Canad. Math. Bull. Vol. 65 (2), 2022, pp. 525–541
http://dx.doi.org/10.4153/S0008439521000461

Galois-theoretic features for 1-smooth pro-p groups

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Abstract. Let p be a prime. A pro-p group G is said to be 1-smooth if it can be endowed with a continuous representation $\theta\colon G\to \mathrm{GL}_1(\mathbb{Z}_p)$ such that every open subgroup H of G, together with the restriction $\theta|_H$, satisfies a formal version of Hilbert 90. We prove that every 1-smooth pro-p group contains a unique maximal closed abelian normal subgroup, in analogy with a result by Engler and Koenigsmann on maximal pro-p Galois groups of fields, and that if a 1-smooth pro-p group is solvable, then it is locally uniformly powerful, in analogy with a result by Ware on maximal pro-p Galois groups of fields. Finally, we ask whether 1-smooth pro-p groups satisfy a "Tits' alternative."

1 Introduction

Throughout the paper p will denote a prime number, and $\mathbb{K}$ a field containing a root of unity of order p. Let $\mathbb{K}(p)$ denote the compositum of all finite Galois p-extensions of $\mathbb{K}$. The maximal pro-p Galois group of $\mathbb{K}$, denoted by $G_{\mathbb{K}}(p)$, is the Galois group $\mathrm{Gal}(\mathbb{K}(p)/\mathbb{K})$, and it coincides with the maximal pro-p quotient of the absolute Galois group of $\mathbb{K}$. Characterising maximal pro-p Galois groups of fields among pro-p groups is one of the most important—and challenging—problems in Galois theory. One of the obstructions for the realization of a pro-p group as maximal pro-p Galois group for some field $\mathbb{K}$ is given by the Artin–Scherier theorem: the only finite group realizable as $G_{\mathbb{K}}(p)$ is the cyclic group of order 2 (cf. [1]).

The proof of the celebrated *Bloch-Kato conjecture*, completed by Rost and Voevodsky with Weibel's "patch" (cf. [12, 27, 29]) provided new tools to study absolute Galois groups of field and their maximal pro-p quotients (see, e.g., [2, 3, 17, 21]). In particular, the now-called Norm Residue Theorem implies that the $\mathbb{Z}/p$-*cohomology algebra* of a maximal pro-p Galois group $G_{\mathbb{K}}(p)$

$$H^\bullet(G_{\mathbb{K}}(p),\mathbb{Z}/p) := \bigoplus_{n\geq 0} H^n(G_{\mathbb{K}}(p),\mathbb{Z}/p),$$

with $\mathbb{Z}/p$ a trivial $G_{\mathbb{K}}(p)$-module and endowed with the cup-product, is a quadratic algebra: i.e., all its elements of positive degree are combinations of products of elements of degree 1, and its defining relations are homogeneous relations of degree 2 (see Section 2.3). For instance, from this property one may recover the Artin-Schreier obstruction (see, e.g., [17, Section 2]).

Received by the editors June 26, 2020; revised June 21, 2021; accepted June 23, 2021.
Published online on Cambridge Core June 29, 2021.
AMS subject classification: **12G05**, 20E18, 20J06, 12F10.
Keywords: Galois cohomology, maximal pro-p Galois groups, Bloch–Kato conjecture, Kummerian pro-p pairs, Tits' alternative.

More recently, a formal version of *Hilbert 90* for pro-p groups was employed to find further results on the structure of maximal pro-p Galois groups (see [9, 19, 21]). A pair $\mathcal{G} = (G, \theta)$ consisting of a pro-p group G endowed with a continuous representation $\theta\colon G \to \mathrm{GL}_1(\mathbb{Z}_p)$ is called a *pro-p pair*. For a pro-p pair $\mathcal{G} = (G, \theta)$ let $\mathbb{Z}_p(1)$ denote the continuous left G-module isomorphic to $\mathbb{Z}_p$ as an abelian pro-p group, with G-action induced by θ (namely, $g.v = \theta(g) \cdot v$ for every $v \in \mathbb{Z}_p(1)$). The pair $\mathcal{G}$ is called a *Kummerian pro-p pair* if the canonical map

$$H^1(G, \mathbb{Z}_p(1)/p^n) \longrightarrow H^1(G, \mathbb{Z}_p(1)/p)$$

is surjective for every $n \geq 1$. Moreover the pair $\mathcal{G}$ is said to be a *1-smooth* pro-p pair if every closed subgroup H, endowed with the restriction $\theta|_H$, gives rise to a Kummerian pro-p pair (see Definition 2.1). By Kummer theory, the maximal pro-p Galois group $G_{\mathbb{K}}(p)$ of a field $\mathbb{K}$, together with the pro-p *cyclotomic character* $\theta_{\mathbb{K}}\colon G_{\mathbb{K}}(p) \to \mathrm{GL}_1(\mathbb{Z}_p)$ (induced by the action of $G_{\mathbb{K}}(p)$ on the roots of unity of order a p-power lying in $\mathbb{K}(p)$) gives rise to a 1-smooth pro-p pair $\mathcal{G}_{\mathbb{K}}$ (see Theorem 2.8).

In [5]—driven by the pursuit of an "explicit" proof of the Bloch–Kato conjecture as an alternative to the proof by Voevodsky—De Clerq and Florence introduced the 1-smoothness property, and formulated the so-called "Smoothness Conjecture": namely, that it is possible to deduce the surjectivity of the norm residue homomorphism (which is acknowledged to be the "hard part" of the Bloch–Kato conjecture) from the fact that $G_{\mathbb{K}}(p)$ together with the pro-p cyclotomic character is a 1-smooth pro-p pair (see [5, Conjecture 14.25] and [15, Section 3.1.6], and Question 2.10).

In view of the Smoothness Conjecture, it is natural to ask which properties of maximal pro-p Galois groups of fields arise also for 1-smooth pro-p pairs. For example, the Artin–Scherier obstruction does: the only finite p-group which may complete into a 1-smooth pro-p pair is the cyclic group C_2 of order 2, together with the nontrivial representation $\theta\colon C_2 \to \{\pm 1\} \subseteq \mathrm{GL}_1(\mathbb{Z}_2)$ (see Example 2.9).

A pro-p pair $\mathcal{G} = (G, \theta)$ comes endowed with a distinguished closed subgroup: the *θ-center* $Z(\mathcal{G})$ of $\mathcal{G}$, defined by

$$Z(\mathcal{G}) = \left\langle\, h \in \mathrm{Ker}(\theta) \;\middle|\; ghg^{-1} = h^{\theta(g)}\ \forall\, g \in G \,\right\rangle.$$

This subgroup is abelian, and normal in G. In [10], Engler and Koenigsmann showed that if the maximal pro-p Galois group $G_{\mathbb{K}}(p)$ of a field $\mathbb{K}$ is not cyclic then it has a unique maximal normal abelian closed subgroup (i.e., one containing all normal abelian closed subgroups of $G_{\mathbb{K}}(p)$), which coincides with the $\theta_{\mathbb{K}}$-center $Z(\mathcal{G}_{\mathbb{K}})$, and the short exact sequence of pro-p groups

$$\{1\} \longrightarrow Z(\mathcal{G}_{\mathbb{K}}) \longrightarrow G_{\mathbb{K}}(p) \longrightarrow G_{\mathbb{K}}(p)/Z(\mathcal{G}_{\mathbb{K}}) \longrightarrow \{1\}$$

splits. We prove a group-theoretic analogue of Engler–Koenigsmann's result for 1-smooth pro-p groups.

