for the rest of this section. In the definitions of U_χ and $h(\mu_K,\chi)$ below, we shall assume that $L=K$. Hence χ is a non-trivial irreducible faithful character of $G=\mathrm{Gal}(K/k)$.

For the general case of $L\neq K$, let ϕ be the character of $\mathrm{Gal}(K/k)$ that inflates to χ. Then ϕ is irreducible and faithful, and we have $E_\chi = E_\phi$. We define $U_\chi := U_\phi$ and $h(\mu_K,\chi) := h(\mu_K,\phi)$.

2.3 Definition of U_χ

We first recall the following construction from [Gre07, § 2]. Let $\mathfrak{p}$ be a finite prime of k and fix a prime $\mathfrak{P}$ of K above $\mathfrak{p}$. We use the standard notation $G_\mathfrak{p}, G_{0,\mathfrak{p}}$ and $\overline{G}_\mathfrak{p} = G_\mathfrak{p}/G_{0,\mathfrak{p}}$ for, respectively, the decomposition group, the inertia group and the residual group of K/k at $\mathfrak{P}$. Choose a lift $F_\mathfrak{p}$ (fixed for the rest of the paper) of the Frobenius element $\mathrm{Fr}_\mathfrak{p}\in\overline{G}_\mathfrak{p}$ to $G_\mathfrak{p}\subset G$. For any subgroup H of G, let $\mathrm{Norm}_H := \sum_{h\in H} h$. We define central idempotents of $\mathbb{Q}[G_\mathfrak{p}]$ as follows:

$$\begin{aligned} e'_\mathfrak{p} &:= |G_{0,\mathfrak{p}}|^{-1}\,\mathrm{Norm}_{G_{0,\mathfrak{p}}}, \quad & e''_\mathfrak{p} &:= 1-e'_\mathfrak{p};\\ \bar{e}_\mathfrak{p} &:= |G_\mathfrak{p}|^{-1}\,\mathrm{Norm}_{G_\mathfrak{p}}, \quad & \bar{\bar{e}}_\mathfrak{p} &:= 1-\bar{e}_\mathfrak{p}. \end{aligned}$$

We define the $\mathbb{Z}[G_\mathfrak{p}]$-modules $U_\mathfrak{p}$ by

$$U_\mathfrak{p} := \langle \mathrm{Norm}_{G_{0,\mathfrak{p}}}, 1-e'_\mathfrak{p}F_\mathfrak{p}^{-1}\rangle_{\mathbb{Z}[G_\mathfrak{p}]}\subset \mathbb{Q}[G_\mathfrak{p}]$$

and note that $U_\mathfrak{p} = \mathbb{Z}[G_\mathfrak{p}]$ if $\mathfrak{p}$ is unramified in K/k.

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Let $\mathrm{nr}_{e_\chi E[G]} : e_\chi E[G] \to E$ be the reduced norm map (see [CR81, § 7D]). More explicitly, this is the determinant map $e_\chi E[G] \cong \mathrm{Mat}_{\chi(1)}(E) \to E$. We define a fractional ideal of $\mathcal{O}$ by setting

$$U_\chi := \prod_{\mathfrak{p} \in \mathcal{S}_{\mathrm{ram}}(K/k)} \mathrm{nr}_{e_\chi E[G]}(e_\chi \mathfrak{M} U_\mathfrak{p})\mathcal{O}.$$

Here we use the following notation: for any finitely generated $\mathbb{Z}[G_\mathfrak{p}]$-submodule $V_\mathfrak{p}$ of $\mathbb{Q}[G_\mathfrak{p}]$, we write $\mathrm{nr}_{e_\chi E[G]}(e_\chi \mathfrak{M} V_\mathfrak{p})\mathcal{O}$ for the $\mathcal{O}$-submodule of E that is generated by the elements $\mathrm{nr}_{e_\chi E[G]}(x)$ as x runs over $e_\chi \mathfrak{M} V_\mathfrak{p}$ and note that this is indeed a fractional ideal of $\mathcal{O}$ since $\mathrm{nr}_{e_\chi E[G]}(e_\chi \mathfrak{M}) = \mathcal{O}$.

Recalling the hypothesis that χ is faithful and non-trivial, it is a straightforward exercise to show that U_χ is the trivial ideal if $G_{0,\mathfrak{p}}$ is normal in G for every $\mathfrak{p} \in \mathcal{S}_{\mathrm{ram}}(K/k)$ (the point is that χ must be non-trivial on $G_{0,\mathfrak{p}}$ and so e_χ annihilates both $\mathrm{Norm}_{G_{0,\mathfrak{p}}}$ and $e'_\mathfrak{p}$). In particular, this is the case when G is abelian or Hamiltonian (i.e. every subgroup of G is normal), or when every $\mathfrak{p} \in \mathcal{S}_{\mathrm{ram}}(K/k)$ is non-split in K/k (i.e. $G = G_\mathfrak{p}$).

2.4 Definition of $h(\mu_K, \chi)$

For any finitely generated $\mathcal{O}$-module M, we let $\mathrm{Fit}_\mathcal{O}(M)$ denote the Fitting ideal of M. We define $h(\mu_K, \chi)$ to be the natural truncated Euler characteristic,

$$h(\mu_K, \chi) := \prod_{i=0}^{i=2} \mathrm{Fit}_\mathcal{O}(H^i(G, \mu_K[\chi]))^{(-1)^i}.$$

Note that if μ_K is cohomologically trivial as a G-module, then $h(\mu_K, \chi) = \mathrm{Fit}_\mathcal{O}(\mu_K[\chi]^G)$.

3. Algebraic K-theory

In this section we summarise some of the necessary background material from algebraic K-theory. Further details can be found in [CR81, CR87], [BB07, § 2] and [Bre04, § 2].

3.1 Relative K-theory

For any integral domain R of characteristic 0, any extension field F of the field of fractions of R and any finite group G, let $K_0(R[G], F[G])$ denote the relative algebraic K-group associated to the ring homomorphism $R[G] \hookrightarrow F[G]$. We write $K_0(R[G])$ for the Grothendieck group of the category of finitely generated projective $R[G]$-modules and $K_1(R[G])$ for the Whitehead group. There is the following long exact sequence of relative K-theory:

$$K_1(R[G]) \longrightarrow K_1(F[G]) \longrightarrow K_0(R[G], F[G]) \longrightarrow K_0(R[G]) \longrightarrow K_0(F[G]). \tag{1}$$

3.2 Reduced norms

Let $\zeta(F[G])^\times$ denote the multiplicative group of the centre of $F[G]$. There exist a reduced norm map $\mathrm{nr}_{F[G]} : (F[G])^\times \to \zeta(F[G])^\times$, with image denoted by $\zeta(F[G])^{\times +}$, and a natural surjective map $(F[G])^\times \to K_1(F[G])$, $x \mapsto (F[G], x_r)$, where x_r denotes right multiplication by x. However, these maps have the same kernel, namely the commutator subgroup $[(F[G])^\times, (F[G])^\times]$, and so we have the commutative diagram

$$\begin{tikzcd}
(F[G])^\times \arrow[r] \arrow[d, "\mathrm{nr}_{F[G]}"'] & K_1(F[G]) \arrow[dl, dashed, "\simeq"', "\overline{\mathrm{nr}}_{F[G]}"] \\
\zeta(F[G])^{\times +} &
\end{tikzcd}$$

where $\overline{\mathrm{nr}}_{F[G]}$ is the induced isomorphism. Note that the inverse map $\overline{\mathrm{nr}}_{F[G]}^{-1} : \zeta(F[G])^{\times +} \to K_1(F[G])$ can be described explicitly by $\mathrm{nr}_{F[G]}(x) \mapsto (F[G], x_r)$. By composing the map $\overline{\mathrm{nr}}_{F[G]}^{-1} : \zeta(F[G])^{\times +} \to K_1(F[G])$ with the boundary map $K_1(F[G]) \to K_0(R[G], F[G])$, we obtain a homomorphism

$$\partial_{R[G],F[G]} : \zeta(F[G])^{\times +} \longrightarrow K_0(R[G], F[G]), \quad \mathrm{nr}_{F[G]}(x) \mapsto (R[G], x_r, R[G]). \tag{2}$$

We note that if F is algebraically closed, then $\mathrm{nr}_{F[G]}$ is surjective, i.e. $\zeta(F[G])^{\times +} = \zeta(F[G])^{\times}$. In any case, we always have $(\zeta(F[G])^{\times})^2 \subseteq \zeta(F[G])^{\times +}$.

3.3 Induction

Let H be a subgroup of G. The functor $M \mapsto R[G] \otimes_{R[H]} M$ from projective $R[H]$-modules to projective $R[G]$-modules and the corresponding functor from $F[H]$-modules to $F[G]$-modules induce induction maps ind_H^G for all K-groups in the exact sequence (1). We also obtain an induction map

$$i_H^G := \overline{\mathrm{nr}}_{F[G]} \circ \mathrm{ind}_H^G \circ \overline{\mathrm{nr}}_{F[H]}^{-1} : \zeta(F[H])^{\times +} \to \zeta(F[G])^{\times +}.$$

Specialising to the case where $R = \mathbb{Z}$ and $F = \mathbb{R}$, we obtain the following commutative diagram.

$$\begin{tikzcd}
(\mathbb{R}[H])^{\times} \arrow[r, "\text{inclusion}"] \arrow[d, "\mathrm{nr}_{\mathbb{R}[H]}"] & (\mathbb{R}[G])^{\times} \arrow[d, "\mathrm{nr}_{\mathbb{R}[G]}"'] \\
\zeta(\mathbb{R}[H])^{\times +} \arrow[r, "i_H^G"] \arrow[d, "\overline{\mathrm{nr}}_{\mathbb{R}[H]}^{-1}", "\simeq"'] \arrow[dd, bend right=60, "\partial_{\mathbb{Z}[H],\mathbb{R}[H]}"'] & \zeta(\mathbb{R}[G])^{\times +} \arrow[d, "\overline{\mathrm{nr}}_{\mathbb{R}[G]}^{-1}"', "\simeq"] \arrow[dd, bend left=60, "\partial_{\mathbb{Z}[G],\mathbb{R}[G]}"] \\
K_1(\mathbb{R}[H]) \arrow[r, "\mathrm{ind}_H^G"] \arrow[d] & K_1(\mathbb{R}[G]) \arrow[d] \\
K_0(\mathbb{Z}[H], \mathbb{R}[H]) \arrow[r, "\mathrm{ind}_H^G"] & K_0(\mathbb{Z}[G], \mathbb{R}[G])
\end{tikzcd} \tag{3}$$

3.4 The extended boundary homomorphism

We recall some properties of the 'extended boundary homomorphism'

$$\hat{\partial}_{\mathbb{Z}[G],\mathbb{R}[G]} : \zeta(\mathbb{R}[G])^{\times} \to K_0(\mathbb{Z}[G], \mathbb{R}[G]),$$

which was first introduced in [BF01, Lemma 9] (a more conceptual description was given in [BB07, Lemma 2.2]). The restriction of $\hat{\partial}_{\mathbb{Z}[G],\mathbb{R}[G]}$ to $\zeta(\mathbb{R}[G])^{\times +}$ is $\partial_{\mathbb{Z}[G],\mathbb{R}[G]}$.

