Multiplication is to addition as addition is to what?

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1. *A new operation*

When children are first taught about multiplication of the natural numbers, it is usually presented as repeated addition. Later, learning about raising to an exponent is presented as repeated multiplication. Then the following analogy is obvious: addition is to multiplication as multiplication is to raising to an exponent. An interesting question is to ask what happens if we go in the other direction. That is, multiplication is to addition as addition is to what? In this paper, we will answer this question, and show that there are several possible operations that could be used to answer the question. Some of them will be presented in connection with algebraic semirings. A semiring is a set, together with two operations, semiring addition and semiring multiplication, satisfying certain properties. We will define several semirings, such that the semiring multiplication is ordinary addition. In each case, the semiring addition will then be an answer to the question in the title of this paper.

In the course of our study, we will define the terms semiring, log semiring, tropical semiring, hyperoperations, and zeration. Much of this material has been done elsewhere. However, I believe that this is the first paper in which the various answers to the question in the title of this paper have been gathered together in one place. Much of this material has already been used in applied mathematics. Original to this paper will be a proof that a log semiring can be extended to a field, and that this field is isomorphic to the field of real numbers under ordinary addition and multiplication. Fields are important because, among other reasons, they allow the usual four operations, addition, subtraction, multiplication, and division.

2. *Pre-addition in terms of logarithms*

At first, it would appear that there is no operation that answers the question in the title of this paper. Suppose there were such an operation. Let us call the operation ‘pre-addition’, and represent it with the symbol $\oplus$. If repeated pre-addition were ordinary addition, we would expect for all a:

$$3 + a = a \oplus a \oplus a \tag{1}$$

$$2 + a = a \oplus a \tag{2}$$

$$1 + a = a. \tag{3}$$

This last equation is obviously impossible for ordinary addition. So it appears that our task is also impossible [1].

However, we do not despair, and instead press on. In a recent blog entry [2], Kasper Müller takes a slightly different approach to the problem. He argues that the operation we seek should satisfy a distributive law of

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ordinary addition over pre-addition, analogous to the fact that multiplication is distributive over addition. That is, we want

$$a + (b \oplus c) = (a + b) \oplus (a + c). \tag{4}$$

In addition, Müller reasons that since the natural logarithm changes multiplication into addition, analogously the natural logarithm ought to change addition into pre-addition. That is, we want

$$\ln(a + b) = \ln a \oplus \ln b. \tag{5}$$

Furthermore, he argues that the operation should be commutative and associative. He then introduces the operation:

$$a \oplus b = \ln\left(e^a + e^b\right). \tag{6}$$

where a and b are real numbers.

First, it is obvious that this operation is commutative, and it is easy to check that it is also associative. That is $a \oplus b = b \oplus a$, and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

The proof of the distributive law (4) is simple:

$$\begin{aligned} a + (b \oplus c) &= a + \ln(e^b + e^c) &&= \ln e^a + \ln(e^b + e^c) &&= \ln\left[e^a\left(e^b + e^c\right)\right] \\ &= \ln(e^{a+b} + e^{a+c}) &&= (a + b) \oplus (a + c), \text{ as desired.} \end{aligned}$$

The logarithm law (5) is easy to prove, but note that it only works if a and b are positive.

Now it is interesting to reconsider the problem discussed above in (1), (2) and (3). Using (6), and the fact that pre-addition is associative, we find that

$$a \oplus a \oplus a = \ln(e^a + e^a + e^a) = \ln(3e^a) = \ln 3 + a.$$

Thus $\qquad \ln 3 + a = a \oplus a \oplus a.$

And analogously, we can show $\qquad \ln 2 + a = a \oplus a$

$$\ln 1 + a = a.$$

We see that they are similar to (1), (2) and (3), but slightly different.