Theorem 1.1 *Let G be a torsion-free pro-p group, $G \not\simeq \mathbb{Z}_p$, endowed with a representation $\theta\colon G \to \mathrm{GL}_1(\mathbb{Z}_p)$ such that $\mathcal{G} = (G, \theta)$ is a 1-smooth pro-p pair. Then $Z(\mathcal{G})$ is the*

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unique maximal normal abelian closed subgroup of G, and the quotient $G/Z(\mathcal{G})$ is a torsion-free pro-p group.

In [28], Ware proved the following result on maximal pro-p Galois groups of fields: if $G_{\mathbb{K}}(p)$ is solvable, then it is locally uniformly powerful, i.e., $G_{\mathbb{K}}(p) \simeq A \rtimes \mathbb{Z}_p$, where A is a free abelian pro-p group, and the right-side factor acts by scalar multiplication by a unit of $\mathbb{Z}_p$ (see Section 3.1). We prove that the same property holds also for 1-smooth pro-p groups.

Theorem 1.2 *Let G be a solvable torsion-free pro-p group, endowed with a representation $\theta\colon G \to \mathrm{GL}_1(\mathbb{Z}_p)$ such that $\mathcal{G} = (G, \theta)$ is a 1-smooth pro-p pair. Then G is locally uniformly powerful.*

This gives a complete description of solvable torsion-free pro-p groups which may be completed into a 1-smooth pro-p pair. Moreover, Theorem 1.2 settles the Smoothness Conjecture positively for the class of solvable pro-p groups.

Corollary 1.3 *If $\mathcal{G} = (G, \theta)$ is a 1-smooth pro-p pair with G solvable, then G is a Bloch–Kato pro-p group, i.e., the $\mathbb{Z}/p$-cohomology algebra of every closed subgroup of G is quadratic.*

Remark 1.4 After the submission of this paper, Snopce and Tanushevski showed in [24] that Theorems 1.2–1.1 hold for a wider class of pro-p groups. A pro-p group is said to be *Frattini-injective* if distinct finitely generated closed subgroups have distinct Frattini subgroups (cf. [24, Definition 1.1]). By [24, Theorem 1.11 and Corollary 4.3], a pro-p group which may complete into a 1-smooth pro-p pair is Frattini-injective. By [24, Theorem 1.4] a Frattini-injective pro-p group has a unique maximal normal abelian closed subgroup, and by [24, Theorem 1.3] a Frattini-injective pro-p group is solvable if, and only if, it is locally uniformly powerful.

A solvable pro-p group does not contain a free nonabelian closed subgroup. For Bloch–Kato pro-p groups—and thus in particular for maximal pro-p Galois groups of fields containing a root of unity of order p—Ware proved the following Tits' alternative: either such a pro-p group contains a free non-abelian closed subgroup; or it is locally uniformly powerful (see [28, Corollary 1] and [17, Theorem B]). We conjecture that the same phenomenon occurs for 1-smooth pro-p groups.

Conjecture 1.5 *Let G be a torsion-free pro-p group which may be endowed with a representation $\theta\colon G \to \mathrm{GL}_1(\mathbb{Z}_p)$ such that $\mathcal{G} = (G, \theta)$ is a 1-smooth pro-p pair. Then either G is locally uniformly powerful, or G contains a closed nonabelian free pro-p group.*

2 Cyclotomic pro-*p* pairs

Henceforth, every subgroup of a pro-p group will be tacitly assumed to be closed, and the generators of a subgroup will be intended in the topological sense.

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In particular, for a pro-p group G and a positive integer n, G^{p^n} will denote the closed subgroup of G generated by the p^nth powers of all elements of G. Moreover, for two elements $g, h \in G$, we set

$$h^g = g^{-1}hg, \qquad \text{and} \qquad [h, g] = h^{-1} \cdot h^g,$$

and for two subgroups H_1, H_2 of G, $[H_1, H_2]$ will denote the closed subgroup of G generated by all commutators $[h, g]$ with $h \in H_1$ and $g \in H_2$. In particular, G' will denote the commutator subgroup $[G, G]$ of G, and the Frattini subgroup $G^p \cdot G'$ of G is denoted by $\Phi(G)$. Finally, $d(G)$ will denote the minimal number of generatord of G, i.e., $d(G) = \dim(G/\Phi(G))$ as a $\mathbb{Z}/p$-vector space.

2.1 Kummerian pro-p pairs

Let $1 + p\mathbb{Z}_p = \{1 + p\lambda \mid \lambda \in \mathbb{Z}_p\} \subseteq \mathrm{GL}_1(\mathbb{Z}_p)$ denote the pro-p Sylow subgroup of the group of units of the ring of p-adic integers $\mathbb{Z}_p$. A pair $\mathcal{G} = (G, \theta)$ consisting of a pro-p group G and a continuous homomorphism

$$\theta\colon G \longrightarrow 1 + p\mathbb{Z}_p$$

is called a *cyclotomic pro-p pair*, and the morphism θ is called an *orientation* of G (cf. [7, Section 3] and [21]).

A cyclotomic pro-p pair $\mathcal{G} = (G, \theta)$ is said to be *torsion-free* if $\mathrm{Im}(\theta)$ is torsion-free: this is the case if p is odd; or if $p = 2$ and $\mathrm{Im}(\theta) \subseteq 1 + 4\mathbb{Z}_2$. Observe that a cyclotomic pro-p pair $\mathcal{G} = (G, \theta)$ may be torsion-free even if G has nontrivial torsion—e.g., if G is the cyclic group of order p and θ is constantly equal to 1. Given a cyclotomic pro-p pair $\mathcal{G} = (G, \theta)$ one has the following constructions:

(a) if H is a subgroup of G, $\mathrm{Res}_H(\mathcal{G}) = (H, \theta|_H)$;
(b) if N is a normal subgroup of G contained in $\mathrm{Ker}(\theta)$, then θ induces an orientation $\bar{\theta}\colon G/N \to 1 + p\mathbb{Z}_p$, and we set $\mathcal{G}/N = (G/N, \bar{\theta})$;
(c) if A is an abelian pro-p group, we set $A \rtimes \mathcal{G} = (A \rtimes G, \theta \circ \pi)$, with $a^g = a^{\theta(g)^{-1}}$ for all $a \in A$, $g \in G$, and π the canonical projection $A \rtimes G \to G$.

Given a cyclotomic pro-p pair $\mathcal{G} = (G, \theta)$, the pro-$p$ group G has two distinguished subgroups:

(a) the subgroup

$$K(\mathcal{G}) = \left\langle h^{-\theta(g)} \cdot h^{g^{-1}} \,\middle|\, g \in G, h \in \mathrm{Ker}(\theta) \right\rangle \tag{2.1}$$

introduced in [9, Section 3];
(b) the θ-center

$$Z(\mathcal{G}) = \left\langle h \in \mathrm{Ker}(\theta) \,\middle|\, ghg^{-1} = h^{\theta(g)} \ \forall\, g \in G \right\rangle \tag{2.2}$$

introduced in [17, Section 1].

