UNISOLVENCE ON MULTIDIMENSIONAL SPACES

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(received February 1, 1968)

1. Introduction. In this note we consider the possibility of unisolvence of a family $\mathcal{G}$ of real continuous functions on a compact subset X of m-dimensional Euclidean space. Such a study is of interest for two reasons. First, an elegant theory of Chebyshev approximation has been constructed for the case where the approximating family $\mathcal{G}$ is unisolvent of degree n on an interval $[\alpha, \beta]$. We study what sort of theory results from unisolvence of degree n on a more general space. Secondly, uniqueness of best Chebyshev approximation on a general compact space to any continuous function on X can be shown if the approximating family $\mathcal{G}$ is unisolvent of degree n and $\mathcal{G}$ satisfies certain convexity conditions. It is therefore of importance to Chebyshev approximation to consider the domains X on which unisolvence can occur. We will also study a more general condition on $\mathcal{G}$ involving a variable degree.

2. Unisolvence.

DEFINITION. A family $\mathcal{G}$ of real continuous functions is called unisolvent of degree n on a space X if for any given distinct points $x_1, \ldots, x_n$, and real numbers $w_1, \ldots, w_n$, there exists a unique element $G \varepsilon \mathcal{G}$ such that

$$G(x_i) = w_i \qquad i = 1, \ldots, n.$$

Such a family is called by Tornheim and Curtis an n-parameter family.

LEMMA 1. Let $\mathcal{G}$ be unisolvent of degree 2 on a closed interval $[x_1, x_2]$, then two distinct approximants G_1 and G_2 cannot have a difference $G_1 - G_2$ with an interior zero at which no sign change occurs.

Proof. Suppose such approximants G_1 and G_2 exist. By definition of unisolvence $G_1 - G_2$ can have no other zeros. Select $G_3 \varepsilon \mathcal{G}$ such that

Canad. Math. Bull. vol. 11, no.3, 1968

https://doi.org/10.4153/CMB-1968-056-6 Published online by Cambridge University Press

$$G_3(x_1) = G_1(x_1), \; G_3(x_2) = (G_1(x_2)+G_2(x_2))/2.$$

By drawing a diagram it is seen that one of the differences G_3-G_1, G_3-G_2 must have two zeros in $[x_1, x_2]$.

A tripod is a set of m-dimensional real space consisting of three non-degenerate line segments S_1, S_2, S_3, joined at and only at one common endpoint, a ramification point. In [1, pp. 16-17] can be found a diagram of a tripod and Haar's ingenious argument that a unisolvent linear family of dimension greater than one cannot exist on a space containing a tripod.

THEOREM 1. *Let $\mathcal{G}$ be unisolvent of degree n on X, $n \geq 2$, then X does not contain a tripod.*

Proof. Assume the theorem is false. In the case $n > 2$ we can take distinct points $x_3, \ldots, x_n$ in the interior of the segment S_1, and if we choose values $w_3, \ldots, w_n$, the requirement that

$$G(x_i) = w_i \qquad i = 3, \ldots, n$$

gives us a subset of $\mathcal{G}$ which is unisolvent of degree 2 on a set comprising S_2, S_3, and a subset S_1' of S_1 connected with the other segments. We therefore have a family $\mathcal{G}'$ unisolvent of degree 2 on the tripod $S_1' \cup S_2 \cup S_3$.

It only remains to show that no family $\mathcal{G}$ unisolvent of degree 2 on a tripod $S_1 \cup S_2 \cup S_3$ can exist. Let G_1 and G_2 be two approximants with the same value at the ramification point and differing on some other fixed point. By definition G_1-G_2 has no other zeros. On two of the three segment interiors, G_1-G_2 has therefore the same sign; assume without loss of generality that these segments are S_1 and S_2. We therefore have the difference of two distinct approximants having an interior zero with no sign change on the interval $S_1 \cup S_2$. By lemma 1 this is impossible and the theorem is proven.