Since our pre-addition operation is commutative and associative, a natural question to ask next is whether there is an identity element. Assume there is an identity element, and call it k. Then, for all a, $a \oplus k = a$. By (6), this becomes $\ln(e^a + e^k) = a$. Raising e to both sides, we get $e^a + e^k = e^a$, and therefore $e^k = 0$. However, there is no real number k satisfying this. Thus, if we want an identity element, we must extend the real numbers to include the ‘number’ $-\infty$, since $e^{-\infty} = 0$. Then the identity element of $\oplus$ is $-\infty$.

Of course, the base of the natural logarithm is the number e. We can generalise what Müller did, and define a pre-addition operation using any legal base of logarithm. That is, if a and c are real numbers, and $b \neq 1$ is a

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positive real number, we define $\oplus_b$ by

$$a \oplus_b c = \log_b \left(b^a + b^c\right). \tag{7}$$

It is easy to show that this operation is commutative and associative, and that it satisfies the analogues of (4) and (5):

$$a + \left(c \oplus_b d\right) = (a + c) \oplus_b (a + d). \tag{8}$$

$$\log_b (a + c) = \log_b a \oplus \log_b c \text{ if } a,\, c > 0. \tag{9}$$

Again the question of an identity element arises. It turns out that if $b > 1$ then the identity element of $\oplus_b$ is $-\infty$, and if $0 < b < 1$ then the identity element is ∞. Both are easy to show.

3. *Rings, semirings and fields*

At this point, let us pause, review the definitions of field and of ring, and introduce the definition of semiring. A ring [3, p. 34] is a set of elements along with two binary operations, the ring addition and the ring multiplication, such that it is an abelian group under the addition (therefore addition is commutative and associative, there is an additive identity element, and every element has an additive inverse), the multiplication is associative, there is a multiplicative identity element, and the multiplication is distributive over the addition. An example of a ring is the set of integers under ordinary addition and multiplication.

A semiring [4, 5, 6] is similar to a ring, except that not every element needs to have an additive inverse. Thus the addition is commutative, associative, and has an identity element, the multiplication is associative and has an identity element, and the multiplication is distributive over the addition. Furthermore, the additive identity annihilates any element of the semiring; that is, if a is any element of the semiring and 0 is the additive identity, then $a \cdot 0 = 0 \cdot a = 0$. Two examples of semirings are the non-negative integers and the non-negative real numbers, both under ordinary addition and multiplication.

A field [3, p. 83] is a ring in which the multiplication is commutative and every element except the additive identity has a multiplicative inverse. Therefore, the field with the additive identity removed is an abelian group under the multiplication. Two examples of fields are the real numbers and the complex numbers, both under ordinary addition and multiplication.

In general, we will use the symbol $\oplus$ to represent the ring (or semiring, or field) addition, and $\otimes$ to represent the ring (or semiring, or field) multiplication. We will use the symbol R to represent the set of real numbers.

4. *Log semiring*

Let us return to the pre-addition operation defined by (7). It turns out that it was known prior to Müller's re-discovery. If we take the set $R \cup \{-\infty, \infty\}$, our pre-addition operation $\oplus_b$, and ordinary addition $+$, we

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get a *log semiring*. [7] Our log semiring's 'addition' is our pre-addition, $\oplus_b$, and our log semiring's 'multiplication', $\otimes$, is ordinary addition, +, extended so that for any real number a we have $a + (\pm\infty) = \pm\infty$, analogous to the fact that $a \cdot 0 = 0$. If $b > 1$, the 'additive' identity is $-\infty$ (the identity for pre-addition), and if $0 < b < 1$, the 'additive' identity is ∞. The 'multiplicative' identity is 0 (the identity for ordinary addition).

Unless otherwise stated, we will assume that $b > 1$, and that therefore the 'additive' identity element is $-\infty$. In that case, we can actually remove ∞ from the log semiring. Results for the case of $0 < b < 1$ are analogous, and can be derived by the reader.

Log semirings are used in such applications as speech recognition and computational biology [8, 9, 10].