Both $Z(\mathcal{G})$ and $K(\mathcal{G})$ are normal subgroups of G, and they are contained in $\mathrm{Ker}(\theta)$. Moreover, $Z(\mathcal{G})$ is abelian, while

$$K(\mathcal{G}) \supseteq \mathrm{Ker}(\theta)', \qquad \text{and} \qquad K(\mathcal{G}) \subseteq \Phi(G).$$

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Thus, the quotient $\mathrm{Ker}(\theta)/K(\mathcal{G})$ is abelian, and if $\mathcal{G}$ is torsion-free one has an isomorphism of pro-p pairs

$$\mathcal{G}/K(\mathcal{G}) \simeq (\mathrm{Ker}(\theta)/K(\mathcal{G})) \rtimes (\mathcal{G}/\mathrm{Ker}(\theta)), \tag{2.3}$$

namely, $G/K(\mathcal{G}) \simeq (\mathrm{Ker}(\theta)/K(\mathcal{G})) \rtimes (G/\mathrm{Ker}(\theta))$ (where the action is induced by θ, in the latter), and both pro-p groups are endowed with the orientation induced by θ (cf. [18, Equation 2.6]).

Definition 2.1 Given a cyclotomic pro-p pair $\mathcal{G} = (G, \theta)$, let $\mathbb{Z}_p(1)$ denote the continuous G-module of rank 1 induced by θ, i.e., $\mathbb{Z}_p(1) \simeq \mathbb{Z}_p$ as abelian pro-p groups, and $g.\lambda = \theta(g) \cdot \lambda$ for every $\lambda \in \mathbb{Z}_p(1)$. The pair $\mathcal{G}$ is said to be *Kummerian* if for every $n \geq 1$ the map

$$H^1(G, \mathbb{Z}_p(1)/p^n) \longrightarrow H^1(G, \mathbb{Z}_p(1)/p), \tag{2.4}$$

induced by the epimorphism of G-modules $\mathbb{Z}_p(1)/p^n \to \mathbb{Z}_p(1)/p$, is surjective. Moreover, $\mathcal{G}$ is *1-smooth* if $\mathrm{Res}_H(\mathcal{G})$ is Kummerian for every subgroup $H \subseteq G$.

Observe that the action of G on $\mathbb{Z}_p(1)/p$ is trivial, as $\mathrm{Im}(\theta) \subseteq 1 + p\mathbb{Z}_p$. We say that a pro-$p$ group G may complete into a Kummerian, or 1-smooth, pro-p pair if there exists an orientation $\theta\colon G \to 1 + p\mathbb{Z}_p$ such that the pair (G, θ) is Kummerian, or 1-smooth.

Kummerian pro-p pairs and 1-smooth pro-p pairs were introduced in [9] and in [5, Section 14] respectively. In [21], if $\mathcal{G} = (G, \theta)$ is a 1-smooth pro-p pair, the orientation θ is said to be *1-cyclotomic*. Note that in [5, Section 14.1], a pro-p pair is defined to be 1-smooth if the maps (2.4) are surjective for every *open* subgroup of G, yet by a limit argument this implies also that the maps (2.4) are surjective also for every *closed* subgroup of G (cf. [21, Corollary 3.2]).

Remark 2.1 Let $\mathcal{G} = (G, \theta)$ be a cyclotomic pro-p pair. Then $\mathcal{G}$ is Kummerian if, and only if, the map

$$H^1_{\mathrm{cts}}(G, \mathbb{Z}_p(1)) \longrightarrow H^1(G, \mathbb{Z}_p(1)/p),$$

induced by the epimorphism of continuous left G-modules $\mathbb{Z}_p(1) \twoheadrightarrow \mathbb{Z}_p(1)/p$, is surjective (cf. [21, Proposition 2.1])—here H^*_{cts} denotes continuous cochain cohomology as introduced by Tate in [26].

One has the following group-theoretic characterization of Kummerian torsion-free pro-p pairs (cf. [9, Theorems 5.6 and 7.1] and [20, Theorem 1.2]).

Proposition 2.2 *A torsion-free cyclotomic pro-p pair* $\mathcal{G} = (G, \theta)$ *is Kummerian if and only if* $\mathrm{Ker}(\theta)/K(\mathcal{G})$ *is a free abelian pro-p group.*

Remark 2.3 Let $\mathcal{G} = (G, \theta)$ be a cyclotomic pro-p pair with $\theta \equiv \mathbf{1}$, i.e., θ is constantly equal to 1. Since $K(\mathcal{G}) = G'$ in this case, $\mathcal{G}$ is Kummerian if and only if the quotient G/G' is torsion-free. Hence, by Proposition 2.2, $\mathcal{G}$ is 1-smooth if and only if H/H' is torsion-free for every subgroup $H \subseteq G$. Pro-p groups with such property are called

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absolutely torsion-free, and they were introduced by Würfel in [30]. In particular, if $\mathcal{G} = (G,\theta)$ is a 1-smooth pro-p pair (with θ nontrivial), then $\mathrm{Res}_{\mathrm{Ker}(\theta)}(\mathcal{G}) = (\mathrm{Ker}(\theta),\mathbf{1})$ is again 1-smooth, and thus $\mathrm{Ker}(\theta)$ is absolutely torsion-free. Hence, a pro-p group which may complete into a 1-smooth pro-p pair is an absolutely torsion-free-by-cyclic pro-p group.

Example 2.4 (a) A cyclotomic pro-p pair (G,θ) with G a free pro-p group is 1-smooth for any orientation $\theta\colon G \to 1+p\mathbb{Z}_p$ (cf. [21, Section 2.2]).

(b) A cyclotomic pro-p pair (G,θ) with G an infinite Demushkin pro-p group is 1-smooth if and only if $\theta\colon G \to 1+p\mathbb{Z}_p$ is defined as in [14, Theorem 4] (cf. [9, Theorem 7.6]). E.g., if G has a minimal presentation

$$G = \left\langle x_1,\dots,x_d \;\middle|\; x_1^{p^f}[x_1,x_2]\cdots[x_{d-1},x_d] = 1 \right\rangle$$

with $f \geq 1$ (and $f \geq 2$ if $p = 2$), then $\theta(x_2) = (1-p^f)^{-1}$, while $\theta(x_i) = 1$ for $i \neq 2$.

(c) For $p \neq 2$ let G be the pro-p group with minimal presentation

$$G = \langle x,y,z \mid [x,y] = z^p \rangle.$$

Then the pro-p pair (G,θ) is not Kummerian for any orientation $\theta\colon G \to 1+p\mathbb{Z}_p$ (cf. [9, Theorem 8.1]).

(d) Let

$$H = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a,b,c \in \mathbb{Z}_p \right\}$$

be the Heisenberg pro-p group. The pair $(H,\mathbf{1})$ is Kummerian, as $H/H' \simeq \mathbb{Z}_p^2$, but H is not absolutely torsion-free. In particular, H can not complete into a 1-smooth pro-p pair (cf. [18, Example 5.4]).

(e) The only 1-smooth pro-p pair (G,θ) with G a finite p-group is the cyclic group of order 2 $G \simeq \mathbb{Z}/2$, endowed with the only nontrivial orientation $\theta\colon G \twoheadrightarrow \{\pm 1\} \subseteq 1+2\mathbb{Z}_2$ (cf. [9, Example 3.5]).

Remark 2.5 By Example 2.4(e), if $\mathcal{G} = (G,\theta)$ is a torsion-free 1-smooth pro-p pair, then G is torsion-free.