Lemma 3.1. *Letting α and β denote the natural inclusions, the diagram*

$$\begin{tikzcd}
\zeta(\mathbb{R}[G])^{\times} \arrow[r, "\hat{\partial}_{\mathbb{Z}[G],\mathbb{R}[G]}"] \arrow[d, "\alpha"] & K_0(\mathbb{Z}[G], \mathbb{R}[G]) \arrow[d, "\beta"] \\
\zeta(\mathbb{C}[G])^{\times} \arrow[r, "\partial_{\mathbb{Z}[G],\mathbb{C}[G]}"] & K_0(\mathbb{Z}[G], \mathbb{C}[G])
\end{tikzcd}$$

commutes up to elements of order two. In other words, given $x \in \zeta(\mathbb{R}[G])^{\times}$, we have

$$\beta(\hat{\partial}_{\mathbb{Z}[G],\mathbb{R}[G]}(x)) = \partial_{\mathbb{Z}[G],\mathbb{C}[G]}(\alpha(x)) \cdot u$$

for some $u \in K_0(\mathbb{Z}[G], \mathbb{C}[G])$ of order at most two.

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Proof. Reduced norms commute with extension of scalars, and squares in $\zeta(\mathbb{R}[G])^\times$ are reduced norms. Thus, for $x \in \zeta(\mathbb{R}[G])^\times$ we have

$$\beta(\hat{\partial}_{\mathbb{Z}[G],\mathbb{R}[G]}(x))^2 = \beta(\hat{\partial}_{\mathbb{Z}[G],\mathbb{R}[G]}(x^2)) = \beta(\partial_{\mathbb{Z}[G],\mathbb{R}[G]}(x^2)) = \partial_{\mathbb{Z}[G],\mathbb{C}[G]}(\alpha(x^2)) = \partial_{\mathbb{Z}[G],\mathbb{C}[G]}(\alpha(x))^2,$$

from which the desired result follows immediately. □

Regarding G as fixed, we henceforth abbreviate $\hat{\partial}_{\mathbb{Z}[G],\mathbb{R}[G]}$ and $\partial_{\mathbb{Z}[G],\mathbb{R}[G]}$ to $\hat{\partial}$ and ∂, respectively.

4. Centres of complex group algebras

Let G be a finite group and let $\mathrm{Irr}(G)$ be the set of irreducible complex characters of G. Recall that there is a canonical isomorphism $\zeta(\mathbb{C}[G]) = \prod_{\chi\in\mathrm{Irr}(G)} \mathbb{C}$. We shall henceforth use this identification without further mention.

4.1 Explicit induction

Let H be a subgroup of G, and define a map

$$i_H^G : \zeta(\mathbb{C}[H]) \to \zeta(\mathbb{C}[G]), \quad (\alpha_\psi)_{\psi\in\mathrm{Irr}(H)} \mapsto \Bigg(\prod_{\psi\in\mathrm{Irr}(H)} \alpha_\psi^{\langle\chi|_H,\psi\rangle_H}\Bigg)_{\chi\in\mathrm{Irr}(G)}, \tag{4}$$

where $\langle\chi|_H,\psi\rangle_H$ denotes the usual inner product of characters of H. The restriction of this map to $\zeta(\mathbb{C}[H])^\times$ is the same as the map $i_H^G : \zeta(\mathbb{C}[H])^\times \to \zeta(\mathbb{C}[G])^\times$ defined in § 3.3 (with $F = \mathbb{C}$) so that using the same name for these maps is justified. This map restricts further to $i_H^G : \zeta(\mathbb{R}[H])^{\times+} \to \zeta(\mathbb{R}[G])^{\times+}$ (as defined in § 3.3 with $F = \mathbb{R}$).

4.2 The involution

We write $\alpha \mapsto \alpha^\#$ for the involution of $\zeta(\mathbb{C}[G])$ induced by the $\mathbb{C}$-linear anti-involution of $\mathbb{C}[G]$ that sends each element of G to its inverse. If $\alpha = (\alpha_\chi)_{\chi\in\mathrm{Irr}(G)}$, then $\alpha^\# = (\alpha_{\bar{\chi}})_{\chi\in\mathrm{Irr}(G)}$. Furthermore, # restricts to an involution of $\zeta(\mathbb{R}[G])^{\times+}$ which is compatible with induction; that is, if $\alpha \in \zeta(\mathbb{R}[H])^{\times+}$, then $i_H^G(x^\#) = i_H^G(x)^\#$.

4.3 Meromorphic $\zeta(\mathbb{C}[G])$-valued functions

A meromorphic $\zeta(\mathbb{C}[G])$-valued function is a function of a complex variable s of the form $s \mapsto g(s) = (g(s,\chi))_{\chi\in\mathrm{Irr}(G)}$ where each function $s \mapsto g(s,\chi)$ is meromorphic. If $r(\chi)$ denotes the order of vanishing of $g(s,\chi)$ at $s = 0$, then we set $g^*(0,\chi) := \lim_{s\to 0} s^{-r(\chi)} g(s,\chi)$ and $g^*(0) := (g^*(0,\chi))_{\chi\in\mathrm{Irr}(G)} \in \zeta(\mathbb{C}[G])^\times$.

5. *L*-functions

Let K/k be a finite Galois extension of number fields with Galois group G, and let $\mathcal{S}$ be a finite set of places of k containing the infinite places $\mathcal{S}_\infty$.

5.1 Artin *L*-functions

Let $\mathfrak{p}$ be a finite prime of k. Let ψ be a complex character of $G_\mathfrak{p}$ and choose a $\mathbb{C}[G_\mathfrak{p}]$-module V_ψ with character ψ. Recalling the notation from § 2.3, we define

$$L_{K_\mathfrak{P}/k_\mathfrak{p}}(s,\psi) := \det_{\mathbb{C}}(1 - F_\mathfrak{p}(\mathrm{N}\mathfrak{p})^{-s} | V_\psi^{G_{0,\mathfrak{p}}})^{-1}, \tag{5}$$

where $\mathrm{N}\mathfrak{p}$ is the cardinality of the residue field of $\mathfrak{p}$. Note that $L_{K_{\mathfrak{P}}/k_{\mathfrak{p}}}(s,\psi)$ depends only on $\mathfrak{p}$ and not on the choice of $\mathfrak{P}$. Furthermore, it is easy to see that

$$L_{K_{\mathfrak{P}}/k_{\mathfrak{p}}}(s,\psi+\psi')=L_{K_{\mathfrak{P}}/k_{\mathfrak{p}}}(s,\psi)L_{K_{\mathfrak{P}}/k_{\mathfrak{p}}}(s,\psi')$$

for any two characters ψ and ψ' of $G_{\mathfrak{p}}$; thus the definition extends to all virtual characters of $G_{\mathfrak{p}}$.

Now let $\chi\in\mathrm{Irr}(G)$ and, for each $\mathfrak{p}$, let $\chi_{\mathfrak{p}}$ denote the restriction of χ to $G_{\mathfrak{p}}$. The Artin L-function attached to $\mathcal{S}$ and χ is defined as an infinite product

$$L_{K/k,\mathcal{S}}(s,\chi):=\prod_{\mathfrak{p}\notin\mathcal{S}}L_{K_{\mathfrak{P}}/k_{\mathfrak{p}}}(s,\chi_{\mathfrak{p}}),\tag{6}$$

which converges for $\mathrm{Re}(s)>1$ and can be extended to the whole complex plane by meromorphic continuation.

5.2 Equivariant L-functions

We define meromorphic $\zeta(\mathbb{C}[G_{\mathfrak{p}}])$-valued functions by

$$L_{K_{\mathfrak{P}}/k_{\mathfrak{p}}}(s):=(L_{K_{\mathfrak{P}}/k_{\mathfrak{p}}}(s,\psi))_{\psi\in\mathrm{Irr}(G_{\mathfrak{p}})}$$

and define the equivariant Artin L-function to be the meromorphic $\zeta(\mathbb{C}[G])$-valued function

$$L_{K/k,\mathcal{S}}(s):=(L_{K/k,\mathcal{S}}(s,\chi))_{\chi\in\mathrm{Irr}(G)}.$$

From (4) and (6) it is straightforward to check that for $\mathrm{Re}(s)>1$ we have

$$L_{K/k,\mathcal{S}}(s)=\prod_{\mathfrak{p}\notin\mathcal{S}}i_{G_{\mathfrak{p}}}^{G}(L_{K_{\mathfrak{P}}/k_{\mathfrak{p}}}(s)).\tag{7}$$

Note that $L^*_{K_{\mathfrak{P}}/k_{\mathfrak{p}}}(0)\in\zeta(\mathbb{R}[G_{\mathfrak{p}}])^{\times+}$ and $L^*_{K/k,\mathcal{S}}(0)\in\zeta(\mathbb{R}[G])^{\times}$ (see [BB07, Lemma 2.7]). We henceforth abbreviate $L_{K/k,\mathcal{S}}(s)$ to $L_{\mathcal{S}}(s)$ and $L_{\mathcal{S}_\infty}(s)$ to $L(s)$.

6. Tate sequences, refined Euler characteristics and the ETNC

Let K/k be a finite Galois extension of number fields with Galois group G, and take S to be a finite G-stable set of places of K containing the set of archimedean places S_∞. We let S_{ram} denote the places of K ramified in K/k. The set of k-places below places in S (respectively, in S_∞ or S_{ram}) will be denoted by $\mathcal{S}$ (respectively, $\mathcal{S}_\infty$ or $\mathcal{S}_{\mathrm{ram}}$). (Note that this is different from the notation used in [Gre07].) Let $E_S=\mathcal{O}_{K,S}^{\times}$ and let ΔS be the kernel of the augmentation map $\mathbb{Z}S\to\mathbb{Z}$. We shall henceforth use the abbreviations 'c.t.' for 'cohomologically trivial' and 'f.g.' for 'finitely generated'. Note that as G is finite, 'G-c.t.' is equivalent to 'of projective dimension at most one over $\mathbb{Z}[G]$'.