The reason that $\langle R \cup \{-\infty\}, \oplus_b, +\rangle$ is a semiring, not a ring, is that no element other than $-\infty$ has a pre-additive inverse, i.e. an inverse under $\oplus_b$. If we attempt to find the inverse c of a real number a, we get

$$a \oplus_b c = -\infty$$

$$\log_b \left(b^a + b^c\right) = -\infty$$

$$b^a + b^c = b^{-\infty}$$

$$b^a + b^c = 0$$

$$b^c = -b^a. \tag{10}$$

This last equation is true if $a = c = -\infty$. However, if a and c are real numbers, then the right side of (10) is negative, and left side is positive, which is impossible.

5. *Turning a log semiring into a group*

Nevertheless, let us see if we can construct a group that uses pre-addition as the group operation. I have not seen this discussed in the literature. To solve (10) for c, we will attempt to take the logarithm of both sides of the equation:

$$\log_b \left(b^c\right) = \log_b \left(-b^a\right)$$

$$c = \log_b \left(-b^a\right). \tag{11}$$

The right side of (11) involves taking the logarithm of a negative number. That is not possible if we are restricted to the real numbers. However, it *is* possible if we use complex numbers. Therefore, we will extend $R \cup \{-\infty\}$ so that it includes some complex numbers.

Of course, we know that in C, the set of complex numbers, the logarithm of a number can have more than one value. Specifically, since

$$-1 = e^{i\pi + 2in\pi} = \exp\left(i\pi + 2in\pi\right),$$

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where n is any integer, (11) becomes

$$\begin{aligned} c &= \log_b(-b^a) \\ &= \log_b(-1) + \log_b b^a \\ &= \log_b\left(\exp\left(i\pi + 2in\pi\right)\right) + a \\ &= \frac{\ln(\exp(i\pi + 2in\pi))}{\ln b} + a, \text{ by the change-of-base formula} \\ c &= \frac{i\pi + 2in\pi}{\ln b} + a. \end{aligned}$$

Thus a single number a has an infinite number of pre-additive inverses. However, we can take addition to be modulo $\dfrac{2i\pi}{\ln b}$. Then $\dfrac{2i\pi n}{\ln b} = 0$, and the pre-additive inverse of a is

$$c = \frac{i\pi}{\ln b} + a. \tag{12}$$

Note then that we have

$$\log_b(-1) = \frac{i\pi}{\ln b} \tag{13}$$

and

$$\log_b(0) = \log_b b^{-\infty} = -\infty. \tag{14}$$

Using (13), we also have

$$b^{(i\pi/\ln b)} = -1, \tag{15}$$

which will be useful.

We now claim that

$$\left\langle \left\{ a + \frac{mi\pi}{\ln b} \mid a \in R, m \in \{0,1\} \right\} \cup \{-\infty\}, \oplus_b \right\rangle$$

is an abelian group. First, it is obvious from (7) that pre-addition is commutative. Showing that pre-addition is associative is easy, and is left as an exercise.

Next, we need to show closure. Let a and c be real numbers. It is easy to see that $a \oplus_b c = \log_b\left(b^a + b^c\right)$ is also a real number, and is therefore in the group.

Next we consider $a \oplus_b \left(c + \dfrac{i\pi}{\ln b}\right)$. This equals

$$\log_b\left(b^a + b^{c + (i\pi/\ln b)}\right) = \log_b\left(b^a + b^c b^{i\pi/\ln b}\right) = \log_b\left(b^a - b^c\right),$$

in which we have used (15).

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If $a > c$, and remembering that we are considering the case for which $b > 1$, then $\log_b(b^a - b^c)$ is a real number and is thus in the group.

If $a < c$ then $\log_b(b^a - b^c) = \log_b[-1(b^c - b^a)] = \dfrac{i\pi}{\ln b} + \ln(b^c - b^a)$, where we have used (13), and this is of the form $d + \dfrac{i\pi}{\ln b}$, where d is a real number. It is thus in the group.

If $a = c$ then $\log_b(b^a - b^c) = \log_b(0) = -\infty$, by (14), and is thus in the group.

Showing that $\left(a + \dfrac{i\pi}{\ln b}\right) \oplus_b \left(c + \dfrac{i\pi}{\ln b}\right)$ is in the group is analogous. And so we have closure.