A torsion-free pro-p pair $\mathcal{G} = (G,\theta)$ is said to be *θ-abelian* if the following equivalent conditions hold:

(i) $\mathrm{Ker}(\theta)$ is a free abelian pro-p group, and $\mathcal{G} \simeq \mathrm{Ker}(\theta) \rtimes (\mathcal{G}/\mathrm{Ker}(\theta))$;
(ii) $Z(\mathcal{G})$ is a free abelian pro-p group, and $Z(\mathcal{G}) = \mathrm{Ker}(\theta)$;
(iii) $\mathcal{G}$ is Kummerian and $K(\mathcal{G}) = \{1\}$

(cf. [17, Proposition 3.4] and [20, Section 2.3]). Explicitly, a torsion-free pro-p pair $\mathcal{G} = (G,\theta)$ is θ-abelian if and only if G has a minimal presentation

$$G = \left\langle x_0, x_i, i \in I \mid [x_0,x_i] = x_i^q, [x_i,x_j] = 1\ \forall\, i,j \in I \right\rangle \simeq \mathbb{Z}_p^I \rtimes \mathbb{Z}_p \qquad (2.5)$$

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for some set I and some p-power q (possibly $q = p^\infty = 0$), and in this case $\mathrm{Im}(\theta) = 1 + q\mathbb{Z}_p$. In particular, a θ-abelian pro-p pair is also 1-smooth, as every open subgroup U of G is again isomorphic to $\mathbb{Z}_p^I \rtimes \mathbb{Z}_p$, with action induced by $\theta|_U$, and therefore $\mathrm{Res}_U(\mathcal{G})$ is $\theta|_U$-abelian.

Remark 2.6 From [9, Theorem 5.6], one may deduce also the following group-theoretic characterization of Kummerian pro-p pairs: a pro-p group G may complete into a Kummerian oriented pro-p group if, and only if, there exists an epimorphism of pro-p groups $\varphi: G \twoheadrightarrow \bar{G}$ such that $\bar{G}$ has a minimal presentation (2.5), and $\mathrm{Ker}(\varphi)$ is contained in the Frattini subgroup of G (cf., e.g., [22, Proposition 3.11]).

Remark 2.7 If $G \simeq \mathbb{Z}_p$, then the pair (G, θ) is θ-abelian, and thus also 1-smooth, for any orientation $\theta: G \to 1 + p\mathbb{Z}_p$.

On the other hand, if $\mathcal{G} = (G, \theta)$ is a θ-abelian pro-p pair with $d(G) \geq 2$, then θ is the only orientation which may complete G into a 1-smooth pro-p pair. Indeed, let $\mathcal{G}' = (G, \theta')$ be a cyclotomic pro-p pair, with $\theta': G \to 1 + p\mathbb{Z}_p$ different to θ, and let $\{x_0, x_i, i \in I\}$ be a minimal generating set of G as in the presentation (2.5)—thus, $\theta(x_i) = 1$ for all $i \in I$, and $\theta(x_0) \in 1 + q\mathbb{Z}_p$. Then for some $i \in I$ one has $\theta'|_H \not\equiv \theta|_H$, with H the subgroup of G generated by the two elements x_0 and x_i. In particular, one has $\theta([x_0, x_i]) = \theta'([x_0, x_i]) = 1$.

Suppose that $\mathcal{G}'$ is 1-smooth. If $\theta'(x_i) \neq 1$, then

$$x_i^q = x_i \cdot x_i^q \cdot x_i^{-1} = (x_i^q)^{\theta'(x_i)} = x_i^{q\theta'(x_i)},$$

hence $x_i^{q(1-\theta'(x_i))} = 1$, a contradiction as G is torsion-free by Remark 2.5. If $\theta'(x_i) = 1$ then necessarily $\theta'(x_0) \neq \theta(x_0)$, and thus

$$x_i^{\theta(x_0)} = x_0 \cdot x_i \cdot x_0^{-1} = x_i^{\theta'(x_0)},$$

hence $x_i^{\theta(x_0)-\theta'(x_0)} = 1$, again a contradiction as G is torsion-free. (See also [21, Corollary 3.4].)

2.2 The Galois case

Let $\mathbb{K}$ be a field containing a root of 1 of order p, and let μ_{p^∞} denote the group of roots of 1 of order a p-power contained in the separable closure of $\mathbb{K}$. Then $\mu_{p^\infty} \subseteq \mathbb{K}(p)$, and the action of the maximal pro-p Galois group $G_{\mathbb{K}}(p) = \mathrm{Gal}(\mathbb{K}(p)/\mathbb{K})$ on μ_{p^∞} induces a continuous homomorphism

$$\theta_{\mathbb{K}}: G_{\mathbb{K}}(p) \longrightarrow 1 + p\mathbb{Z}_p$$

—called the *pro-p cyclotomic character of* $G_{\mathbb{K}}(p)$—as the group of the automorphisms of μ_{p^∞} which fix the roots of order p is isomorphic to $1 + p\mathbb{Z}_p$ (see, e.g., [8, p. 202] and [9, Section 4]). In particular, if $\mathbb{K}$ contains a root of 1 of order p^k for $k \geq 1$, then $\mathrm{Im}(\theta_{\mathbb{K}}) \subseteq 1 + p^k\mathbb{Z}_p$.

Set $\mathcal{G}_{\mathbb{K}} = (G_{\mathbb{K}}(p), \theta_{\mathbb{K}})$. Then by Kummer theory one has the following (see, e.g., [9, Theorem 4.2]).

Theorem 2.8 *Let $\mathbb{K}$ be a field containing a root of 1 of order p. Then $\mathcal{G}_{\mathbb{K}} = (G_{\mathbb{K}}(p), \theta_{\mathbb{K}})$ is 1-smooth.*

1-smooth pro-p pairs share the following properties with maximal pro-p Galois groups of fields.

Example 2.9 (a) The only finite p-group which occurs as maximal pro-p Galois group for some field $\mathbb{K}$ is the cyclic group of order 2, and this follows from the pro-p version of the Artin–Schreier Theorem (cf. [1]). Likewise, the only finite p-group which may complete into a 1-smooth pro-p pair, is the cyclic group of order 2 (endowed with the only nontrivial orientation onto $\{\pm 1\}$), as it follows from Example 2.4(e) and Remark 2.5.

(b) If x is an element of $G_{\mathbb{K}}(2)$ for some field $\mathbb{K}$ and x has order 2, then x self-centralizes (cf. [4, Proposition 2.3]). Likewise, if x is an element of a pro- 2 group G which may complete into a 1-smooth pro-2 pair, then x self-centralizes (cf. [21, Section 6.1]).

2.3 Bloch–Kato and the Smoothness Conjecture

A non-negatively graded algebra $A_\bullet = \bigoplus_{n\geq 0} A_n$ over a field $\mathbb{F}$, with $A_0 = \mathbb{F}$, is called a *quadratic algebra* if it is one-generated—i.e., every element is a combination of products of elements of degree 1—and its relations are generated by homogeneous relations of degree 2. One has the following definitions (cf. [5, Definition 14.21] and [17, Section 1]).

Definition 2.2 Let G be a pro-p group, and let $n \geq 1$. Cohomology classes in the image of the natural cup-product

$$H^1(G, \mathbb{Z}/p) \times \ldots \times H^1(G, \mathbb{Z}/p) \xrightarrow{\cup} H^n(G, \mathbb{Z}/p)$$

are called *symbols* (relative to $\mathbb{Z}/p$, wieved as trivial G-module).