Now let S' denote a finite G-stable set of places of K that is 'large', i.e. $S_\infty\cup S_{\mathrm{ram}}\subseteq S'$ and $\mathrm{cl}_{K,S'}=0$. Tate defined a canonical class $\tau=\tau_{S'}\in\mathrm{Ext}^2_{\mathbb{Z}[G]}(\Delta S',E_{S'})$; see [Tat66] or [Tat84, ch. II]. The fundamental properties of τ ensure the existence of so-called Tate sequences, i.e. four-term exact sequences of f.g. $\mathbb{Z}[G]$-modules

$$0\longrightarrow E_{S'}\longrightarrow A\longrightarrow B\longrightarrow\Delta S'\longrightarrow 0\tag{8}$$

that represent τ, with A G-c.t. and B projective. In [RW96], Ritter and Weiss constructed a Tate sequence for S 'small':

$$0\longrightarrow E_S\longrightarrow A\longrightarrow B\longrightarrow\nabla\longrightarrow 0,$$

where ∇ is given by a short exact sequence $0\to\mathrm{cl}_{K,S}\to\nabla\to\overline{\nabla}\to 0$.

In order to take full advantage of these sequences, we shall use refined Euler characteristics, which we now briefly review in the special case of interest to us. For a full account, we refer the reader to [Bur04] (also see [Bur01, § 1.2]). Following [Gre07, § 3], we adopt a slight change from the usual convention (this results in merely a sign change). A 'metrised' complex over $\mathbb{Z}[G]$ consists of a complex in degrees zero and one,

$$A \longrightarrow B,$$

together with an $\mathbb{R}[G]$-isomorphism

$$\varphi : \mathbb{R} \otimes U \longrightarrow \mathbb{R} \otimes V, \tag{9}$$

with both A and B being f.g. and c.t. over G and where U (respectively, V) is the kernel (respectively, cokernel) of $A \to B$. To every metrised complex $E = (A \to B, \varphi)$ we can associate a refined Euler characteristic $\chi_{\mathrm{ref}}(E) \in K_0(\mathbb{Z}[G], \mathbb{R}[G])$ as follows. We can write down a four-term exact sequence

$$0 \longrightarrow U \longrightarrow A \longrightarrow B \longrightarrow V \longrightarrow 0, \tag{10}$$

which gives rise to the tautological exact sequences

$$\begin{gathered} 0 \longrightarrow \ker(\mathbb{R} \otimes B \to \mathbb{R} \otimes V) \longrightarrow \mathbb{R} \otimes B \longrightarrow \mathbb{R} \otimes V \longrightarrow 0, \\ 0 \longrightarrow \mathbb{R} \otimes U \longrightarrow \mathbb{R} \otimes A \longrightarrow \mathrm{im}(\mathbb{R} \otimes A \to \mathbb{R} \otimes B) \longrightarrow 0. \end{gathered}$$

We choose splittings for these sequences and obtain an isomorphism $\tilde{\varphi} : \mathbb{R} \otimes A \to \mathbb{R} \otimes B$,

$$\begin{aligned} \mathbb{R} \otimes A &\cong \mathrm{im}(\mathbb{R} \otimes A \to \mathbb{R} \otimes B) \oplus (\mathbb{R} \otimes U) = \ker(\mathbb{R} \otimes B \to \mathbb{R} \otimes V) \oplus (\mathbb{R} \otimes U) \\ &\cong \ker(\mathbb{R} \otimes B \to \mathbb{R} \otimes V) \oplus (\mathbb{R} \otimes V) \\ &\cong \mathbb{R} \otimes B, \end{aligned}$$

where the first and third maps are obtained from the chosen splittings and the second map is induced by φ. (We refer to $\tilde{\varphi}$ as a 'transpose' of φ.) If A and B are both $\mathbb{Z}[G]$-projective, we define $\chi_{\mathrm{ref}}(A \to B, \varphi) = (A, \tilde{\varphi}, B) \in K_0(\mathbb{Z}[G], \mathbb{R}[G])$. This definition can be extended to the more general case where A and B are c.t. over G. In all cases, $\chi_{\mathrm{ref}}(A \to B, \varphi)$ can be shown to be independent of the choice of splittings.

We note several properties of χ_{ref}. First, $\chi_{\mathrm{ref}}(A \to B, \varphi)$ remains unchanged if φ is composed with an automorphism of determinant 1 on either side; see [Bur01, Proposition 1.2.1(ii)]. Second, if the metrisation (9) is given, then the class of the exact sequence (10) in $\mathrm{Ext}^2_{\mathbb{Z}[G]}(V, U)$ uniquely determines $\chi_{\mathrm{ref}}(A \to B, \varphi)$; see [Bur01, Proposition 1.2.2 and Remark 1.2.3]. Finally, it is straightforward to show that χ_{ref} is compatible with induction, i.e. if H is a subgroup of G and $A \to B$ is an appropriate complex of $\mathbb{Z}[H]$-modules with metrisation φ, then

$$\mathrm{ind}_H^G(\chi_{\mathrm{ref}}(A \to B, \varphi)) = \chi_{\mathrm{ref}}(\mathrm{ind}_H^G A \to \mathrm{ind}_H^G B, \mathrm{ind}_H^G \varphi). \tag{11}$$

Now let E be the complex formed by the middle two terms of the Tate sequence (8), and metrise it by setting $U = E_{S'}$, $V = \Delta S'$ and $\varphi^{-1} : \mathbb{R}E_{S'} \to \mathbb{R}\Delta S'$ to be the negative of the usual Dirichlet map, i.e. $\varphi^{-1}(u) = -\sum_{v \in S'} \log |u|_v \cdot v$. The equivariant Tamagawa number is defined to be

$$T\Omega(K/k, 0) := \psi_G^*(\hat{\partial}(L^*_{\mathcal{S}'}(0)^{\#}) - \chi_{\mathrm{ref}}(E)) \in K_0(\mathbb{Z}[G], \mathbb{R}[G]),$$

where ψ_G^* is a certain involution of $K_0(\mathbb{Z}[G], \mathbb{R}[G])$ which can be ignored for our purposes. (Note that we have $-\chi_{\mathrm{ref}}(E)$ rather than $+\chi_{\mathrm{ref}}(E)$ here because, as mentioned above, our definition of χ_{ref} is slightly different from the usual convention, resulting in a sign change.) The equivariant

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Tamagawa number conjecture (ETNC) in this context (i.e. for the motive $h^0(K)$ with coefficients in $\mathbb{Z}[G]$) simply states that $T\Omega(K/k,0)$ is zero (see [Bur01, BF01]). One can also re-interpret other well-known conjectures using this framework. For example, Stark's main conjecture is equivalent to the statement that $T\Omega(K/k,0)$ belongs to $K_0(\mathbb{Z}[G],\mathbb{Q})$. If $\mathcal{M}$ is a maximal $\mathbb{Z}$-order in $\mathbb{Q}[G]$ containing $\mathbb{Z}[G]$, then the strong Stark conjecture can be interpreted as

$$T\Omega(K/k,0)\in K_0(\mathbb{Z}[G],\mathbb{Q}[G])_{\mathrm{tors}}=\ker(K_0(\mathbb{Z}[G],\mathbb{Q}[G])\to K_0(\mathcal{M},\mathbb{Q}[G])),$$

i.e. as saying that 'the ETNC holds modulo torsion' (see [Bur01, § 2.2]).

7. Metrised commutative diagrams

We briefly review Greither's construction of certain metrised commutative diagrams which we shall use in a crucial way; we leave the reader to consult [Gre07] for further details. The principal tool used in this construction is the aforementioned 'Tate sequence for small S' of Ritter and Weiss [RW96]. Although the only application in [Gre07] is to the case where G is abelian, the explicit construction given there is in fact valid for the general case, once one makes some minor modifications to the 'core diagram' of [Gre07, § 5] as described in the proof of [Nic09, Proposition 4.4].

We adopt the setup and notation of § 6 and add the further hypotheses that k is totally real, K is a CM field and S' is 'larger' in the sense of [RW96], i.e. $S_\infty\cup S_{\mathrm{ram}}\subseteq S'$, $\mathrm{cl}_{K,S'}=0$ and $G=\bigcup_{\mathfrak{p}\in S'}G_{\mathfrak{p}}$. We write j for the unique complex conjugation in G and define $R:=\mathbb{Z}[G][1/2]/(1+j)$. For every G-module M we let $M^-:=R\otimes_{\mathbb{Z}[G]}M$ (this notation, which includes inversion of 2, is used in [Gre07] and is non-standard but practical). Note that the construction of the refined Euler characteristic also works for complexes over R.

Let C be the free $\mathbb{Z}[G]$-module with basis elements $x_{\mathfrak{p}}$, where $\mathfrak{p}$ runs over $\mathcal{S}'\backslash\mathcal{S}_\infty$. Using the Tate sequences for S 'larger' and for S 'small', Greither constructed the following diagrams.

$$\begin{tikzcd}
E_{S'} \arrow[r] & A \arrow[r] & B_{S'} \arrow[r] & \Delta S' \\
C\oplus E_{S_\infty} \arrow[r] \arrow[u] & C\oplus A \arrow[r] \arrow[u] & B_{S_\infty} \arrow[r] \arrow[u] & \nabla \arrow[u] \\
Z' \arrow[r] \arrow[u] & C \arrow[r] \arrow[u] & C \arrow[r] \arrow[u] & Z'' \arrow[u]
\end{tikzcd} \tag{D1}$$

$$\begin{tikzcd}
E_{S_\infty} \arrow[r] & A \arrow[r] & \tilde{B} \arrow[r] & \nabla/\delta(C) \\
C\oplus E_{S_\infty} \arrow[r] \arrow[u] & C\oplus A \arrow[r] \arrow[u] & B_{S_\infty} \arrow[r] \arrow[u] & \nabla \arrow[u] \\
C \arrow[r, "\mathrm{id}"] \arrow[u] & C \arrow[r, "0"] \arrow[u] & C \arrow[r, "\mathrm{id}"] \arrow[u] & C \arrow[u, "\delta"]
\end{tikzcd} \tag{D2}$$

In the original construction, S was used in place of S_∞, and only later did the author specialise to the case of $S=S_\infty$. Note that the middle rows of (D1) and (D2) are identical and that the middle map of the bottom row of (D1) is, in general, far from the identity. The 'minus part' of each diagram is denoted by $(\mathrm{D1})^-$ or $(\mathrm{D2})^-$, as appropriate.