The group identity element is $-\infty$. If a is a real number, then

$$a \oplus_b -\infty = \log_b\left(b^a + b^{-\infty}\right) = \log_b\left(b^a + 0\right) = a,$$

as desired. Furthermore,

$$\left(a + \frac{i\pi}{\ln b}\right) \oplus_b -\infty = \log_b\left(b^{a + (i\pi/\ln b)} + b^{-\infty}\right) = \log_b\left(-b^a + 0\right) = a + \frac{i\pi}{\ln b},$$

by (15) and (13), as desired.

Next,

$$-\infty \oplus_b -\infty = \log_b\left(b^{-\infty} + b^{-\infty}\right) = \log_b(0 + 0) = \log_b 0 = -\infty, \tag{16}$$

by (14), as desired.

Lastly, we want to show the existence of inverse elements. We claim that if a is real then its inverse is $a + \dfrac{i\pi}{\ln b}$. Note that

$$a \oplus_b \left(a + \frac{i\pi}{\ln b}\right) = \log_b\left(b^a + b^{a + (i\pi/\ln b)}\right) = \log_b\left(b^a - b^a\right) = \log_b 0 = -\infty,$$

as desired, where we have used (15) and (14). By commutativity, we also have that the inverse of $a + \dfrac{i\pi}{\ln b}$ is a. Finally, from (16), the inverse of $-\infty$ is itself.

We have now shown that

$$\left\langle \left\{ a + \frac{mi\pi}{\ln b} \mid a \in R,\ m \in \{0,1\} \right\} \cup \{-\infty\},\ \oplus_b \right\rangle$$

is an abelian group. Note, too, that the logarithm law analogous to (5), $\log_b(a + c) = (\log_b a) \oplus_b (\log_b c)$ is now satisfied even if a or c or both are negative or 0.

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6. *Turning a log semiring into a field*

Next, we will see if we can turn our group into a field by including ordinary addition modulo $\frac{2i\pi}{\ln b}$ as the field 'multiplication'.

So we examine

$$\left\langle \left\{ a + \frac{mi\pi}{\ln b} \mid a \in R, m \in \{0,1\} \right\} \cup \{-\infty\}, \oplus_b, + \left(\text{mod } \frac{2i\pi}{\ln b} \right) \right\rangle,$$

where $b > 1$.

We know from above that this is a group under the 'addition' operation $\oplus_b$. To show it is a field, we need to show that the field 'multiplication', which is addition modulo $\frac{2i\pi}{\ln b}$, satisfies closure, is commutative, associative, and has an identity element, that the 'multiplication' is distributive over the 'addition', and that every element except the 'additive' identity has a 'multiplicative' inverse.

Obviously addition mod $\frac{2i\pi}{\ln b}$ is commutative and associative. Next, we see if we have closure.

The sum of two real numbers is a real number. The sum of two complex numbers of the form $a + \frac{i\pi}{\ln b}$ and $c + \frac{i\pi}{\ln b}$ is the real number $a + c$, modulo $\frac{2i\pi}{\ln b}$. The sum of a real number a and a complex number $c + \frac{i\pi}{\ln b}$ is the complex number $(a + c) + \frac{i\pi}{\ln b}$, which is in our field. Finally, if we define $\frac{i\pi}{\ln b} + (-\infty) = -\infty$, then the sum of any element of the field with $-\infty$ is $-\infty$, which is in the field. This is analogous to the fact that the product of any real number with 0 is 0.

There is an identity element for ordinary addition, namely 0. As for inverse elements, the additive inverse of a real number a is of course $-a$. The additive inverse of $a + \frac{i\pi}{\ln b}$ is $-a + \frac{i\pi}{\ln b}\left(\text{mod } \frac{2i\pi}{\ln b}\right)$. Observe that $-\infty$ does not have an additive inverse, but that is fine, as it is analogous to the fact that 0 has no multiplicative inverse.