(i) If for every open subgroup $U \subseteq G$ every element $\alpha \in H^n(U, \mathbb{Z}/p)$, for every $n \geq 1$, can be written as

$$\alpha = \mathrm{cor}^n_{V_1,U}(\alpha_1) + \cdots + \mathrm{cor}^n_{V_r,U}(\alpha_r),$$

with $r \geq 1$, where $\alpha_i \in H^n(V_i, \mathbb{Z}/p)$ is a symbol and

$$\mathrm{cor}^n_{V_i,U}\colon H^n(V_i, \mathbb{Z}/p) \longrightarrow H^n(U, \mathbb{Z}/p)$$

is the *corestriction map* (cf. [16, Chapter I, Section 5]), for some open subgroups $V_i \subseteq U$, then G is called a *weakly Bloch–Kato pro- p group*.

(ii) If for every closed subgroup $H \subseteq G$ the $\mathbb{Z}/p$-cohomology algebra

$$H^\bullet(H, \mathbb{Z}/p) = \bigoplus_{n\geq 0} H^n(H, \mathbb{Z}/p),$$

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endowed with the cup-product, is a quadratic algebra over $\mathbb{Z}/p$, then G is called a *Bloch–Kato pro- p group*. As the name suggests, a Bloch–Kato pro-p group is also weakly Bloch-Kato.

By the Norm Residue Theorem, if $\mathbb{K}$ contains a root of unity of order p, then the maximal pro-p Galois group $G_{\mathbb{K}}(p)$ is Bloch–Kato. The pro-p version of the "Smoothness Conjecture," formulated by De Clerq and Florence, states that being 1-smooth is a sufficient condition for a pro-p group to be weakly Bloch–Kato (cf. [5, Conjugation 14.25]).

Conjecture 2.10 *Let $\mathcal{G} = (G, \theta)$ be a 1-smooth pro-p pair. Then G is weakly Bloch–Kato.*

In the case of $\mathcal{G} = \mathcal{G}_{\mathbb{K}}$ for some field $\mathbb{K}$ containing a root of 1 of order p, using Milnor K-theory one may show that the weak Bloch–Kato condition implies that $H^\bullet(G, \mathbb{Z}/p)$ is one-generated (cf. [5, Rem. 14.26]). In view of Theorem 2.8, a positive answer to the Smoothness Conjecture would provide a new proof of the surjectivity of the norm residue isomorphism, i.e., the "surjectivity" half of the Bloch–Kato conjecture (cf. [5, Section 1.1]).

Conjecture 2.10 has been settled positively for the following classes of pro-p groups.

(a) Finite p-groups: indeed, if $\mathcal{G} = (G, \theta)$ is a 1-smooth pro-p pair with G a finite (nontrivial) p-group, then by Example 2.4–(e) $p = 2$, G is a cyclic group of order two and $\theta: G \twoheadrightarrow \{\pm 1\}$, so that $\mathcal{G} \simeq (\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \theta_{\mathbb{R}})$, and G is Bloch–Kato.

(b) Analytic pro-p groups: indeed if $\mathcal{G} = (G, \theta)$ is a 1-smooth pro-p pair with G a p-adic analytic pro-p group, then by [18, Theorem 1.1] G is locally uniformly powerful and thus Bloch–Kato (see §3.1 below).

(c) Pro-p completions of right-angled Artin groups: indeed, in [25], it is shown that if $\mathcal{G} = (G, \theta)$ is a 1-smooth pro-p pair with G the pro-p completion of a right-angled Artin group induced by a simplicial graph Γ, then necessarily θ is trivial and Γ has the diagonal property—namely, G may be constructed starting from free pro-p groups by iterating the following two operations: free pro-p products, and direct products with $\mathbb{Z}_p$—and thus G is Bloch–Kato (cf. [25, Theorem 1.2]).

3 Normal abelian subgroups

3.1 Powerful pro-*p* groups

Definition 3.1 A finitely generated pro-p group G is said to be *powerful* if one has $G' \subseteq G^p$, and also $G' \subseteq G^4$ if $p = 2$. A powerful pro-p group which is also torsion-free and finitely generated is called a *uniformly powerful* pro-p group.

For the properties of powerful and uniformly powerful pro-p groups, we refer to [6, Chapter 4].

A pro-p group whose finitely generated subgroups are uniformly powerful, is said to be *locally uniformly powerful*. As mentioned in Section 1, a pro-p group G is locally

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uniformly powerful if, and only if, G has a minimal presentation (2.5)—i.e., G is locally powerful if, and only if, there exists an orientation $\theta\colon G \to 1 + p\mathbb{Z}_p$ such that (G, θ) is a torsion-free θ-abelian pro-p pair (cf. [17, Theorem A] and [3, Proposition 3.5]).

Therefore, a locally uniformly powerful pro-p group G comes endowed automatically with an orientation $\theta\colon G \to 1 + p\mathbb{Z}_p$ such that $\mathcal{G} = (G, \theta)$ is a 1-smooth pro-p pair. In fact, finitely generated locally uniformly powerful pro-p groups are precisely those uniformly powerful pro-p groups which may complete into a 1-smooth pro-p pair (cf. [18, Proposition 4.3]).

Proposition 3.1 *Let $\mathcal{G} = (G, \theta)$ be a 1-smooth torsion-free pro-p pair. If G is locally powerful, then $\mathcal{G}$ is θ-abelian, and thus G is locally uniformly powerful.*

It is well-known that the $\mathbb{Z}/p$-cohomology algebra of a pro-p group G with minimal presentation (2.5) is the exterior $\mathbb{Z}/p$-algebra

$$H^\bullet(H, \mathbb{Z}/p) \simeq \bigwedge_{n \geq 0} H^1(H, \mathbb{Z}/p)$$

—if $p = 2$ then $\bigwedge_{n\geq 0} V$ is defined to be the quotient of the tensor algebra over $\mathbb{Z}/p$ generated by V by the two-sided ideal generated by the elements $v \otimes v$, $v \in V$—so that $H^\bullet(G, \mathbb{Z}/p)$ is quadratic. Moreover, every subgroup $H \subseteq G$ is again locally uniformly powerful, and thus also $H^\bullet(H, \mathbb{Z}/p)$ is quadratic. Hence, a locally uniformly powerful pro-p group is Bloch–Kato.

3.2 Normal abelian subgroups of maximal pro-p Galois groups

Let $\mathbb{K}$ be a field containing a root of 1 of order p (and also $\sqrt{-1}$ if $p = 2$). In Galois theory, one has the following result, due to Engler et al. (cf. [11] and [10]).

Theorem 3.2 *Let $\mathbb{K}$ be a field containing a root of 1 of order p (and also $\sqrt{-1}$ if $p = 2$), and suppose that the maximal pro-p Galois group $G_{\mathbb{K}}(p)$ of $\mathbb{K}$ is not isomorphic to $\mathbb{Z}_p$. Then $G_{\mathbb{K}}(p)$ contains a unique maximal abelian normal subgroup.*

By [21, Theorem 7.7], such a maximal abelian normal subgroup coincides with the $\theta_{\mathbb{K}}$-center $Z(\mathcal{G}_{\mathbb{K}})$ of the pro-p pair $\mathcal{G}_{\mathbb{K}} = (G_{\mathbb{K}}(p), \theta_{\mathbb{K}})$ induced by the pro-p cyclotomic character $\theta_{\mathbb{K}}$ (cf. §2.2). Moreover, the field $\mathbb{K}$ admits a *p-Henselian valuation* with residue characteristic not p and non-p-divisible value group, such that the *residue field* κ of such a valuation gives rise to the cyclotomic pro-p pair $\mathcal{G}_\kappa$ isomorphic to $\mathcal{G}_{\mathbb{K}}/Z(\mathcal{G}_{\mathbb{K}})$, and the induced short exact sequence of pro-p groups

$$\{1\} \longrightarrow Z(\mathcal{G}_{\mathbb{K}}) \longrightarrow G_{\mathbb{K}}(p) \longrightarrow G_\kappa(p) \longrightarrow \{1\} \tag{3.1}$$

splits (cf. [10, Section 1] and [8, Example 22.1.6]—for the definitions related to p-henselian valuations of fields, we direct the reader to [8, Section 15.3]). In particular, $G_{\mathbb{K}}(p)/Z(\mathcal{G}_{\mathbb{K}})$ is torsion-free.

