

Forum

Orthodromes and Loxodromes in Marine Navigation

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Procedures based on solutions of the direct and inverse geodetic problems are presented and their application in orthodromic and loxodromic navigation is analysed. They can be used in traditional navigation during a route planning and during sailing on the planned route in real time. The optimal division of orthodromes during planning and locally predicted short loxodromes during sailing – which allow orthodromic navigation also in manual steering – are proposed.

KEYWORDS

1. Direct and Inverse Geodesic Solutions.
2. Orthodromic Navigation.
3. Loxodromic Navigation.
4. Route Planning.
5. Path Optimisation.

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1. INTRODUCTION. In traditional navigation, to reduce the number of calculations, calculations are done with many simplifications, for example on a plane or a sphere instead of an ellipsoid. These simplifications have been necessary and justified in times of manual mechanical or electronic calculators, but are completely unnecessary and unjustified in times of computer calculations. Additionally, contemporary position fixing devices allow more accurate and more sophisticated navigation than in traditional navigation on a sphere where errors up to $\pm 0.5\%$ of distances can be expected (Earle (2006) and Lenart (2013)). Also orthodromes and loxodromes are calculated on a sphere as well as intermediate points (waypoints) to navigate between them along loxodromes.

In Lenart (2011) and Lenart (2013) a set of procedures for calculating orthodromes (defined as the path of the shortest distance on any surface) and loxodromes by the application of solutions of the problems known in geodesy as the direct and the inverse geodetic problems have been presented.

In formal notations:

$$S = \text{IGP}(\varphi_1, \lambda_1, \varphi_2, \lambda_2) \tag{1}$$

$$C_{gs}, C_{ge} = \text{IGP}(\varphi_1, \lambda_1, \varphi_2, \lambda_2) \tag{2}$$

$$S_{lx} = \text{LX}(\varphi_1, \lambda_1, \varphi_2, \lambda_2) \tag{3}$$

$$C_{glx} = \text{LX}(\varphi_1, \lambda_1, \varphi_2, \lambda_2) \tag{4}$$

$$\varphi_2, \lambda_2 = \text{DGP}(\varphi_1, \lambda_1, S, C_{gs}) \tag{5}$$

$$\varphi_i, \lambda_i = \text{DGPN}(\varphi_1, \lambda_1, \varphi_2, \lambda_2, n) \tag{6}$$

where $P_1(\varphi_1, \lambda_1)$ and $P_2(\varphi_2, \lambda_2)$ are the departure point and the destination point respectively (Figure 1), S is the orthodromic distance, C_{gs} is the Course Over the Ground (COG) at the departure point of the orthodrome and C_{ge} is the COG at the destination point of the orthodrome, S_{lx} is the loxodromic distance, C_{glx} is the loxodromic COG, φ_i, λ_i are coordinates of intermediate points on the orthodrome, IGP is the procedure of the inverse geodetic problem solution, LX is the procedure of loxodromic calculations, DGP is the procedure of the direct geodetic problem solution and DGPN is the procedure of the direct geodetic problem solution for n suborthodromes.

In this paper these procedures, with results based on full accuracy Sodano’s solutions (Sodano, 1958; 1965; 1967) on WGS-84 reference ellipsoid (as in Lenart (2011) and Lenart (2013)) will be used in general formal form and in two modes – during planning of a voyage route (Section 2) and during sailing on the planned route (Section 3).

2. ROUTE PLANNING.

2.1. *Exemplary voyage.* An exemplary route from Cape Horn ($-55^\circ 59', -67^\circ 17'$) to Sydney ($-33^\circ 50', 151^\circ 17'$) is assumed – where north latitudes and east longitudes are positive and south latitudes and west longitudes are negative. This route is intentionally at higher latitudes to magnify all differences. For this route

$$S = 5077.7 \text{ NM}$$

$$S_{lx} = 6063.8 \text{ NM}$$

$$\Delta S_{lx} = S_{lx} - S = 986.1 \text{ NM or } 19.42\% \text{ of } S$$

2.2. *Intermediate points on the orthodrome.* The orthodrome of the length S will be divided for n suborthodromes (Figure 1) of any length S_i such that

$$\sum_{i=1}^n S_i = S \tag{7}$$

In traditional navigation, n intermediate points are calculated during a route planning to navigate between them along loxodromes i.e:

$$\varphi_i, \lambda_i = \text{DGPN}(\varphi_1, \lambda_1, \varphi_2, \lambda_2, n) \tag{8}$$

$$S_{lxi} = \text{LX}(\varphi_i, \lambda_i, \varphi_{i+1}, \lambda_{i+1}) \tag{9}$$

$$S_{lxn} = \sum_{i=1}^n S_{lxi} \tag{10}$$

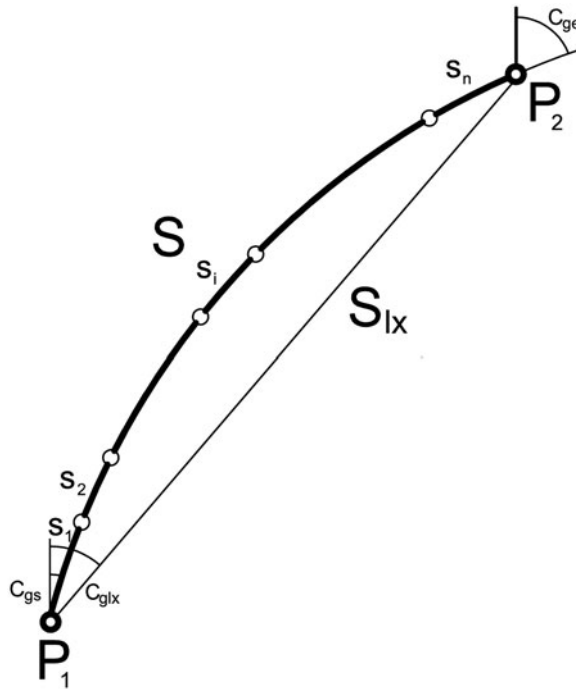


Figure 1. Orthodrome and loxodrome parameters.

These intermediate points will be calculated for $n = 5, 10, 15, 20$ in two options – for n uniform suborthodromes or S_i such as to obtain

$$\Delta S_{lxn} = S_{lxn} - S = \min. \quad (11)$$

The solution in the second option is theoretically very complicated (optimisation with n variables) but with a suitable programming tool (for example Microsoft Excel add-in Solver) is very fast and easy. Results of ΔS_{lxn} for different n are given in Table 1 and illustrated in Figure 2 and all intermediate points in two options for $n = 10$ in Tables 2 and 3. As can be seen in Table 1 this optimisation gives almost 40% lower ΔS_{lxn} for the same n for this exemplary voyage. This difference can be bigger for orthodromes with bigger ΔS_{lx} and smaller for orthodromes with smaller ΔS_{lx} (e.g. 15% for a quite average orthodrome with $\Delta S_{lx} = 5.7\%$ and maximum latitude 50°) but always exists (unless $\Delta S_{lx} = 0$). Table 3 shows that in optimal distribution shorter segments are at higher latitudes (the shortest at maximum latitude) and longer at lower latitudes but such are results, not assumptions.

3. SAILING ON THE PLANNED ROUTE

3.1. *Course and course over the ground.* The most important thing when sailing on the planned route in manual steering is to follow the calculated C_{glxi} whereas the course set and stabilised by an autopilot is the course through the water. Fortunately COG can be obtained from a Global Positioning System (GPS) receiver, therefore

Table 1. ΔS_{ixn} for uniform and optimal division of the orthodrome.

n	5	10	15	20	
Uniform	1.143	0.294	0.131	0.074	%
Optimal	0.724	0.179	0.079	0.045	%
Optimal/uniform	63.342	60.884	60.305	60.811	%

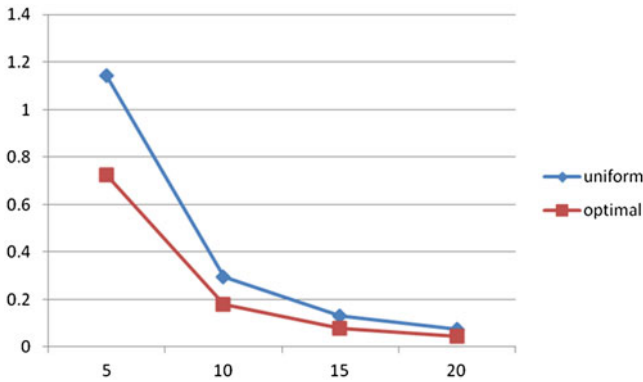


Figure 2. ΔS_{ixn} for uniform and optimal division of the orthodrome (%).