Lastly, we have the distributive law (8) above. Thus,

$$\left\langle \left\{ a + \frac{mi\pi}{\ln b} \mid a \in R, m \in \{0,1\} \right\} \cup \{-\infty\}, \oplus_b, + \left(\text{mod } \frac{2i\pi}{\ln b} \right) \right\rangle$$

is a field.

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7. *The field is isomorphic to the real numbers under ordinary addition and multiplication*

Furthermore, this field is actually isomorphic to $\langle R, +, \cdot \rangle$, the field of the real numbers under ordinary addition and multiplication. Given two fields, F and G, with addition and multiplication operations $+_F$, $+_G$, $\cdot_F$ and $\cdot_G$, and with multiplicative identity elements 1_F and 1_G, a field isomorphism is a map $f : F \to G$ such that for any $x, y \in F$, we have

$$f(x) +_G f(y) = f(x +_F y), \tag{17}$$

$$f(x) \cdot_G f(y) = f(x \cdot_F y), \tag{18}$$

$$\text{and} \qquad f(1_F) = 1_G \tag{19}$$

and such that f is bijective, that is, one-to-one and onto, [3, p. 353].

The isomorphism $f(x) = b^x$, takes elements $a + \dfrac{mi\pi}{\ln b}$ of

$$\left\{a + \frac{mi\pi}{\ln b} \mid a \in R, m \in \{0,1\}\right\}$$

and sends them to real numbers $b^{a + (mi\pi/\ln b)}$. It sends the pre-additive identity $-\infty$ to $b^{-\infty} = 0$, the additive identity.

First, we need to show (17). Let $b\hat{}\,(x)$ denote b^x. We need to show that

$$f\left(a + \frac{ni\pi}{\ln b}\right) + f\left(c + \frac{mi\pi}{\ln b}\right) = f\left[\left(a + \frac{ni\pi}{\ln b}\right) \oplus_b \left(c + \frac{mi\pi}{\ln b}\right)\right],$$

$$\text{i.e. } b\hat{}\left(a + \frac{ni\pi}{\ln b}\right) + b\hat{}\left(c + \frac{mi\pi}{\ln b}\right) = b\hat{}\left[\left(a + \frac{ni\pi}{\ln b}\right) \oplus_b \left(c + \frac{mi\pi}{\ln b}\right)\right], \tag{20}$$

where a and c are real numbers, and n and m take on the values 0 or 1. The right side of (20) is

$$b\hat{}\left[\left(a + \frac{ni\pi}{\ln b}\right) \oplus_b \left(c + \frac{mi\pi}{\ln b}\right)\right] = b\hat{}\left[\log_b \left[b\hat{}\left(a + \frac{ni\pi}{\ln b}\right) + b\hat{}\left(c + \frac{mi\pi}{\ln b}\right)\right]\right]$$

$$= b\hat{}\left(a + \frac{ni\pi}{\ln b}\right) + b\hat{}\left(c + \frac{mi\pi}{\ln b}\right),$$

which is the left side of (20), as desired.

We also need $f\left(a + \dfrac{ni\pi}{\ln b}\right) + f(-\infty) = f\left[\left(a + \dfrac{ni\pi}{\ln b}\right) \oplus_b (-\infty)\right]$, i.e.

$$b\hat{}\left(a + \frac{ni\pi}{\ln b}\right) + b\hat{}\,(-\infty) = b\hat{}\left[\left(a + \frac{ni\pi}{\ln b}\right) \oplus_b (-\infty)\right],$$

which is easy to show, and

$$f(-\infty) + f(-\infty) = f[(-\infty) \oplus_b (-\infty)], \text{ i.e. } b\hat{}\,(-\infty) + b\hat{}\,(-\infty) = b\hat{}\,[(-\infty) \oplus_b (-\infty)],$$

which is also easy to show.