Remark 3.3 By [21, Theorems 1.2 and 7.7], Theorem 3.2 and the splitting of (3.1) generalize to 1-smooth pro-p pairs whose underlying pro-p group is Bloch–Kato.

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Namely, if $\mathcal{G} = (G, \theta)$ is a 1-smooth pro-p pair with G a Bloch–Kato pro-p group, then $Z(\mathcal{G})$ is the unique maximal abelian normal subgroup of G, and it has a complement in G.

3.3 Proof of Theorem 1.1

In order to prove Theorem 1.1 (and also Theorem 1.2 later on), we need the following result.

Proposition 3.4 *Let $\mathcal{G} = (G, \theta)$ be a torsion-free 1-smooth pro-p pair, with $d(G) = 2$ and $G = \langle x, y\rangle$. If $[[x, y], y] = 1$, then* $\mathrm{Ker}(\theta) = \langle y\rangle$ *and*

$$xyx^{-1} = y^{\theta(x)}.$$

Proof Let H be the subgroup of G generated by y and $[x, y]$. Recall that by Remark 2.5, G (and hence also H) is torsion-free.

If $d(H) = 1$ then $H \simeq \mathbb{Z}_p$, as H is torsion-free. Moreover, H is generated by y and $x^{-1}yx$, and thus $xHx^{-1} \subseteq H$. Therefore, x acts on $H \simeq \mathbb{Z}_p$ by multiplication by $1 + p\lambda$ for some $\lambda \in \mathbb{Z}_p$. If $\lambda = 0$ then G is abelian, and thus $G \simeq \mathbb{Z}_p^2$ as it is absolutely torsion-free, and $\theta \equiv \mathbf{1}$ by Remark 2.7. If $\lambda \neq 0$ then x acts nontrivially on the elements of H, and thus $\langle x\rangle \cap H = \{1\}$ and $G = H \rtimes \langle x\rangle$: by (2.5), (G, θ') is a θ'-abelian pro-p pair, with $\theta' \colon G \to 1 + p\mathbb{Z}_p$ defined by $\theta'(x) = 1 + p\lambda$ and $\theta'(y) = 1$. By Remark 2.7, one has $\theta' \equiv \theta$, and thus $\theta(x) = 1 + p\lambda$ and $\theta(y) = 1$.

If $d(H) = 2$, then H is abelian by hypothesis, and torsion-free, and thus (H, θ') is θ'-abelian, with $\theta' \equiv \mathbf{1} \colon H \to 1 + p\mathbb{Z}_p$ trivial. By Remark 2.7, one has $\theta' = \theta|_H$, and thus $y, [x, y] \in \mathrm{Ker}(\theta)$. Now put $z = [x, y]$ and $t = y^p$, and let U be the open subgroup of G generated by x, z, t. Clearly, $\mathrm{Res}_U(\mathcal{G})$ is again 1-smooth. By hypothesis one has $z^y = z$, and hence commutator calculus yields

$$[x, t] = [x, y^p] = z \cdot z^y \cdots z^{y^{p-1}} = z^p. \tag{3.2}$$

Put $\lambda = 1 - \theta(x)^{-1} \in p\mathbb{Z}_p$. Since $t \in \mathrm{Ker}(\theta)$, by (2.1) $[x, t] \cdot t^{-\lambda}$ lies in $K(\mathrm{Res}_U(\mathcal{G}))$. Since t and z commute, from (3.2) one deduces

$$[x, t]t^{-\lambda} = z^p t^{-\lambda} = z^p t^{-\frac{\lambda}{p}p} = \left(zt^{-\lambda/p}\right)^p \in K(\mathrm{Res}_U(\mathcal{G})). \tag{3.3}$$

Moreover, $zt^{-\lambda/p} \in \mathrm{Ker}(\theta|_U)$. Since $\mathrm{Res}_U(\mathcal{G})$ is 1-smooth, by Proposition 2.2, the quotient $\mathrm{Ker}(\theta|_U)/K(\mathrm{Res}_U(\mathcal{G}))$ is a free abelian pro-p group, and therefore (3.3) implies that also $zt^{-\lambda/p}$ is an element of $K(\mathrm{Res}_U(\mathcal{G}))$.

Since $K(\mathrm{Res}_U(\mathcal{G})) \subseteq \Phi(U)$, one has $z \equiv t^{\lambda/p} \bmod \Phi(U)$. Then by [6, Proposition 1.9] $d(U) = 2$ and U is generated by x and t. Since $[x, t] \in U^p$ by (3.2), the pro-p group U is powerful. Therefore, $\mathrm{Res}_U(\mathcal{G})$ is $\theta|_U$-abelian by Proposition 3.1. In particular, the subgroup $K(\mathrm{Res}_U(\mathcal{G}))$ is trivial, and thus

$$[x, y] = z = t^{\lambda/p} = y^{1-\theta(x)^{-1}},$$

and the claim follows. ∎

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Proposition 3.4 is a generalization of [18, Proposition 5.6].

Theorem 3.5 *Let $\mathcal{G} = (G, \theta)$ be a torsion-free 1-smooth pro-p pair, with $d(G) \geq 2$.*

(i) *The θ-center $Z(\mathcal{G})$ is the unique maximal abelian normal subgroup of G.*
(ii) *The quotient $G/Z(\mathcal{G})$ is a torsion-free pro-p group.*

Proof Recall that G is torsion-free by Remark 2.5. Since $Z(\mathcal{G})$ is an abelian normal subgroup of G by definition, in order to prove (i) we need to show that if A is an abelian normal subgroup of G, then $A \subseteq Z(\mathcal{G})$.

First, we show that $A \subseteq \mathrm{Ker}(\theta)$. If $A \simeq \mathbb{Z}_p$, let y be a generator of A. For every $x \in G$ one has $xyx^{-1} \in A$, and thus $xyx^{-1} = y^{\lambda}$, for some $\lambda \in 1 + p\mathbb{Z}_p$. Let H be the subgroup of G generated by x and y, for some $x \in G$ such that $d(H) = 2$. Then the pair (H, θ') is θ'-abelian for some orientation $\theta' : H \to 1 + p\mathbb{Z}_p$ such that $y \in \mathrm{Ker}(\theta')$, as H has a presentation as in (2.5). Since both $\mathrm{Res}_H(\mathcal{G})$ and (H, θ') are 1-smooth pro-p pairs, by Remark 2.7, one has $\theta' = \theta|_H$, and thus $A \subseteq \mathrm{Ker}(\theta)$.