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In [Gre07, § 7] each row is given a metrisation, and we label the corresponding refined Euler characteristics as follows:

$$\begin{aligned}
X_{S'} &= \chi_{\mathrm{ref}}(\text{top row of (D1)}),\\
X_1 &= \chi_{\mathrm{ref}}(\text{middle row of (D1)}) = \chi_{\mathrm{ref}}(\text{middle row of (D2)}),\\
X_C &= \chi_{\mathrm{ref}}(\text{bottom row of (D1)}),\\
X_\infty^- &= \chi_{\mathrm{ref}}(\text{top row of (D2)}^-),\\
X_2 &= \chi_{\mathrm{ref}}(\text{bottom row of (D2)}).
\end{aligned}$$

(Note that there is a typo in the definition of X_∞^- in [Gre07, top of p. 1418].) The metrisations are chosen to be 'compatible' within the minus part of each diagram (see [Gre07, Lemmas 7.3 and 7.4]) so that we have

$$X_1^- = X_{S'}^- + X_C^- \quad \text{and} \quad X_1^- = X_\infty^- + X_2^- \quad \text{in } K_0(R, \mathbb{R}[G]^-),$$

where we denote the natural map $K_0(\mathbb{Z}[G], \mathbb{R}[G]) \to K_0(R, \mathbb{R}[G]^-)$ by a minus exponent. Putting these two equations together gives the following result.

Proposition 7.1. *We have $X_\infty^- = X_{S'}^- + X_C^- - X_2^-$ in $K_0(R, \mathbb{R}[G]^-)$.*

8. Computing refined Euler characteristics

We compute the refined Euler characteristics of the metrised commutative diagrams of § 7 (i.e. of [Gre07, § 3]) in the non-abelian case. Recall the definitions of $e'_{\mathfrak{p}}, e''_{\mathfrak{p}}$ and $\bar{e}_{\mathfrak{p}}, \bar{\bar{e}}_{\mathfrak{p}}$ from § 2.3, and note that even though $\bar{\bar{e}}_{\mathfrak{p}} e''_{\mathfrak{p}} = e''_{\mathfrak{p}}$, we sometimes retain $\bar{\bar{e}}_{\mathfrak{p}}$ for clarity.

Recall that h is a positive integer multiple of $|\mathrm{cl}_K|$ (see [Gre07, p. 1411]).

Lemma 8.1. *Let $v_{\mathfrak{p}} = h|G_{\mathfrak{p}}| \cdot \bar{e}_{\mathfrak{p}} + \bar{\bar{e}}_{\mathfrak{p}} \in \mathbb{Q}[G_{\mathfrak{p}}]$. Then in $K_0(\mathbb{Z}[G], \mathbb{R}[G])$ we have*

$$X_C = \sum_{\mathfrak{p} \in \mathcal{S}' \setminus \mathcal{S}_\infty} \partial(\mathrm{nr}_{\mathbb{R}[G]}(v_{\mathfrak{p}})).$$

Proof. This was proved in the abelian case in [Gre07, Lemma 7.6]. The key point is that the 'transposed isomorphism' at the local level is multiplication by $v_{\mathfrak{p}}$ (note that this is central in $\mathbb{R}[G_{\mathfrak{p}}]$), and this part of the proof holds without change in the non-abelian case. Hence

$$\begin{aligned}
X_C &= \sum_{\mathfrak{p} \in \mathcal{S}' \setminus \mathcal{S}_\infty} \mathrm{ind}_{G_{\mathfrak{p}}}^G((\mathbb{Z}[G_{\mathfrak{p}}], (v_{\mathfrak{p}})_r, \mathbb{Z}[G_{\mathfrak{p}}])) = \sum_{\mathfrak{p} \in \mathcal{S}' \setminus \mathcal{S}_\infty} (\mathbb{Z}[G], (v_{\mathfrak{p}})_r, \mathbb{Z}[G])\\
&= \sum_{\mathfrak{p} \in \mathcal{S}' \setminus \mathcal{S}_\infty} \partial(\mathrm{nr}_{\mathbb{R}[G]}(v_{\mathfrak{p}})),
\end{aligned}$$

where the equalities are due, respectively, to the compatibility of χ_{ref} with induction (i.e. equation (11) with $H = G_{\mathfrak{p}}$), the commutativity of diagram (3) (again with $H = G_{\mathfrak{p}}$) and the explicit formula (2) (with $R = \mathbb{Z}$ and $F = \mathbb{R}$). □

Recall that $h_{\mathfrak{p}} = g_{\mathfrak{p}} \cdot e'_{\mathfrak{p}} + e''_{\mathfrak{p}}$ where $g_{\mathfrak{p}} = |G_{0,\mathfrak{p}}| + 1 - F_{\mathfrak{p}}^{-1}$ (see [Gre07, p. 1420]).

Lemma 8.2. *Let*

$$t_{\mathfrak{p}} = h \log \mathrm{N}\mathfrak{P} \cdot \bar{e}_{\mathfrak{p}} + \frac{1 - F_{\mathfrak{p}}^{-1}}{h_{\mathfrak{p}}} \cdot \bar{\bar{e}}_{\mathfrak{p}} e'_{\mathfrak{p}} + \bar{\bar{e}}_{\mathfrak{p}} e''_{\mathfrak{p}}.$$

Then in $K_0(R, \mathbb{R}[G]^-)$ *we have*

$$X_2^- = \sum_{\mathfrak{p} \in \mathcal{S}' \setminus \mathcal{S}_\infty} \partial(\mathrm{nr}_{\mathbb{R}[G]}(t_\mathfrak{p}))^-.$$

Proof. Recall that the bottom row of (D2) is metrised with the map $\psi : \mathbb{R}C \to \mathbb{R}C$ (see [Gre07, p. 1417]) and that X_2 is the associated refined Euler characteristic (see § 7). Because of the zero map in the middle of the row, the transpose of ψ is just ψ itself. In [Gre07, Lemma 8.6] it was shown that in the minus part, ψ is the direct sum of maps induced from endomorphisms $\psi_\mathfrak{p}$ of $\mathbb{R}[G_\mathfrak{p}] \cdot x_\mathfrak{p}$ given by multiplication by $t_\mathfrak{p}$ (note that this is central in $\mathbb{R}[G_\mathfrak{p}]$), and this holds without change in the non-abelian case. The result then follows in the same way as in the proof of Lemma 8.1 above, but with $v_\mathfrak{p}$ replaced by $t_\mathfrak{p}$. □

Definition 8.3. Fix a finite prime $\mathfrak{p}$ of k. Let $\psi \in \mathrm{Irr}(G_\mathfrak{p})$ and let e_ψ be the primitive central idempotent of $\mathbb{C}[G_\mathfrak{p}]$ attached to ψ. In the spirit of [Gre07, p. 1421], we say that:

(i) $\psi \in T_1(\mathfrak{p})$ if ψ is trivial, i.e. $e_\psi = \bar{e}_\mathfrak{p}$;

(ii) $\psi \in T_2(\mathfrak{p})$ if ψ is non-trivial but trivial on $G_{0,\mathfrak{p}}$, i.e. $e_\psi \bar{e}_\mathfrak{p} = 0$ but $e_\psi e'_\mathfrak{p} = e_\psi$;

(iii) $\psi \in T_3(\mathfrak{p})$ if ψ is non-trivial on $G_{0,\mathfrak{p}}$, i.e. $e_\psi e'_\mathfrak{p} = 0$.

This division into types corresponds to the decomposition of 1 into orthogonal idempotents,

$$1 = \bar{e}_\mathfrak{p} + \bar{\bar{e}}_\mathfrak{p} e'_\mathfrak{p} + \bar{\bar{e}}_\mathfrak{p} e''_\mathfrak{p}$$

in $\mathbb{Q}[G_\mathfrak{p}] \subseteq \mathbb{C}[G_\mathfrak{p}]$, where $\psi \in T_i(\mathfrak{p})$ corresponds to e_ψ sending the ith of the right-hand summands to e_ψ and the other two to 0, for $i = 1, 2, 3$. Note that if $\psi \in T_2(\mathfrak{p})$, then ψ factors through the cyclic group $\overline{G}_\mathfrak{p} = G_\mathfrak{p}/G_{0,\mathfrak{p}}$ and so ψ is linear.

Lemma 8.4. *Fix a prime* $\mathfrak{p} \in \mathcal{S}' \setminus \mathcal{S}_\infty$ *and let* $\psi \in \mathrm{Irr}(G_\mathfrak{p})$. *We have*

$$e_\psi(v_\mathfrak{p} t_\mathfrak{p}^{-1} h_\mathfrak{p}^{-1}) = \begin{cases} (e_\psi(\log \mathrm{N}\mathfrak{p}))^{-1} & \text{if } \psi \in T_1(\mathfrak{p}); \\ (e_\psi(1 - F_\mathfrak{p}^{-1}))^{-1} & \text{if } \psi \in T_2(\mathfrak{p}); \\ e_\psi & \text{if } \psi \in T_3(\mathfrak{p}). \end{cases}$$

Proof. Suppose $\psi \in T_1(\mathfrak{p})$. Then $e_\psi v_\mathfrak{p} = e_\psi h |G_\mathfrak{p}|$, $e_\psi t_\mathfrak{p} = e_\psi h \log \mathrm{N}\mathfrak{P}$ and $e_\psi h_\mathfrak{p} = e_\psi |G_{0,\mathfrak{p}}|$. Hence

$$e_\psi(v_\mathfrak{p} t_\mathfrak{p}^{-1} h_\mathfrak{p}^{-1}) = e_\psi [G_\mathfrak{p} : G_{0,\mathfrak{p}}](\log \mathrm{N}\mathfrak{P})^{-1} = (e_\psi(\log \mathrm{N}\mathfrak{p}))^{-1}.$$

Suppose $\psi \in T_2(\mathfrak{p})$. Then $e_\psi v_\mathfrak{p} = e_\psi$ and $e_\psi t_\mathfrak{p} = e_\psi h_\mathfrak{p}^{-1}(1 - F_\mathfrak{p}^{-1})$. Hence

$$e_\psi(v_\mathfrak{p} t_\mathfrak{p}^{-1} h_\mathfrak{p}^{-1}) = (e_\psi(1 - F_\mathfrak{p}^{-1}))^{-1}.$$

If $\psi \in T_3(\mathfrak{p})$, then $e_\psi v_\mathfrak{p} = e_\psi t_\mathfrak{p} = e_\psi h_\mathfrak{p} = e_\psi$ and so $e_\psi(v_\mathfrak{p} t_\mathfrak{p}^{-1} h_\mathfrak{p}^{-1}) = e_\psi$. □

Lemma 8.5. *We have*

$$L^*_{\mathcal{S}'}(0)^\# \mathrm{nr}_{\mathbb{R}[G]}\bigg(\prod_{\mathfrak{p} \in \mathcal{S}' \setminus \mathcal{S}_\infty} v_\mathfrak{p} t_\mathfrak{p}^{-1} h_\mathfrak{p}^{-1}\bigg) = L^*(0)^\# \quad \text{in } \zeta(\mathbb{R}[G])^\times.$$