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Next, we need to show (18), i.e. that

$$f\left(a+\frac{ni\pi}{\ln b}\right)\cdot f\left(c+\frac{mi\pi}{\ln b}\right)=f\left[\left(a+\frac{ni\pi}{\ln b}\right)+\left(c+\frac{mi\pi}{\ln b}\right)\operatorname{mod}\frac{2i\pi}{\ln b}\right],$$

$$\text{i.e. } b\hat{}\left(a+\frac{ni\pi}{\ln b}\right)\cdot b\hat{}\left(c+\frac{mi\pi}{\ln b}\right)=b\hat{}\left[\left(a+\frac{ni\pi}{\ln b}\right)+\left(c+\frac{mi\pi}{\ln b}\right)\operatorname{mod}\frac{2i\pi}{\ln b}\right]. \quad (21)$$

If n and m are both 0, then (21) is obvious. Now consider the case for which $n = 0$ and $m = 1$. In that case, (21) becomes

$$(b\hat{}a)\cdot b\hat{}\left(c+\frac{i\pi}{\ln b}\right) = b\hat{}\left[(a)+\left(c+\frac{i\pi}{\ln b}\right)\right]. \quad (22)$$

The left side of (22) becomes $b^a \cdot (-b^c) = -b\hat{}(a + c)$, where we have used (15), and it is clear that the right side of (22) gives the same quantity. Proving (21) for the case for which $n = m = 1$ is similarly easy.

Showing the analogue of (21) for which one or both arguments are $-\infty$ is also easy.

We need to show (19), that is, $f(0) = 1$. That is obvious since $f(x) = b^x$.

Now we need to show that f is bijective, i.e. one-to-one and onto. First, $f(x) = b^x$ from R to R^+, the positive real numbers, is bijective. Analogously, $f(x) = b^x$ from $\left\{a + \frac{i\pi}{\ln b} \mid a \in R\right\}$ to R^-, the negative real numbers, is bijective. Lastly $f(x) = b^x$ sends $-\infty$ to 0. This completes the proof, and it concludes the material that is original to this paper.

8. *Tropical mathematics*

Let us return to the log semiring $\langle R \cup \{-\infty\} . \oplus_b, +\rangle$, where $a \oplus_b c = \log_b(b^a + b^c)$. We can use this to generate two more pre-addition operations.

Let x and y be two non-negative real numbers. Consider the obvious inequality, $\max(x, y) \leqslant x + y \leqslant 2\max(x, y)$, [11]. Let $x = b^a$ and $y = b^c$, where $b > 1$. Then we have $\max(b^a, b^c) \leqslant b^a + b^c \leqslant 2\max(b^a, b^c)$. We can take a logarithm with base b of each side, and the inequality still holds, because the logarithm function is increasing. We get

$$\max(a, c) \leqslant a \oplus_b c \leqslant \log_b 2 + \max(a, c).$$

Now, if we take the limit of each side as b approaches infinity, $\log_b 2$ approaches 0, and so by the squeeze theorem or sandwich theorem, we have $\lim_{b\to\infty}(a \oplus_b c) = \max(a, c)$. Analogously, one can show that, for $0 < b < 1$, $\lim_{b\to 0^+}(a \oplus_b c) = \min(a, c)$.

Using these results, we can define two new semirings. The *max-plus semiring* is $\langle R \cup \{-\infty\}, \oplus, \otimes\rangle$, where the semiring addition is the max operation, $a \oplus b = \max(a, b)$, and the semiring multiplication is standard addition, $a \otimes b = a + b$. Analogously, we have the *min-plus semiring*,

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$\langle R \cup \{\infty\}, \oplus, \otimes \rangle$, where $a \oplus b = \min(a, b)$, and $a \otimes b = a + b$. Together, these two semirings are called the *tropical semirings* [12]. The word 'tropical' was chosen to honour the Brazilian mathematician Imre Simon, since Brazil is a tropical country. Obviously in both cases the tropical 'multiplication' is associative and has an identity element, 0. It is also obvious that the tropical 'addition' is commutative. It is left as an exercise for the reader to show that the 'addition' is associative, that it has an identity element (either $-\infty$ or ∞), and that the 'multiplication' is distributive over the 'addition'. It is clear why we have semirings rather than rings. Most elements of the max-plus semiring do not have 'additive inverses', because if $a \neq -\infty$ then the equation $a \oplus x = \max(a, x) = -\infty$ has no solution for x.