If $A \not\simeq \mathbb{Z}_p$, then A is a free abelian pro-p group with $d(A) \geq 2$, as G is torsion-free. Therefore, by Remark 2.3 the pro-p pair $(A, \mathbf{1})$ is 1-smooth. Since also $\mathrm{Res}_A(\mathcal{G})$ is 1-smooth, Remark 2.7 implies that $\theta|_A = \mathbf{1}$, and hence $A \subseteq \mathrm{Ker}(\theta)$.

Now, for arbitrary elements $x \in G$ and $y \in A$, put $z = [x, y]$. Since A is normal in G, one has $z \in A$, and since A is abelian, one has $[z, y] = 1$. Then Proposition 3.4 applied to the subgroup of G generated by $\{x, y\}$ yields $xyx^{-1} = x^{\theta(x)}$, and this completes the proof of statement (i).

In order to prove statement (ii), suppose that $y^p \in Z(\mathcal{G})$ for some $y \in G$. Then $y^p \in \mathrm{Ker}(\theta)$, and since $\mathrm{Im}(\theta)$ has no nontrivial torsion, also y lies in $\mathrm{Ker}(\theta)$. Since G is torsion-free by Remark 2.5, $y^p \neq 1$. Let H be the subgroup of G generated by y and x, for some $x \in G$ such that $d(H) \geq 2$. Since $xy^px^{-1} = (y^p)^{\theta(x)}$, commutator calculus yields

$$y^{p(1-\theta(x)^{-1})} = [x, y^p] = [x, y] \cdot [x, y]^y \cdots [x, y]^{y^{p-1}}. \tag{3.4}$$

Put $z = [x, y]$, and let S be the subgroup of H generated by y, z. Clearly, $\mathrm{Res}_S(\mathcal{G})$ is 1-smooth, and since $y, z \in \mathrm{Ker}(\theta)$, one has $\theta|_S = \mathbf{1}$, and thus S/S' is a free abelian pro-p group by Remark 2.3. From (3.4) one deduces

$$y^{p(1-\theta(x)^{-1})} \cdot z^{-p} \equiv \left(y^{1-\theta(x)^{-1}} \cdot z^{-1}\right)^p \equiv 1 \bmod S'. \tag{3.5}$$

Since S/S' is torsion-free, (3.5) implies that $z \equiv y^{1-\theta(x)^{-1}} \bmod \Phi(S)$, so that S is generated by y, and $S \simeq \mathbb{Z}_p$, as G is torsion-free. Therefore, $S' = \{1\}$, and (3.5) yields $[x, y] = y^{1-\theta(x)^{-1}}$, and this completes the proof of statement (ii). ∎

Remark 3.6 Let G be a pro-p group isomorphic to $\mathbb{Z}_p$, and let $\theta : G \to 1 + p\mathbb{Z}_p$ be a nontrivial orientation. Then by Example 2.4(a), $\mathcal{G} = (G, \theta)$ is 1-smooth. Since G is abelian and $\theta(x) \neq 1$ for every $x \in G$, $x \neq 1$, $Z(\mathcal{G}) = \{1\}$, still every subgroup of G is normal and abelian.

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In view of the splitting of (3.1) (and in view of Remark 3.3), it seems natural to ask the following question.

Question 3.7 *Let $\mathcal{G} = (G, \theta)$ be a torsion-free 1-smooth pro-p pair, with $d(G) \geq 2$. Is the pro-p pair $\mathcal{G}/Z(\mathcal{G}) = (G/Z(\mathcal{G}), \bar{\theta})$ 1-smooth? Does the short exact sequence of pro-p groups*

$$\{1\} \longrightarrow Z(\mathcal{G}) \longrightarrow G \longrightarrow G/Z(\mathcal{G}) \longrightarrow \{1\}$$

split?

If $\mathcal{G} = (G, \theta)$ is a torsion-free pro-p pair, then either $\operatorname{Ker}(\theta) = G$, or $\operatorname{Im}(\theta) \simeq \mathbb{Z}_p$, hence in the former case one has $G \simeq \operatorname{Ker}(\theta) \rtimes (G/\operatorname{Ker}(\theta))$, as the right-side factor is isomorphic to $\mathbb{Z}_p$, and thus p-projective (cf. [16, Chapter III, Section 5]). Since $Z(\mathcal{G}) \subseteq Z(\operatorname{Ker}(\theta))$ (and $Z(\mathcal{G}) = Z(G)$ if $\operatorname{Ker}(\theta) = G$), and since $\operatorname{Ker}(\theta)$ is absolutely torsion-free if $\mathcal{G}$ is 1-smooth, Question 3.7 is equivalent to the following question (of its own group-theoretic interest): if G is an absolutely torsion-free pro-p group, does G split as direct product

$$G \simeq Z(G) \times (G/Z(G))\,?$$

One has the following partial answer (cf. [30, Proposition 5]): if G is absolutely torsion-free, and $Z(G)$ is finitely generated, then $\Phi_n(G) = Z(\Phi_n(G)) \times H$, for some $n \geq 1$ and some subgroup $H \subseteq \Phi_n(G)$ (here $\Phi_n(G)$ denotes the iterated Frattini series of G, i.e., $\Phi_1(G) = G$ and $\Phi_{n+1}(G) = \Phi(\Phi_n(G))$ for $n \geq 1$).

4 Solvable pro-*p* groups

4.1 Solvable pro-*p* groups and maximal pro-*p* Galois groups

Recall that a (pro-p) group G is said to be meta-abelian if there is a short exact sequence

$$\{1\} \longrightarrow N \longrightarrow G \longrightarrow \bar{G} \longrightarrow \{1\}$$

such that both N and $\bar{G}$ are abelian; or, equivalently, if the commutator subgroup G' is abelian. Moreover, a pro-p group G is solvable if the derived series $(G^{(n)})_{n\geq 1}$ of G—i.e., $G^{(1)} = G$ and $G^{(n+1)} = [G^{(n)}, G^{(n)}]$—is finite, namely $G^{(N+1)} = \{1\}$ for some finite N.

Example 4.1 A nonabelian locally uniformly powerful pro-p group G is meta-abelian: if $\theta\colon G \to 1 + p\mathbb{Z}_p$ is the associated orientation, then $G' \subseteq \operatorname{Ker}(\theta)^p$, and thus G' is abelian.

In Galois theory, one has the following result by Ware (cf. [28, Theorem 3], see also [13] and [17, Theorem 4.6]).

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Theorem 4.2 *Let $\mathbb{K}$ be a field containing a root of 1 of order p (and also $\sqrt{-1}$ if $p = 2$). If the maximal pro-p Galois group $G_{\mathbb{K}}(p)$ is solvable, then $\mathcal{G}_{\mathbb{K}}$ is $\theta_{\mathbb{K}}$-abelian.*

4.2 Proof of Theorem 1.2 and Corollary 1.3

In order to prove Theorem 1.2, we prove first the following intermediate results—a consequence of Würfel's result [30, Proposition 2] —, which may be seen as the "1-smooth analogue" of [28, Theorem 2].

Proposition 4.3 *Let $\mathcal{G} = (G, \theta)$ be a torsion-free 1-smooth pro-p pair. If G is meta-abelian, then $\mathcal{G}$ is θ-abelian.*

Proof Assume first that $\theta \equiv \mathbf{1}$—i.e., G is absolutely torsion-free (cf. Remark 2.3). Then G is a free abelian pro-p group by [30, Proposition 2].