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Proof. First, recall the definitions from §§ 4 and 5. From (7), we have

$$L^*(0)^{\#}(L^*_{\mathcal{S}'}(0)^{\#})^{-1} = (L^*(0)L^*_{\mathcal{S}'}(0)^{-1})^{\#} = \prod_{\mathfrak{p}\in\mathcal{S}'\setminus\mathcal{S}_\infty} i^G_{G_\mathfrak{p}}(L^*_{K_\mathfrak{P}/k_\mathfrak{p}}(0))^{\#} \quad \text{in } \zeta(\mathbb{R}[G])^\times.$$

Since $\#$ is compatible with induction, we are reduced to showing that

$$\prod_{\mathfrak{p}\in\mathcal{S}'\setminus\mathcal{S}_\infty} \mathrm{nr}_{\mathbb{R}[G]}(v_\mathfrak{p} t_\mathfrak{p}^{-1} h_\mathfrak{p}^{-1}) = \prod_{\mathfrak{p}\in\mathcal{S}'\setminus\mathcal{S}_\infty} i^G_{G_\mathfrak{p}}(L^*_{K_\mathfrak{P}/k_\mathfrak{p}}(0)^{\#}) \quad \text{in } \zeta(\mathbb{R}[G])^{\times+}.$$

Fix $\mathfrak{p}\in\mathcal{S}'\setminus\mathcal{S}_\infty$. We have

$$\mathrm{nr}_{\mathbb{R}[G]}(v_\mathfrak{p} t_\mathfrak{p}^{-1} h_\mathfrak{p}) = i^G_{G_\mathfrak{p}}(\mathrm{nr}_{\mathbb{R}[G_\mathfrak{p}]}(v_\mathfrak{p} t_\mathfrak{p}^{-1} h_\mathfrak{p}))$$

from the commutative diagram (3) (with $H = G_\mathfrak{p}$). Moreover, Lemma 8.4 shows that

$$\mathrm{nr}_{\mathbb{R}[G_\mathfrak{p}]}(v_\mathfrak{p} t_\mathfrak{p}^{-1} h_\mathfrak{p}^{-1}) = v_\mathfrak{p} t_\mathfrak{p}^{-1} h_\mathfrak{p}^{-1},$$

since ψ is linear if $\psi \in T_1(\mathfrak{p}) \cup T_2(\mathfrak{p})$. Therefore we are further reduced to verifying that

$$v_\mathfrak{p} t_\mathfrak{p}^{-1} h_\mathfrak{p}^{-1} = L^*_{K_\mathfrak{P}/k_\mathfrak{p}}(0)^{\#} \quad \text{in } \zeta(\mathbb{R}[G_\mathfrak{p}])^{\times+}.$$

However, this holds because for each $\psi \in \mathrm{Irr}(G_\mathfrak{p})$ we have

$$e_\psi L^*_{K_\mathfrak{P}/k_\mathfrak{p}}(0)^{\#} = e_\psi L^*_{K_\mathfrak{P}/k_\mathfrak{p}}(\bar{\psi}, 0) = e_\psi(v_\mathfrak{p} t_\mathfrak{p}^{-1} h_\mathfrak{p}^{-1}),$$

where the second equality follows from Lemma 8.4 and a direct computation using the definition of local L-function (5). □

Definition 8.6. We say that a character χ of G is *odd* if j (the unique complex conjugation in G) acts as -1 on a $\mathbb{C}[G]$-module V_χ with character χ or, equivalently, $e_\chi e_- = e_\chi$ in $\mathbb{C}[G]$ where $e_- := (1-j)/2$.

Proposition 8.7. *Assume that ETNC holds for the motive $h^0(K)$ with coefficients in R. Let $h_{\mathrm{glob}} := \prod_{\mathfrak{p}\in\mathcal{S}'\setminus\mathcal{S}_\infty} h_\mathfrak{p}$ (as in [Gre07, § 8]). Then $\hat{\partial}(L(0)^{\#}\mathrm{nr}_{\mathbb{R}[G]}(h_{\mathrm{glob}}))^- = X_\infty^-$ in $K_0(R, \mathbb{R}[G]^-)$.*

Proof. The ETNC for $h^0(K)$ with coefficients in $\mathbb{Z}[G]$ gives $\hat{\partial}(L^*_{S'}(0)^{\#}) = X_{S'}$ (recall the exposition in § 6 and note that $X_{S'} = \chi_{\mathrm{ref}}(E)$). Hence the ETNC for $h^0(K)$ with coefficients in R gives $\hat{\partial}(L^*_{S'}(0)^{\#})^- = X^-_{S'}$. Let

$$f := L^*_{S'}(0)^{\#} \prod_{\mathfrak{p}\in\mathcal{S}'\setminus\mathcal{S}_\infty} \mathrm{nr}_{\mathbb{R}[G]}(v_\mathfrak{p} t_\mathfrak{p}^{-1}) \in \zeta(\mathbb{R}[G])^\times.$$

Then, combining Lemmas 8.1 and 8.2 with Proposition 7.1 gives $\hat{\partial}(f)^- = X_\infty^-$. However, from Lemma 8.5 we deduce that

$$f = L^*(0)^{\#}\mathrm{nr}_{\mathbb{R}[G]}\Bigg(\prod_{\mathfrak{p}\in\mathcal{S}'\setminus\mathcal{S}_\infty} v_\mathfrak{p}^{-1} t_\mathfrak{p} h_\mathfrak{p}\Bigg) \prod_{\mathfrak{p}\in\mathcal{S}'\setminus\mathcal{S}_\infty} \mathrm{nr}_{\mathbb{R}[G]}(v_\mathfrak{p} t_\mathfrak{p}^{-1}) = L^*(0)^{\#}\mathrm{nr}_{\mathbb{R}[G]}(h_{\mathrm{glob}})$$

in the minus part. A standard argument shows that $L(0,\chi) = L^*(0,\chi)$ for every odd irreducible character χ of G (this is a straightforward exercise once one has the order of vanishing formula (22) used in § 12), and so we have $L^*(0)^- = L(0)^-$. Therefore $f = L(0)^{\#}\mathrm{nr}_{\mathbb{R}[G]}(h_{\mathrm{glob}})$ in the minus part, and the desired result now follows from applying $\hat{\partial}$ to both sides. □

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9. Computing fitting ideals

Proposition 9.1. *Let $\chi \in \mathrm{Irr}(G)$ with χ odd. Then, ignoring 2-parts, we have*

$$\mathrm{nr}_{e_\chi E[G]}(e_\chi h_{\mathrm{glob}}^{-1})U_\chi \frac{\mathrm{Fit}_{\mathcal{O}}(H^2(G,\mu_K[\chi]))}{\mathrm{Fit}_{\mathcal{O}}(H^1(G,\mu_K[\chi]))}\mathrm{Fit}_{\mathcal{O}}((\nabla/\delta(C))[\chi]_G) \subseteq \mathrm{Fit}_{\mathcal{O}}(\mathrm{cl}_K[\chi]^G).$$

Remark 9.2. If μ_K^- is R-c.t., then $(\nabla/\delta(C))^-$ is also R-c.t. and our argument shows that

$$\mathrm{nr}_{e_\chi E[G]}(e_\chi h_{\mathrm{glob}}^{-1})U_\chi \mathrm{Fit}_{\mathcal{O}}((\nabla/\delta(C))[\chi]_G) = \mathrm{Fit}_{\mathcal{O}}(\mathrm{cl}_K[\chi]^G).$$

Proof. We shall abuse notation by systematically ignoring 2-parts. We begin by following the proof of [Gre07, Lemma 8.2]. Recall that the map $\delta : C \to \nabla$ is injective (see [Gre07, § 6]) and that there is the crucial short exact sequence $0 \to \mathrm{cl}_K \to \nabla \to \bar{\nabla} \to 0$. With an abuse of notation, we also use δ to denote the map $C \to \bar{\nabla}$ and note that this is still injective since C is free and thus torsion-free. Since $(\nabla/\delta(C))^-$ is finite, we may choose a natural number x such that $x\nabla^- \subset \delta(C)^-$. Therefore we have two short exact sequences

$$0 \to \mathrm{cl}_K^- \to \frac{\nabla^-}{\delta(C)^-} \to \frac{\bar{\nabla}^-}{\delta(C)^-} \to 0, \quad 0 \to \frac{\bar{\nabla}^-}{\delta(C)^-} \to \frac{x^{-1}\delta(C)^-}{\delta(C)^-} \to \frac{x^{-1}\delta(C)^-}{\bar{\nabla}^-} \to 0.$$

These combine into the four-term exact sequence

$$0 \longrightarrow \mathrm{cl}_K^- \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0,$$

where

$$M_1 := \frac{\nabla^-}{\delta(C)^-}, \quad M_2 := \frac{x^{-1}\delta(C)^-}{\delta(C)^-}, \quad M_3 := \frac{x^{-1}\delta(C)^-}{\bar{\nabla}^-}.$$

The functor $M \mapsto M[\chi] := T_\chi \otimes_{\mathbb{Z}} M$ is exact as T_χ is free over $\mathbb{Z}$; so we obtain the exact sequence

$$0 \longrightarrow \mathrm{cl}_K[\chi] \longrightarrow M_1[\chi] \longrightarrow M_2[\chi] \longrightarrow M_3[\chi] \longrightarrow 0.$$

Since x is a natural number and $\delta(C)^-$ is free, M_2 is of projective dimension one over R and so is R-c.t.; therefore $M_2[\chi]$ is also R-c.t. and hence we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{cl}_K[\chi]^G \arrow[r] & M_1[\chi]^G \arrow[r, "\alpha"] & M_2[\chi]^G & & \\
& & M_1[\chi]_G \arrow[u] \arrow[r, "\beta"] & M_2[\chi]_G \arrow[u, "\simeq"'] \arrow[r] & M_3[\chi]_G \arrow[r] & 0
\end{tikzcd}$$

where the rows are exact and the vertical maps are induced by Norm_G. If we identify $M_2[\chi]_G$ with $M_2[\chi]^G$, then $\mathrm{im}(\beta) \subseteq \mathrm{im}(\alpha)$. Therefore we have

$$\begin{aligned}
\mathrm{Fit}_{\mathcal{O}}(\mathrm{cl}_K[\chi]^G) &= \mathrm{Fit}_{\mathcal{O}}(M_1[\chi]^G)\mathrm{Fit}_{\mathcal{O}}(\mathrm{im}(\alpha))^{-1} \\
&\supseteq \mathrm{Fit}_{\mathcal{O}}(M_1[\chi]^G)\mathrm{Fit}_{\mathcal{O}}(\mathrm{im}(\beta))^{-1} \\
&= \mathrm{Fit}_{\mathcal{O}}(M_1[\chi]^G)\mathrm{Fit}_{\mathcal{O}}(M_2[\chi]_G)^{-1}\mathrm{Fit}_{\mathcal{O}}(M_3[\chi]_G).
\end{aligned} \tag{12}$$