Table 2. Intermediate points for n = 10 and the uniform division.

i	φ_i			λ_i			S_i NM	C_{gsi} °	S_{lxi} NM	C_{glxi} °
	°	'	"	°	'	"				
	-55	59	0	-67	17	0		211.3		215.3
1	-62	52	49	-76	54	4	507.8	219.6	508.2	226.1
2	-68	44	19	-91	49	43	507.8	233.3	509.0	243.7
3	-72	29	25	-114	47	29	507.8	255.0	510.8	268.5
4	-72	42	33	-143	12	35	507.8	282.1	512.5	293.8
5	-69	17	0	-167	5	22	507.8	304.8	511.1	312.3
6	-63	35	35	177	12	30	507.8	319.2	509.1	323.8
7	-56	46	41	167	8	19	507.8	327.9	508.3	330.8
8	-49	23	52	160	16	16	507.8	333.4	508.0	335.3
9	-41	42	27	155	13	57	507.8	337.1	507.9	338.3
10	-33	50	0	151	17	0	507.8	C_{ge} 339.5	507.8	
						S	5 077.7		S_{ixn} 5 092.6	
									ΔS_{ixn} 14.9	
									0.294	%

COG must be carefully observed and the course set on an autopilot must be accordingly corrected to obtain COG equal to the required C_{glxi} for each subloxodrome.

3.2. *Return to the required route.* If due to set and drift our current position is off the route (even if Section 3.1 is applied), then a new corrected C_{glxi} should be calculated to the nearest intermediate point. This additional loss of distance can be calculated

Table 3. Intermediate points for n = 10 and the optimal division.

i	φ _i			λ _i			S _i	C _{gsi}	S _{lxi}	C _{glxi}
	°	'	"	°	'	"	NM	°	NM	°
	-55	59	0	-67	17	0		211.3		216.4
1	-64	17	56	-79	39	17	620.6	222.1	621.5	228.2
2	-69	10	56	-93	31	54	440.4	234.9	441.4	241.9
3	-71	53	49	-108	47	24	346.0	249.3	346.9	256.9
4	-73	1	58	-124	58	18	300.8	264.7	301.7	272.6
5	-72	48	57	-141	24	10	290.3	280.4	291.2	288.1
6	-71	12	14	-157	20	57	311.3	295.6	312.2	302.8
7	-67	52	10	-172	11	34	370.3	309.5	371.2	315.9
8	-62	3	14	174	25	39	487.0	321.7	487.9	327.0
9	-52	7	17	162	29	52	711.1	331.7	712.0	336.0
10	-33	50	0	151	17	0	1 199.8	C _{ge} 339.5	1200.7	
						S	5 077.7		S _{lxn} 5 086.8	
									ΔS _{lxn} 9.1	
									0.179	%

from Equation (12) (Figure 3)

$$\Delta s = \frac{s_1 + s_2}{l_1 + l_2} - 1 = \frac{k}{\cos \alpha} + \sqrt{(1 - k)^2 + (k \tan \alpha)^2} - 1 \tag{12}$$

where

$$k = \frac{l_1}{l_1 + l_2} \tag{13}$$

For example for α = 5° and k = 0.5 Δs = 0.38%, for k = 0.75 Δs = 1.13%, for k = 1 Δs = 9.13% (although in the last case it would be better to steer to the next intermediate point and not to the nearest). It can be seen that this loss of distance can be many times bigger than the loss between loxodromes and orthodromes.

3.3. *Orthodromic navigation.* Let us assume that we can also use the set of procedures as in Section 2 during the voyage. Section 3.1 is still valid and very important. During the voyage if we enter as P₁ the current position and P₂ is constantly the destination point then Equation (2) gives the C_{gs} for the orthodrome to the destination point, even if we, for example, due to set and drift are off the route, and not to an intermediate point. We can calculate a new course for the orthodrome after each position fix and always navigate along the current orthodrome without these calculated intermediate points and without loss of Δs and ΔS_{lxn}.