THEOREM 2. *Let $\mathcal{G}$ be unisolvent of degree $n > 1$ on X containing a subset $\hat{X}$ homeomorphic to the circumference of a circle in 2-space, then $X = \hat{X}$ and n is odd.*

Proof. First we consider the case where n is even. Let $x_1, \ldots, x_n$ be n distinct points on $\hat{X}$ and let G_1 and G_2 be chosen

https://doi.org/10.4153/CMB-1968-056-6 Published online by Cambridge University Press

in $\mathcal{G}$ such that $G_1(x_i) = G_2(x_i)$ $(i = 1, \ldots, n-1)$ and $G_1(x_n) \neq G_2(x_n)$. G_1 and G_2 are distinct elements of $\mathcal{G}$ which agree on the $n-1$ points $\{x_1, \ldots, x_{n-1}\}$; by definition they can agree on no other point. From the circularity of $\hat{X}$, oddness of $n-1$, and continuity of G_1-G_2, the difference G_1-G_2 cannot change sign at all its zeros; hence there is a zero x_j of G_1-G_2 at which no sign change occurs. Let S be a segment of $\hat{X}$ containing x_j as an interior point and no other zeros of G_1-G_2. Let $\mathcal{G}_2$ be the family of elements G of $\mathcal{G}$ such that $G(x_i) = G_1(x_i)$ $(i = 1, \ldots, n-1,\ i \neq j)$; then $\mathcal{G}_2$ is unisolvent of degree 2 on S and contains G_1 and G_2. Applying the lemma, we obtain a contradiction, and so n must be odd. Next suppose $X \sim \hat{X}$ is a non-empty set containing x_0 and n is odd, then define $\mathcal{G}_n$ to be the set of elements G of $\mathcal{G}$ such that $G(x_0) = 0$. $\mathcal{G}_n$ is unisolvent of degree $n-1$, which is even, on $\hat{X}$, which is impossible by the previous argument. The theorem is proven.

If we consider compact arcwise-connected subsets X of m-dimensional Euclidean space, we find there are only three possibilities topologically. First, X can be homeomorphic to a closed interval, which is topologically equivalent to a proper arcwise-connected subset of the circumference of the circle. Secondly, X can be homeomorphic to the circumference of the circle. Thirdly X can contain a tripod. Applying the previous results we obtain

THEOREM 3. *Let $\mathcal{G}$ be unisolvent of degree n greater than one on a compact arcwise-connected subset X of m-dimensional Euclidean space; then X is homeomorphic to an arcwise-connected subset of the circumference of the circle in 2-space. If n is even the subset of the circumference must be a proper subset.*

COROLLARY. *Let X be a compact arcwise-connected subset of finite dimensional m-space and $\mathcal{G}$ be unisolvent of degree n on X. If $\{G_k\} \varepsilon \mathcal{G}$ converges pointwise to $G_0 \varepsilon \mathcal{G}$ on a set of n distinct points, then $\{G_k\}$ converges uniformly to G_0.*

Proof of Corollary. The result is easily seen to be true in the case where $n = 1$. In the case $n > 1$ we can assume without loss of generality that X is an arcwise-connected subset of the circumference of the circle, and it is readily shown by the arguments of Tornheim [4, pp. 450-462], for the case X is an interval and the argument of Curtis [2, p. 1014], for the case X is the circumference, that the result is true. If the hypotheses of the corollary are satisfied it follows that closed bounded subsets of $\mathcal{G}$ are compact.

We obtain the standard alternating theory [4, pp. 463-466] of Chebyshev approximation if we insist that X be arcwise-connected and $\mathcal{G}$ be unisolvent of degree n on X.

In case X is not connected, unisolvent families may no longer possess the topological properties given in the corollary. For example, let X be a space of 2 points, x_1, x_2, and let ρ be a 1-1 function from reals to reals, then the family of functions $\mathcal{G} = \{F(a, .)\}$,

$$F(a, x_1) = a \qquad F(a, x_2) = \rho(a),$$

is unisolvent of degree 1 on X. $\mathcal{G}$ need have no topological structure if ρ is highly discontinuous. If ρ is selected such that for any reals $b, c, b < c$, $\{\rho(a): b < a < c\}$ is dense in the real line, then the only functions on X with best Chebyshev approximations are elements of $\mathcal{G}$. Similar pathological cases can be given for unisolvence of higher degree and X consisting of a finite point set plus an interval. It is an open question whether the corollary is true when $n > 1$ and X consists of non-degenerate intervals.