We now compute the two rightmost terms explicitly. Let $n = |\mathcal{S}' \backslash \mathcal{S}_\infty|$. Then

$$M_2[\chi]_G = T_\chi \otimes_R \frac{x^{-1}\delta(C)^-}{\delta(C)^-} \cong T_\chi \otimes_R \left(\frac{x^{-1}R^n}{R^n}\right) \cong \frac{x^{-1}T_\chi^n}{T_\chi^n} \cong \frac{T_\chi^n}{xT_\chi^n},$$

so recalling that T_χ is locally free of rank $\chi(1)$ over $\mathcal{O}$ gives

$$\mathrm{Fit}_{\mathcal{O}}(M_2[\chi]_G) = x^{n\chi(1)}\mathcal{O}. \tag{13}$$

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In [Gre07, bottom of p. 1419] it is noted that

$$\bar{\nabla}^{-} = \bigoplus_{\mathfrak{p} \in \mathcal{S}' \backslash \mathcal{S}_\infty} (\mathrm{ind}_{G_\mathfrak{p}}^{G} W_\mathfrak{p}^{0})^{-}. \tag{14}$$

Recall from [Gre07, proof of Lemma 6.1] that $g_\mathfrak{p} := |G_{0,\mathfrak{p}}| + 1 - F_\mathfrak{p}^{-1}$ maps to a non-zero divisor $\bar{g}$ of $\mathbb{Z}[G_\mathfrak{p}/G_{0,\mathfrak{p}}]$ and that $g_\mathfrak{p}^{-1}$ stands for any lift of $\bar{g}^{-1}$ to $\mathbb{Q}[G_\mathfrak{p}]$. This uniquely defines the element $g_\mathfrak{p}^{-1}\,\mathrm{Norm}_{G_{0,\mathfrak{p}}}$ (note that this is central in $\mathbb{Q}[G_\mathfrak{p}]$). From [Gre07, top of p. 1420] we have

$$W_\mathfrak{p}^{0} = \delta(x_\mathfrak{p}) \cdot \langle 1, g_\mathfrak{p}^{-1}\ \mathrm{Norm}_{G_{0,\mathfrak{p}}} \rangle_{\mathbb{Z}[G_\mathfrak{p}]}. \tag{15}$$

Now recall that $h_\mathfrak{p} := e'_\mathfrak{p} g_\mathfrak{p} + e''_\mathfrak{p}$ (again this is central in $\mathbb{Q}[G_\mathfrak{p}]$). In [Gre07, Lemma 8.3] it is shown that

$$\langle 1, g_\mathfrak{p}^{-1}\ \mathrm{Norm}_{G_{0,\mathfrak{p}}} \rangle_{\mathbb{Z}[G_\mathfrak{p}]} = h_\mathfrak{p}^{-1} \langle \mathrm{Norm}_{G_{0,\mathfrak{p}}}, 1 - e'_\mathfrak{p} F_\mathfrak{p}^{-1} \rangle_{\mathbb{Z}[G_\mathfrak{p}]} = h_\mathfrak{p}^{-1} U_\mathfrak{p}, \tag{16}$$

with the proof being valid for the non-abelian case as well since the relevant elements are central in $\mathbb{Q}[G_\mathfrak{p}]$. Combining equations (14), (15) and (16) yields

$$\bar{\nabla}^{-} = \bigoplus_{\mathfrak{p} \in \mathcal{S}' \backslash \mathcal{S}_\infty} (\delta(x_\mathfrak{p})\mathrm{ind}_{G_\mathfrak{p}}^{G} h_\mathfrak{p}^{-1} U_\mathfrak{p})^{-} = \bigoplus_{\mathfrak{p} \in \mathcal{S}' \backslash \mathcal{S}_\infty} (\delta(x_\mathfrak{p}) \mathbb{Z}[G](h_\mathfrak{p}^{-1} U_\mathfrak{p}))^{-},$$

where the second equality follows from the fact that $\bar{\nabla}$ is torsion-free. Hence

$$M_3 = \frac{x^{-1}\delta(C)^{-}}{\bar{\nabla}^{-}} \cong \bigoplus_{\mathfrak{p} \in \mathcal{S}' \backslash \mathcal{S}_\infty} \frac{R}{xR(h_\mathfrak{p}^{-1} U_\mathfrak{p})}$$

and so

$$M_3[\chi]_G = T_\chi \otimes_R M_3 \cong \bigoplus_{\mathfrak{p} \in \mathcal{S}' \backslash \mathcal{S}_\infty} \frac{T_\chi}{xT_\chi(h_\mathfrak{p}^{-1} U_\mathfrak{p})} = \bigoplus_{\mathfrak{p} \in \mathcal{S}' \backslash \mathcal{S}_\infty} \frac{T_\chi}{xT_\chi(e_\chi \mathfrak{M})(h_\mathfrak{p}^{-1} U_\mathfrak{p})}.$$

Recall that $h_{\mathrm{glob}} := \prod_{\mathfrak{p} \in \mathcal{S}' \backslash \mathcal{S}_\infty} h_\mathfrak{p}$ and

$$U_\chi := \prod_{\mathfrak{p} \in \mathcal{S}_{\mathrm{ram}}(K/k)} \mathrm{nr}_{e_\chi E[G]}(e_\chi \mathfrak{M} U_\mathfrak{p})\mathcal{O} = \prod_{\mathfrak{p} \in \mathcal{S}' \backslash \mathcal{S}_\infty} \mathrm{nr}_{e_\chi E[G]}(e_\chi \mathfrak{M} U_\mathfrak{p})\mathcal{O},$$

where the second equality holds because $U_\mathfrak{p} = \mathbb{Z}[G_\mathfrak{p}]$; hence $\mathrm{nr}_{e_\chi E[G]}(e_\chi \mathfrak{M} U_\mathfrak{p})\mathcal{O} = \mathcal{O}$ if $\mathfrak{p} \notin \mathcal{S}_{\mathrm{ram}}(K/k)$. Note that $\Lambda := e_\chi \mathfrak{M}$ is a maximal $\mathcal{O}$-order in $e_\chi E[G]$; so if $\mathcal{O}'$ is the localisation of $\mathcal{O}$ at any prime ideal, then $\Lambda' := \mathcal{O}' \otimes_\mathcal{O} \Lambda$ is a maximal $\mathcal{O}'$-order and every (left) Λ'-ideal is principal (see [Rei03, Theorem 18.7(ii)]). In particular, for each $\mathfrak{p} \in \mathcal{S}' \backslash \mathcal{S}_\infty$, there exists an element $y_\mathfrak{p}$ such that $\Lambda' \otimes_\Lambda x(e_\chi \mathfrak{M})(h_\mathfrak{p}^{-1} U_\mathfrak{p}) = \Lambda' y_\mathfrak{p}$ and hence there is an exact sequence of $\mathcal{O}'$-modules of the form

$$\mathcal{O}' \otimes_\mathcal{O} T_\chi \xrightarrow{y_\mathfrak{p}} \mathcal{O}' \otimes_\mathcal{O} T_\chi \longrightarrow \mathcal{O}' \otimes_\mathcal{O} \frac{T_\chi}{xT_\chi(e_\chi \mathfrak{M})(h_\mathfrak{p}^{-1} U_\mathfrak{p})} \longrightarrow 0. \tag{17}$$

Now, $\mathcal{O}' \otimes_\mathcal{O} T_\chi$ is free of rank $\chi(1)$ over $\mathcal{O}'$. Thus, since $\mathrm{nr}_{e_\chi E[G]}$ is the determinant map $e_\chi E[G] \cong \mathrm{Mat}_{\chi(1)}(E) \to E$, the definition of Fitting ideal combines with (17) and our definition of the fractional ideal $\mathrm{nr}_{e_\chi E[G]}(x(e_\chi \mathfrak{M})(h_\mathfrak{p}^{-1} U_\mathfrak{p}))\mathcal{O}$ to imply that

$$\begin{aligned} \mathrm{Fit}_{\mathcal{O}'}\left(\mathcal{O}' \otimes_\mathcal{O} \frac{T_\chi}{xT_\chi(e_\chi \mathfrak{M})(h_\mathfrak{p}^{-1} U_\mathfrak{p})}\right) &= \mathcal{O}' \cdot \det_{\mathcal{O}'}(y_\mathfrak{p}) \\ &= \mathcal{O}' \otimes_\mathcal{O} \mathrm{nr}_{e_\chi E[G]}(x(e_\chi \mathfrak{M})(h_\mathfrak{p}^{-1} U_\mathfrak{p}))\mathcal{O}. \end{aligned}$$

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However, Fitting ideals over $\mathcal{O}$ can be computed by localising and so, upon taking the product over all primes $\mathfrak{p} \in \mathcal{S}'\backslash\mathcal{S}_\infty$, we obtain

$$\mathrm{Fit}_{\mathcal{O}}(M_3[\chi]_G) = \prod_{\mathfrak{p} \in \mathcal{S}'\backslash\mathcal{S}_\infty} \mathrm{nr}_{e_\chi E[G]}(x(e_\chi \mathfrak{M})(h_{\mathfrak{p}}^{-1} U_{\mathfrak{p}}))\mathcal{O} = x^{n\chi(1)} \mathrm{nr}_{e_\chi E[G]}(e_\chi h_{\mathrm{glob}}^{-1}) U_\chi. \tag{18}$$

Substituting equations (13) and (18) into the containment (12) gives

$$\mathrm{nr}_{e_\chi E[G]}(e_\chi h_{\mathrm{glob}}^{-1}) U_\chi \mathrm{Fit}_{\mathcal{O}}(M_1[\chi]^G) \subseteq \mathrm{Fit}_{\mathcal{O}}(\mathrm{cl}_K[\chi]^G). \tag{19}$$

We have an exact sequence

$$0 \longrightarrow \widehat{H}^{-1}(G, M_1[\chi]) \longrightarrow M_1[\chi]_G \longrightarrow M_1[\chi]^G \longrightarrow \widehat{H}^0(G, M_1[\chi]) \longrightarrow 0,$$

where $\widehat{H}^i(G, M)$ denotes Tate cohomology. Hence

$$\mathrm{Fit}_{\mathcal{O}}(M_1[\chi]^G) = \mathrm{Fit}_{\mathcal{O}}(M_1[\chi]_G) \frac{\mathrm{Fit}_{\mathcal{O}}(\widehat{H}^0(G, M_1[\chi]))}{\mathrm{Fit}_{\mathcal{O}}(\widehat{H}^{-1}(G, M_1[\chi]))}. \tag{20}$$