Orthodromic navigation, although very promising, is possible only in automatic steering systems where an autopilot (course controller which stabilises the required course) is connected with a track controller (which nearly continuously fixes the position and calculates C_{gs} from the current position to the destination point) but such systems are very rare in marine navigation. Most common is a separate autopilot and for example a GPS receiver which sometimes can calculate the orthodromic or loxodromic course (mainly on a sphere) and this course must be manually set on an autopilot.

Let us assume that we will be steering with a manually set autopilot and we will be fixing position at Δt intervals where Δt is for example equal to 1, 2 or 4 hours.

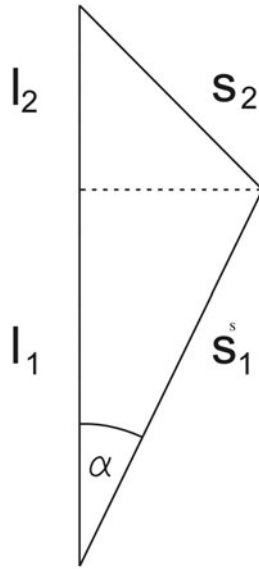


Figure 3. Return to the required route.

Table 4. Δs_{dev} [NM] for $V_g \Delta t = 20$ NM.

C_{gs} [°]	0	10	20	30	40	50	60	70	80	90
φ_1 [°]										
0	0-00	0-00	0-00	0-00	0-00	0-00	0-00	0-00	0-00	0-00
10	0-00	0-00	0-00	0-01	0-01	0-01	0-01	0-01	0-01	0-01
20	0-00	0-00	0-01	0-01	0-01	0-02	0-02	0-02	0-02	0-02
30	0-00	0-01	0-01	0-02	0-02	0-03	0-03	0-03	0-03	0-03
40	0-00	0-01	0-02	0-02	0-03	0-04	0-04	0-05	0-05	0-05
50	0-00	0-01	0-02	0-03	0-04	0-05	0-06	0-07	0-07	0-07
60	0-00	0-02	0-03	0-05	0-06	0-08	0-09	0-09	0-10	0-10
70	0-00	0-03	0-06	0-08	0-10	0-12	0-14	0-15	0-16	0-16
80	0-00	0-06	0-12	0-17	0-22	0-26	0-29	0-31	0-32	0-33

For $\Delta t = 1$ h and our speed $V_g = 20$ kn for 20 NM the orthodrome at the maximum latitude of our exemplary voyage (-73.1°) $C_{ge} - C_{gs} = 1.1^\circ$. Let us assume that, to be very accurate, we will be changing our course in 0.1° steps, therefore our course should be changed 11 times in 1 hour to obtain 56 cm (the difference between the loxodromic and the orthodromic distance). On the other hand if we steer for 1 hour on the constant C_{gs} course we will be off the route by 0.19 NM. This deviation is given by equation

$$\Delta s_{dev} = (C_{glx} - C_{gs}) V_g \Delta t \tag{14}$$

Table 4 gives these deviations for different latitudes and different C_{gs} for $V_g \Delta t = 20$ NM (for example $\Delta t = 1$ hour at $V_g = 20$ kn). These deviations increase very quickly with longer $V_g \Delta t$ as Table 5 shows ($V_g \Delta t = 80$ NM e. g. $\Delta t = 4$ hours at $V_g = 20$ kn).

Table 5. ΔS_{dev} [NM] for $V_g \Delta t = 80$ NM.

C_{gs} [°]	0	10	20	30	40	50	60	70	80	90
Φ_1 [°]										
0	0-00	0-00	0-00	0-00	0-00	0-00	0-00	0-00	0-00	0-00
10	0-00	0-03	0-06	0-09	0-11	0-13	0-15	0-16	0-16	0-16
20	0-00	0-06	0-12	0-17	0-22	0-26	0-30	0-32	0-33	0-34
30	0-00	0-10	0-19	0-27	0-35	0-42	0-47	0-51	0-53	0-54
40	0-00	0-14	0-27	0-40	0-51	0-61	0-68	0-74	0-77	0-78
50	0-00	0-20	0-39	0-57	0-73	0-86	0-97	1-05	1-09	1-10
60	0-00	0-29	0-57	0-83	1-06	1-26	1-42	1-53	1-59	1-60
70	0-00	0-47	0-92	1-34	1-71	2-02	2-27	2-44	2-53	2-54
80	0-00	1-02	2-01	2-90	3-68	4-32	4-80	5-11	5-25	5-23

Table 6. ΔS_{lx} [m] for $V_g \Delta t = 20$ NM.