Assume now that $\theta \not\equiv \mathbf{1}$. Since $\mathcal{G}$ is 1-smooth, also $\mathrm{Res}_{\mathrm{Ker}(\theta)}(\mathcal{G})$ and $\mathrm{Res}_{\mathrm{Ker}(\theta)'}(\mathcal{G})$ are 1-smooth pro-p pairs, and thus $\mathrm{Ker}(\theta)$ and $\mathrm{Ker}(\theta)'$ are absolutely torsion-free. Moreover, $\mathrm{Ker}(\theta)' \subseteq G'$, and since the latter is abelian, also $\mathrm{Ker}(\theta)'$ is abelian, i.e., $\mathrm{Ker}(\theta)$ is meta-abelian. Thus $\mathrm{Ker}(\theta)$ is a free abelian pro-p group by [30, Proposition 2]. Consequently, for arbitrary $y \in \mathrm{Ker}(\theta)$ and $x \in G$, the commutator $[x, y]$ lies in $\mathrm{Ker}(\theta)$ and $[[x, y], y] = 1$. Therefore, Proposition 3.4 implies that $xyx^{-1} = y^{\theta(y)}$ for every $x \in G$ and $y \in \mathrm{Ker}(\theta)$, namely, $\mathcal{G}$ is θ-abelian. ■

Note that Proposition 4.3 generalizes [30, Proposition 2] from absolutely torsion-free pro-p groups to 1-smooth pro-p groups. From Proposition 4.3, we may deduce Theorem 1.2.

Proposition 4.4 *Let $\mathcal{G} = (G, \theta)$ be a torsion-free 1-smooth pro-p pair. If G is solvable, then G is locally uniformly powerful.*

Proof Let N be the positive integer such that $G^{(N)} \neq \{1\}$ and $G^{(N+1)} = \{1\}$. Then for every $1 \leq n \leq N$, the pro-p pair $\mathrm{Res}_{G^{n}}(\mathcal{G})$ is 1-smooth, and $G^{(n)}$ is solvable, and moreover $\theta|_{G^{(n)}} \equiv \mathbf{1}$ if $n \geq 2$.

Suppose that $N \geq 3$. Since $G^{(N-1)}$ is meta-abelian and $\theta|_{G^{(N-1)}} \equiv \mathbf{1}$, Proposition 4.3 implies that $G^{(N-1)}$ is a free abelian pro-p group, and therefore $G^{(N)} = \{1\}$, a contradiction. Thus, $N \leq 2$, and G is meta-abelian. Therefore, Proposition 4.3 implies that the pro-p pair $\mathcal{G}$ is θ-abelian, and hence G is locally uniformly powerful (cf. §3.1). ■

Proposition 4.4 may be seen as the 1-smooth analogue of Ware's Theorem 4.2. Corollary 1.3 follows from Proposition 4.4 and from the fact that a locally uniformly powerful pro-p group is Bloch–Kato (cf. §3.1).

Corollary 4.5 *Let $\mathcal{G} = (G, \theta)$ be a torsion-free 1-smooth pro-p pair. If G is solvable, then G is Bloch–Kato.*

This settles the Smoothness Conjecture for the class of solvable pro-p groups.

4.3 A Tits' alternative for 1-smooth pro-p groups

For maximal pro-p Galois groups of fields one has the following Tits' alternative (cf. [28, Corollary 1]).

Theorem 4.6 *Let $\mathbb{K}$ be a field containing a root of 1 of order p (and also $\sqrt{-1}$ if $p = 2$). Then either $\mathcal{G}_{\mathbb{K}}$ is $\theta_{\mathbb{K}}$-abelian, or $G_{\mathbb{K}}(p)$ contains a closed nonabelian free pro-p group.*

Actually, the above Tits' alternative holds also for the class of Bloch–Kato pro-p groups, with p odd: if a Bloch–Kato pro-p group G does not contain any free nonabelian subgroups, then it can complete into a θ-abelian pro-p pair $\mathcal{G} = (G, \theta)$ (cf. [17, Theorem B], this Tits' alternative holds also for $p = 2$ under the further assumption that the Bockstein morphism $\beta\colon H^1(G, \mathbb{Z}/2) \to H^2(G, \mathbb{Z}/2)$ is trivial, see [17, Theorem 4.11]).

Clearly, a solvable pro-p group contains no free nonabelian subgroups.

A pro-p group is *p-adic analytic* if it is a p-adic analytic manifold and the map $(x, y) \mapsto x^{-1}y$ is analytic, or, equivalently, if it contains an open uniformly powerful subgroup (cf. [6, Theorem 8.32])—e.g., the Heisenberg pro-p group is analytic. Similarly to solvable pro-p groups, a p-adic analytic pro-p group does not contain a free nonabelian subgroup (cf. [6, Corollary 8.34]).

Even if there are several p-adic analytic pro-p groups which are solvable (e.g., finitely generated locally uniformly powerful pro-p groups), none of these two classes of pro-p groups contains the other one: e.g.,

(a) the wreath product $\mathbb{Z}_p \wr \mathbb{Z}_p \simeq \mathbb{Z}_p^{\mathbb{Z}_p} \rtimes \mathbb{Z}_p$ is a meta-abelian pro-p group, but it is not p-adic analytic (cf. [23]) and

(b) if G is a pro-p-Sylow subgroup of $\mathrm{SL}_2(\mathbb{Z}_p)$, then G is a p-adic analytic pro-p group, but it is not solvable.

In addition, it is well-known that also for the class of pro-p completions of right-angled Artin pro-p groups one has a Tits' alternative: the pro-p completion of a right-angled Artin pro-p group contains a free nonabelian subgroup unless it is a free abelian pro-p group (i.e., unless the associated graph is complete)—and thus it is locally uniformly powerful.

In [18], it is shown that analytic pro-p groups which may complete into a 1-smooth pro-p pair are locally uniformly powerful. Therefore, after the results in [18] and [25], and Theorem 1.2, it is natural to ask whether a Tits' alternative, analogous to Theorem 4.6 (and its generalization to Bloch–Kato pro-p groups), holds also for all torsion-free 1-smooth pro-p pairs.

Question 4.7 *Let $\mathcal{G} = (G, \theta)$ be a torsion-free 1-smooth pro-p pair, and suppose that $\mathcal{G}$ is not θ-abelian. Does G contain a closed nonabelian free pro-p group?*

In other words, we are asking whether there exists torsion-free 1-smooth pro-p pairs $\mathcal{G} = (G, \theta)$ such that G is not analytic nor solvable, and yet it contains no free nonabelian subgroups. In view of Theorem 4.6 and of the Tits' alternative for

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Bloch–Kato pro-p groups [17, Theorem B], a positive answer to Question 4.7 would corroborate the Smoothness Conjecture.

Observe that—analogously to Question 3.7—Question 4.7 is equivalent to asking whether an absolutely torsion-free pro-p group which is not abelian contains a closed nonabelian free subgroup. Indeed, by Proposition 3.4 (in fact, just by [18, Proposition 5.6]), if $\mathcal{G} = (G, \theta)$ is a torsion-free 1-smooth pro-p pair and $\mathrm{Ker}(\theta)$ is abelian, then $\mathcal{G}$ is θ-abelian.

Acknowledgment The author thanks I. Efrat, J. Minac, N.D. Tân, and Th. Weigel for working together on maximal pro-p Galois groups and their cohomology; and P. Guillot and I. Snopce for the interesting discussions on 1-smooth pro-p groups. Also, the author wishes to thank the editors of CMB-BMC, for their helpfulness, and the anonymous referee.

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https://doi.org/10.4153/S0008439521000461 Published online by Cambridge University Press