Recall that the top row of (D2)$^-$ is an exact sequence

$$0 \longrightarrow \mu_K^- \longrightarrow A^- \longrightarrow \tilde{B}^- \longrightarrow M_1 \longrightarrow 0$$

of f.g. R-modules, where A^- and $\tilde{B}^-$ are R-c.t. Hence

$$\widehat{H}^0(G, M_1[\chi]) \cong H^2(G, \mu_K[\chi]) \quad \text{and} \quad \widehat{H}^{-1}(G, M_1[\chi]) \cong H^1(G, \mu_K[\chi]). \tag{21}$$

Combining (19), (20) and (21) now gives the desired result. $\square$

10. Fitting ideals from refined Euler characteristics

Let $I_{\mathcal{O}}$ denote the multiplicative group of invertible $\mathcal{O}$-modules in $\mathbb{C}$. There exists a natural isomorphism $\iota : K_0(\mathcal{O}, \mathbb{C}) \xrightarrow{\sim} I_{\mathcal{O}}$ with $\iota((P, \tau, Q)) = \tilde{\tau}(\det_{\mathcal{O}}(P) \otimes_{\mathcal{O}} \det_{\mathcal{O}}(Q)^{-1})$, where $\tilde{\tau}$ is the isomorphism

$$\mathbb{C} \otimes_{\mathcal{O}} (\det_{\mathcal{O}}(P) \otimes_{\mathcal{O}} \det_{\mathcal{O}}(Q)^{-1}) \cong \det_{\mathbb{C}}(\mathbb{C} \otimes_{\mathcal{O}} Q) \otimes_{\mathbb{C}} \det_{\mathbb{C}}(\mathbb{C} \otimes_{\mathcal{O}} Q)^{-1} \cong \mathbb{C}$$

induced by τ. Indeed, ι is induced by the exact sequence

$$K_1(\mathcal{O}) \longrightarrow K_1(\mathbb{C}) \longrightarrow K_0(\mathcal{O}, \mathbb{C}) \longrightarrow K_0(\mathcal{O}) \longrightarrow K_0(\mathbb{C})$$

and the canonical isomorphisms $K_1(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}^\times$ and $K_1(\mathcal{O}) \xrightarrow{\sim} \mathcal{O}^\times$. Under this identification, the boundary map $\mathbb{C}^\times \cong K_1(\mathbb{C}) \to K_0(\mathcal{O}, \mathbb{C}) \cong I_{\mathcal{O}}$ simply sends x to the lattice $x\mathcal{O}$.

In what follows, φ_{triv} denotes the only metrisation possible, namely the unique isomorphism from the complex vector space 0 to itself.

Lemma 10.1. *If $0 \to U \to A \to B \to V \to 0$ is an exact sequence of f.g. $\mathcal{O}$-modules with U and V finite, then $\iota(\chi_{\mathrm{ref}}(A \to B, \varphi_{\mathrm{triv}})) = \mathrm{Fit}_{\mathcal{O}}(U)^{-1}\mathrm{Fit}_{\mathcal{O}}(V)$.*

Proof. We have a distinguished triangle of perfect metrised complexes of $\mathcal{O}$-modules

$$\mathcal{C}_0 \longrightarrow \mathcal{C}_1 \longrightarrow \mathcal{C}_2 \longrightarrow \mathcal{C}_0[1],$$

where $\mathcal{C}_0$ is the complex $U[0]$, $\mathcal{C}_2$ the complex $V[-1]$ and $\mathcal{C}_1$ the complex $A \to B$ with the first term placed in degree zero. To see this, write $\widetilde{\mathcal{C}_2}$ for the complex $A/U \to B$ with first term placed in degree zero and map induced by $A \to B$; then $\widetilde{\mathcal{C}_2}$ is naturally quasi-isomorphic to $\mathcal{C}_2$ and also

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lies in the obvious short exact sequence of complexes of the form

$$0 \longrightarrow \mathcal{C}_0 \longrightarrow \mathcal{C}_1 \longrightarrow \widetilde{\mathcal{C}}_2 \longrightarrow 0.$$

Furthermore, since $\mathcal{O}$ is a Dedekind domain, every f.g. $\mathcal{O}$-module is of projective dimension at most one. Therefore the refined Euler characteristics in question are additive (see [Bur04, Theorem 2.8]) and so we are reduced to showing that for a finite $\mathcal{O}$-module M,

$$\iota(\chi_{\mathrm{ref}}(M[i], \varphi_{\mathrm{triv}})) = \mathrm{Fit}_{\mathcal{O}}(M)^{(-1)^{i+1}}.$$

Now, $\chi_{\mathrm{ref}}(M[i], \varphi_{\mathrm{triv}}) = (-1)^i \chi_{\mathrm{ref}}(M[0], \varphi_{\mathrm{triv}})$, so we are further reduced to considering the case where $i = 0$. There exists an exact sequence of f.g. $\mathcal{O}$-modules

$$0 \longrightarrow P \xrightarrow{d} F \longrightarrow M \longrightarrow 0,$$

with P projective and F free of equal rank r. We fix an isomorphism $F \cong \mathcal{O}^r$ and hence an identification of $\det_{\mathcal{O}}(F) = \wedge^r_{\mathcal{O}}(F)$ with $\mathcal{O}$. Under this identification, $\mathrm{Fit}_{\mathcal{O}}(M)$ is by definition the image of the homomorphism $\wedge^r_{\mathcal{O}}(d) : \wedge^r_{\mathcal{O}}(P) \to \wedge^r_{\mathcal{O}}(F) = \mathcal{O}$. Setting $\tau := \mathbb{C} \otimes_{\mathcal{O}} d$ and noting that $M[0]$ is quasi-isomorphic to the complex $P \to F$ with the second term placed in degree zero, we therefore have

$$\begin{aligned}\iota(\chi_{\mathrm{ref}}(M[0], \varphi_{\mathrm{triv}})) &= \iota((F, \tau^{-1}, P)) = \iota((P, \tau, F))^{-1} = \tilde{\tau}(\det_{\mathcal{O}}(P) \otimes_{\mathcal{O}} \det_{\mathcal{O}}(F)^{-1})^{-1} \\ &= \mathrm{im}(\wedge^r_{\mathcal{O}}(d))^{-1} = \mathrm{Fit}_{\mathcal{O}}(M)^{-1}\end{aligned}$$

as required. □

11. Two annihilation lemmas

Let χ be an irreducible character of a finite group G and let M be a $\mathbb{Z}[G]$-module.

LEMMA 11.1. *If $x \in \mathrm{Ann}_{\mathcal{O}}(M[\chi]^G)$, then $x \cdot \mathrm{pr}_\chi \in \mathrm{Ann}_{\mathcal{O}[G]}(\mathcal{O} \otimes_{\mathbb{Z}} M)$.*

Proof. It suffices to show that the result holds after localising at $\mathfrak{p}$ for all primes $\mathfrak{p}$ of $\mathcal{O}$. We set $n := \chi(1)$ and recall that T_χ is locally free of rank n over $\mathcal{O}$. In what follows, we abuse notation by omitting subscripts $\mathfrak{p}$ (i.e. all of the following $\mathcal{O}$-modules are localised at $\mathfrak{p}$).

We fix an $\mathcal{O}$-basis $\{t_i : 1 \leqslant i \leqslant n\}$ of T_χ and write $\rho_\chi : G \to \mathrm{GL}_n(\mathcal{O})$ for the associated representation. Then, for each $m \in M$ and each index i, the element $T_i(m) := \sum_{g \in G} g(t_i \otimes m)$ belongs to $M[\chi]^G = (T_\chi \otimes_{\mathbb{Z}} M)^G$. Now, in $M[\chi] = T_\chi \otimes_{\mathbb{Z}} M = T_\chi \otimes_{\mathcal{O}} (\mathcal{O} \otimes_{\mathbb{Z}} M)$ we have

$$\begin{aligned}T_i(m) &= \sum_{g \in G} t_i g^{-1} \otimes g(m) = \sum_{g \in G} \sum_{j=1}^{j=n} \rho_\chi(g^{-1})_{ij} t_j \otimes g(m) \\ &= \sum_{g \in G} \sum_{j=1}^{j=n} \rho_{\overline{\chi}}(g)_{ji} t_j \otimes g(m) = \sum_{j=1}^{j=n} t_j \otimes \left(\sum_{g \in G} \rho_{\overline{\chi}}(g)_{ji} g(m) \right).\end{aligned}$$

However, x annihilates $T_i(m) \in M[\chi]^G$ and $\{t_j : 1 \leqslant j \leqslant n\}$ is an $\mathcal{O}$-basis of T_χ, so the above equation implies that $x \cdot \sum_{g \in G} \rho_{\overline{\chi}}(g)_{ji} g(m) = 0$ for all i and j. Hence each element $c(x)_{ij} := x \cdot \sum_{g \in G} \rho_{\overline{\chi}}(g)_{ji} g$ belongs to $\mathrm{Ann}_{\mathcal{O}[G]}(\mathcal{O} \otimes_{\mathbb{Z}} M)$. In particular, the element

$$\sum_{i=1}^{i=n} c(x)_{ii} = \sum_{i=1}^{i=n} x \cdot \sum_{g \in G} \rho_{\overline{\chi}}(g)_{ii} g = x \cdot \sum_{g \in G} \left(\sum_{i=1}^{i=n} \rho_{\overline{\chi}}(g)_{ii} \right) g = x \cdot \sum_{g \in G} \overline{\chi}(g) g = x \cdot \mathrm{pr}_\chi$$

belongs to $\mathrm{Ann}_{\mathcal{O}[G]}(\mathcal{O} \otimes_{\mathbb{Z}} M)$, as required. □

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Lemma 11.2. *If $x \in \mathcal{O}$ such that $x \cdot \mathrm{pr}_\chi \in \mathrm{Ann}_{\mathcal{O}[G]}(\mathcal{O} \otimes_{\mathbb{Z}} M)$, then for every $y \in \mathcal{D}_{E/\mathbb{Q}}^{-1}$ we have $\sum_{\omega \in \mathrm{Gal}(E/\mathbb{Q})} y^\omega x^\omega \cdot \mathrm{pr}_{\chi^\omega} \in \mathrm{Ann}_{\mathbb{Z}[G]}(M)$.*

Proof. The hypotheses imply that $yx \cdot \mathrm{pr}_\chi$ belongs to

$$\mathcal{D}_{E/\mathbb{Q}}^{-1} \cdot \mathrm{Ann}_{\mathcal{O}[G]}(\mathcal{O} \otimes_{\mathbb{Z}} M) = \mathcal{D}_{E/\mathbb{Q}}^{-1} \otimes_{\mathbb{Z}} \mathrm{Ann}_{\mathbb{Z}[G]}(M).$$

The element

$$\sum_{\omega \in \mathrm{Gal}(E/\mathbb{Q})} y^\omega x^\omega \cdot \mathrm{pr}_{\chi^\omega} = \sum_{\omega \in \mathrm{Gal}(E/\mathbb{Q})} (yx \cdot \mathrm{pr}_\chi)^\omega$$

therefore belongs to $\mathrm{Tr}_{E/\mathbb{Q}}(\mathcal{D}_{E/\mathbb{Q}}^{-1}) \otimes_{\mathbb{Z}} \mathrm{Ann}_{\mathbb{Z}[G]}(M) \subseteq \mathrm{Ann}_{\mathbb{Z}[G]}(M)$, as required. □

12. Proofs of the main results

Proof of Theorem 1.2. Let ϕ be the character of $\mathrm{Gal}(K/k)$ whose inflation to $G = \mathrm{Gal}(L/k)$ is χ. For each $x \in \mathrm{cl}_L$, we have $\mathrm{pr}_\chi(x) = \mathrm{pr}_\phi(\mathrm{Norm}_{\mathrm{Gal}(L/K)}(x))$ and $\mathrm{Norm}_{\mathrm{Gal}(L/K)}(x) \in \mathrm{cl}_K$. However, we also have $L(s, \chi) = L(s, \phi)$ (see [Tat84, § 4.2]), so we are reduced to the case of $L = K$. Note that $\chi = \phi$ remains irreducible.