C_{gs} [°]	0	10	20	30	40	50	60	70	80	90
Φ_1 [°]										
0	0-00	0-00	0-00	0-00	0-00	0-00	0-00	0-00	0-00	0-00
10	0-00	0-00	0-00	0-00	0-00	0-00	0-00	0-00	0-00	0-00
20	0-00	0-00	0-00	0-00	0-00	0-00	0-01	0-01	0-01	0-01
30	0-00	0-00	0-00	0-00	0-01	0-01	0-01	0-02	0-02	0-02
40	0-00	0-00	0-00	0-01	0-02	0-02	0-03	0-03	0-04	0-04
50	0-00	0-00	0-01	0-02	0-03	0-04	0-06	0-07	0-07	0-07
60	0-00	0-00	0-02	0-04	0-07	0-09	0-12	0-14	0-15	0-16
70	0-00	0-01	0-05	0-10	0-17	0-23	0-30	0-35	0-38	0-39
80	0-00	0-05	0-21	0-44	0-72	1-02	1-29	1-50	1-63	1-66

Table 7. ΔS_{lx} [m] for $V_g \Delta t = 80$ NM.

C_{gs} [°]	0	10	20	30	40	50	60	70	80	90
Φ_1 [°]										
0	0-00	0-00	0-00	0-00	0-00	0-00	0-00	0-01	0-01	0-00
10	0-00	0-00	0-01	0-03	0-05	0-07	0-09	0-10	0-11	0-11
20	0-00	0-01	0-06	0-12	0-20	0-28	0-35	0-40	0-44	0-45
30	0-00	0-04	0-14	0-29	0-48	0-68	0-86	1-00	1-09	1-11
40	0-00	0-08	0-29	0-62	1-02	1-43	1-82	2-11	2-30	2-34
50	0-00	0-15	0-59	1-26	2-06	2-90	3-67	4-27	4-63	4-71
60	0-00	0-33	1-27	2-70	4-42	6-20	7-81	9-06	9-79	9-93
70	0-00	0-87	3-33	7-05	11-47	16-01	20-04	23-07	24-76	24-93
80	0-00	4-25	16-23	33-86	54-21	74-11	90-75	102-08	106-97	105-27

The solution is to steer the loxodromic course C_{glx} to the locally predicted position after Δt on the orthodrome from the current position to the destination point, that is

$$\varphi_{\Delta t}, \lambda_{\Delta t} = \text{DGP}(\varphi_c, \lambda_c, V_g \Delta t, C_{\text{gs}}) \quad (15)$$

$$C_{\text{glx}} = \text{LX}(\varphi_c, \lambda_c, \varphi_{\Delta t}, \lambda_{\Delta t}) \quad (16)$$

where: φ_c, λ_c is the current position. Such a combination of the orthodrome and short loxodromes allows orthodromic navigation in manual steering without ΔS_{dev} and ΔS losses and quite negligible ΔS_{lx} . For example, for 20 NM loxodromes, the maximal $\Delta S_{lx} = 1.66$ m and for 80 NM loxodromes, the maximal $\Delta S_{lx} = \approx 107$ m - for maximal latitude 80° (Tables 6 and 7).

4. CONCLUSIONS. The presented procedures are quite general and universal. They can be used for any geodetic solutions and required accuracy, on any reference ellipsoid. For orthodromic navigation they can be used as well during route planning with additional optimally divided orthodromes as during sailing on the planned route with locally predicted short loxodromes.

The proposed locally predicted short loxodromes – calculated in real time during sailing at position fix intervals – also allow orthodromic navigation in manual steering without intermediate points and losses of returning to these points, without deviations caused by constant orthodromic courses and quite negligible differences in distances between loxodromes and orthodromes. In fact a dozen or so constant planned loxodromes are replaced by a few hundred short loxodromes - locally adopted to set and drift - on the orthodrome from the current position to the destination point.

It is worth mentioning that although analysis has been done on an ellipsoid, nothing prevents using the results - optimal division and locally predicted short loxodromes - in calculations even on a sphere.

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