3. Families with variable degree. Properties more general than unisolvence are also important in Chebyshev approximation. Among these properties is Rice's unisolvence of variable degree [3]; we introduce a more general property including Rice's property. It is convenient to give the space of continuous functions on X the norm

$$||g|| = \max \{|g(x)|: x \in X\} .$$

DEFINITION. $\mathcal{G}$ has degree n at G if

(i) $G-G_1$ having n zeros implies $G = G_1$,

(ii) for given n distinct points, real $\varepsilon > 0$, and real numbers $w_1, \ldots, w_n$ taking values $-1, 0, 1$, there is an element $G_1 \in \mathcal{G}$ such that $||G_1 - G|| < \varepsilon$ and $\operatorname{sgn}[G_1(x_i) - G(x_i)] = w_i$, $i = 1, \ldots, n$.

LEMMA 2. *Let $\mathcal{G}$ have positive degree at all elements and X contain a non-degenerate interval I, then for any element G at which $\mathcal{G}$ has degree n, there exists $\delta > 0$ such that $||G-G_1|| < \delta$ implies $\mathcal{G}$ has degree at least n at G_1.*

Proof. Let I contain an ordered set $x_1, \ldots, x_n$ of points. Select $G_1 \in \mathcal{G}$ such that

$$\operatorname{sgn}(G(x_i) - G_1(x_i)) = (-1)^i \qquad i = 1, \ldots, n.$$

https://doi.org/10.4153/CMB-1968-056-6 Published online by Cambridge University Press

Let $\delta = \inf \{|G(x_i)-G_1(x_i)|: i = 1, \ldots, n\}$. Select $G_2 \varepsilon \mathcal{G}$ such that $||G_2-G||< \delta$. It is easily seen that

$$\operatorname{sgn}(G_2(x_i)-G_1(x_i)) = (-1)^i \qquad i = 1, \ldots, n,$$

and hence G_2-G_1 has at least $n-1$ zeros in $[x_1, x_n]$. Hence the degree of $\mathcal{G}$ at G_2 is at least n, and the lemma is proven.

Let $\mathcal{G}$ have a maximum degree n and the hypotheses of lemma 2 hold; then it is easily seen from lemma 2 and the definition that the family $\mathcal{G}_n$, the set of elements of $\mathcal{G}$ at which $\mathcal{G}$ has degree n, has degree n at each of its elements on X. We can easily obtain analogues of lemma 1, Theorem 1, and Theorem 2, using similar arguments on $\mathcal{G}_n$ and from these we obtain the analogue of Theorem 3.

THEOREM 4. Let $\mathcal{G}$ have positive degree at all of its elements and have maximum degree $n > 1$ on a compact arcwise-connected subset X of m-dimensional Euclidean space, then X is homeomorphic to an arcwise-connected subset of the circumference of the unit circle in 2-space. If n is even the subset of the circumference must be a proper subset.

In the case where X is an interval or circumference of a circle and $\mathcal{G}$ has a positive degree at all of its elements, it is readily shown by arguments similar to those of Rice [3, pp. 300-301] that a necessary and sufficient condition for an element G at which $\mathcal{G}$ has degree n to be a best Chebyshev approximation to f is that f-G alternate n times. It follows that we obtain the standard alternating theory if we require that X be arcwise-connected and $\mathcal{G}$ have positive degree at all elements on X.

It should be noted that results stronger than those of this paper have been established for linear unisolvent families by Mairhuber and Curtis (these results apply also to rational families with a degree). The proofs of this paper are however much more elementary.

The author would like to thank Dr. E. Barbeau and Dr. E. W. Cheney for their suggestions.

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https://doi.org/10.4153/CMB-1968-056-6 Published online by Cambridge University Press