We now repeat a reduction argument given in [Tat84, top of p. 71]. The order of vanishing of $L(s, \chi) = L_{\mathcal{S}_\infty}(s, \chi)$ at $s = 0$ is given by

$$r_{\mathcal{S}_\infty}(\chi) = \sum_{v \in \mathcal{S}_\infty} \dim_{\mathbb{C}} V_\chi^{G_v} - \dim_{\mathbb{C}} V_\chi^G, \tag{22}$$

where V_χ is a $\mathbb{C}[G]$-module with character χ (see [Tat84, ch. I, Proposition 3.4]). If $r_{\mathcal{S}_\infty}(\chi) > 0$, then $L(0, \chi) = 0$ and the result is trivial. Hence we may suppose that $r_{\mathcal{S}_\infty}(\chi) = 0$. Since χ is non-trivial, we have $V_\chi^G = \{0\}$ and so (22) gives $V_\chi^{G_v} = \{0\}$ for each $v \in \mathcal{S}_\infty$. In particular, G_v is non-trivial for $v \in \mathcal{S}_\infty$, so k is totally real and K is totally complex. Now $G_v = \{1, j_w\}$ for a complex place w of K, and j_w acts as -1 on V_χ since $j_w^2 = 1$ and $V_\chi^{j_w} = \{0\}$. Thus, since the representation V_χ is faithful, all the j_w are equal to the same $j \in G$. Hence K is a totally imaginary quadratic extension of the totally real subfield $K^{\langle j \rangle}$, i.e. K is a CM field. Furthermore, χ is odd because j acts as -1 on V_χ.

For the rest of this proof, we abuse notation and consider only p-parts. Recall from Proposition 8.7 that under the assumption that the ETNC holds for the motive $h^0(K)$ with coefficients in R, we have

$$\hat{\partial}(L(0)^{\#}\mathrm{nr}_{\mathbb{R}[G]}(h_{\mathrm{glob}}))^- = X_\infty^- \quad \text{in } K_0(R, \mathbb{R}[G]^-). \tag{23}$$

As χ is odd, base change gives a natural homomorphism

$$\mu_\chi : K_0(R, \mathbb{R}[G]^-) \to K_0(\mathcal{O}, \mathbb{C}), \quad (P, f, Q) \mapsto (T_\chi \otimes_R P, \mathrm{id} \otimes f, T_\chi \otimes_R Q) = (P[\chi]_G, f_\chi, Q[\chi]_G).$$

Under condition $(*)$ it can be shown that the strong Stark conjecture at p for χ follows from Wiles's proof of the main conjecture for totally real fields (for details see [Nic, Corollary 2, p. 24], for example). Since the strong Stark conjecture can be interpreted as the 'ETNC modulo torsion' and $K_0(\mathcal{O}, \mathbb{C})$ is torsion-free, the image under μ_χ of equation (23) holds under our hypotheses; that is,

$$\mu_\chi(\hat{\partial}(L(0)^{\#}\mathrm{nr}_{\mathbb{R}[G]}(h_{\mathrm{glob}}))^-) = \mu_\chi(X_\infty^-) \quad \text{in } K_0(\mathcal{O}, \mathbb{C}).$$

Since μ_χ factors via $K_0(\mathbb{Z}[G], \mathbb{C}[G])$ and $K_0(\mathcal{O}, \mathbb{C})$ is torsion-free, Lemma 3.1 then gives

$$(\mathcal{O}, \mathrm{nr}_{e_\chi E[G]}(e_\chi h_{\mathrm{glob}}) L(0, \bar{\chi}), \mathcal{O}) = \mu_\chi(X_\infty^-) \quad \text{in } K_0(\mathcal{O}, \mathbb{C}). \tag{24}$$

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Recall that the top row of (D2)$^-$ is an exact sequence of f.g. R-modules

$$0 \longrightarrow \mu_K^- \longrightarrow A^- \longrightarrow \tilde{B}^- \longrightarrow (\nabla/\delta(C))^- \longrightarrow 0. \tag{25}$$

This gives the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mu[\chi]^G \arrow[r] & A[\chi]^G \arrow[r] & \tilde{B}[\chi]^G & & \\
 & & A[\chi]_G \arrow[r] \arrow[u, "\simeq"'] & \tilde{B}[\chi]_G \arrow[r] \arrow[u, "\simeq"'] & (\nabla/\delta(C))[\chi]_G \arrow[r] & 0
\end{tikzcd}$$

where the vertical maps induced by Norm_G are isomorphisms since A^- and $\tilde{B}^-$ are R-c.t. Hence we have an exact sequence

$$0 \longrightarrow \mu[\chi]^G \longrightarrow A[\chi]_G \longrightarrow \tilde{B}[\chi]_G \longrightarrow (\nabla/\delta(C))[\chi]_G \longrightarrow 0. \tag{26}$$

Now recall that X_∞^- is the refined Euler characteristic of (25). Since μ_K^- and $(\nabla/\delta(C))^-$ are finite, the only possible metrisation is φ_{triv} (i.e. $0 \xrightarrow{\sim} 0$), and so (24) becomes

$$(\mathcal{O}, \mathrm{nr}_{e_\chi E[G]}(e_\chi h_{\mathrm{glob}})L(0,\bar{\chi}), \mathcal{O}) = \chi_{\mathrm{ref}}(A[\chi]_G \to \tilde{B}[\chi]_G, \varphi_{\mathrm{triv}}) \quad \text{in } K_0(\mathcal{O}, \mathbb{C}). \tag{27}$$

It follows from Lemma 10.1 that (26) and (27) give an equality of $\mathcal{O}$-lattices of the form

$$\mathrm{nr}_{e_\chi E[G]}(e_\chi h_{\mathrm{glob}})L(0,\bar{\chi})\mathcal{O} = \mathrm{Fit}_{\mathcal{O}}((\mu_K[\chi])^G)^{-1}\mathrm{Fit}_{\mathcal{O}}((\nabla/\delta(C))[\chi]_G).$$

Combining this equality with Proposition 9.1 gives

$$L(0,\bar{\chi})U_\chi \prod_{i=0}^{i=2} \mathrm{Fit}_{\mathcal{O}}(H^i(G, \mu_K[\chi]))^{(-1)^i} \subseteq \mathrm{Fit}_{\mathcal{O}}(\mathrm{cl}_K^-[\chi]^G).$$

Recalling that

$$h(\mu_K, \chi) := \prod_{i=0}^{i=2} \mathrm{Fit}_{\mathcal{O}}(H^i(G, \mu_K[\chi]))^{(-1)^i},$$

we then obtain

$$L(0,\bar{\chi})U_\chi h(\mu_K,\chi) \subseteq \mathrm{Fit}_{\mathcal{O}}(\mathrm{cl}_K^-[\chi]^G) \subseteq \mathrm{Ann}_{\mathcal{O}}(\mathrm{cl}_K^-[\chi]^G).$$

Hence, for any $x \in U_\chi \cdot h(\mu_K, \chi)$, Lemma 11.1 implies that

$$xL(0,\bar{\chi}) \cdot \mathrm{pr}_\chi \in \mathrm{Ann}_{\mathcal{O}[G]^-}(\mathcal{O} \otimes_{\mathbb{Z}} \mathrm{cl}_K^-) \subseteq \mathrm{Ann}_{\mathcal{O}[G]}(\mathcal{O} \otimes_{\mathbb{Z}} \mathrm{cl}_K). \tag{28}$$

Now, by applying Lemma 11.2, the desired result follows. □

Proof of Corollary 1.5. Let χ be a non-trivial irreducible character of G and let $K := L^{\ker(\chi)}$. As every inertia subgroup is normal in G, every inertia subgroup of $\mathrm{Gal}(K/k)$ is normal. Choose E_χ such that $d_\chi = [E_\chi : \mathbb{Q}(\chi)]$, and let $G \cdot \chi$ denote the orbit of χ in $\mathrm{Irr}(G)$. Then, taking into account Remark 1.3 and applying Theorem 1.2 with $x = 1$ shows that

$$\sum_{\omega \in \mathrm{Gal}(E_\chi/\mathbb{Q})} L(0, \bar{\chi}^\omega) \cdot \mathrm{pr}_{\chi^\omega} = d_\chi \sum_{\psi \in G\cdot\chi} L(0,\bar{\psi}) \cdot \mathrm{pr}_\psi$$

belongs to the centre of $\mathbb{Z}_{(p)}[G]$ and annihilates $\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} \mathrm{cl}_L$. Hence, summing over all non-trivial irreducible characters of G gives the desired result in the case where $\mathcal{S} = \mathcal{S}_\infty$.

If $\mathcal{S} \supsetneqq \mathcal{S}_\infty$, then $L_{\mathcal{S}}(0,\chi)$ is $L(0,\chi)$ multiplied by factors of the form

$$L_{K_{\mathfrak{P}}/k_{\mathfrak{p}}}(0,\psi)^{-1} = \lim_{s\to 0} \det_{\mathbb{C}}(1 - F_{\mathfrak{p}}(\mathrm{N}\mathfrak{p})^{-s} | V_\psi^{G_{0,\mathfrak{p}}}),$$

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each of which is an element of $\mathcal{O}$ (possibly zero). Hence the containment (28) is still valid when $L(0,\chi)$ is replaced by $L_{\mathcal{S}}(0,\chi)$, giving the analogous version of Theorem 1.2. The desired result then follows from the same argument as above. $\square$

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