So the maximum and minimum operations are two more answers to the question in the title of this paper.

Tropical mathematics has many applications, both within mathematics and beyond. It is used in geometric combinatorics, algebraic geometry, number theory, symplectic geometry, mathematical physics, computational biology, and more [12].

It is interesting to compare visually the addition operation in a log semiring with tropical addition. We compare the graph of the log semiring addition function $y = x \oplus_e 3 = \ln(e^x + e^3)$ with the graph of the tropical addition function $y = x \oplus 3 = \max(x, 3)$. See Figure 1.

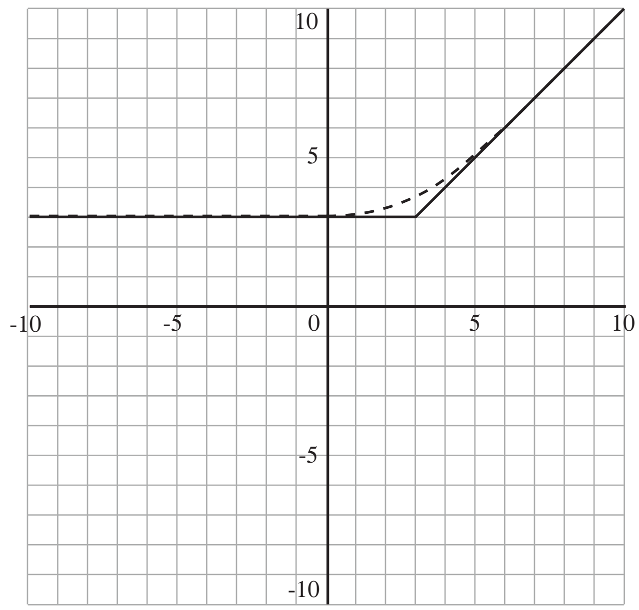

FIGURE 1: Graph of $y = \max(x, 3)$ and $y = \ln(e^x + e^3)$. The graph of the latter is the smooth curve.

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9. *Zeration*

There is yet another way of answering the question in the title of this Article. We introduce *hyperoperations* $H_n(a, b)$ [13], such that

$$H_1(a, b) = a + b$$

$$H_2(a, b) = a \cdot b$$

$$H_3(a, b) = a^b.$$

Then the operation *zeration* [13] is the successor operation given by $H_0(a, b) = b + 1$.

Note that repeated zeration gives ordinary addition. Furthermore, addition is distributive over zeration; it is easy to check that

$$a + H_0(b, c) = H_0(a + b, a + c).$$

On the other hand, zeration is neither commutative nor associative, and there is no identity element for it.

A more sophisticated zeration operation [14] can be defined by

$$a \circ b = \left(\begin{array}{ll} a + 1 & \text{if } a > b \\ b + 1 & \text{if } a < b \\ a + 2 = b + 2 & \text{if } a = b \end{array} \right. . \qquad (23)$$

Note that if $a \neq b$, then zeration is closely linked to the operation of finding a maximum, so we see a similarity to the tropical mathematics described above. Specifically, $a \circ b = \max(a, b) + 1$.

This version of zeration is commutative, but is still not associative. It is easy to show that

$$a \circ a = a + 2,$$

$$a \circ a \circ a = a + 3,$$

and so forth, in analogy with (1) and (2).

It satisfies a distributive law of addition over zeration. It is easy to show that $a + (b \circ c) = (a + b) \circ (a + c)$.

10. *Summary*

We have shown that there are several different operations that answer the question ‘Multiplication is to addition as addition is to what?’. They include the log semiring addition operation, $a \oplus_b c = \log_b\left(b^a + b^c\right)$, the tropical addition operations, $a \oplus b = \max(a, b)$ and $a \oplus b = \min(a, b)$, and the zeration operations, given by $H_0(a, b) = b + 1$ and by (23). We have also shown that the log semiring can be extended to a field, and that this field is isomorphic to the field of real numbers under ordinary addition and multiplication.

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