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MINIMAL GENERATION OF FINITE SOLUBLE GROUPS BY PROJECTORS AND NORMALIZERS

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1. Introduction. In this paper G denotes a non-identity finite soluble group. If A is an irreducible G-module, $\mathrm{End}_G A$ is a division ring by Schur's Lemma, actually a field, since G finite forces A to be finite. Moreover A is a vector space over $\mathrm{End}_G A$ with respect to $\alpha a := \alpha(a)$, $\alpha \in \mathrm{End}_G A$, $a \in A$. We let $\varphi_G(A) := \dim_{\mathrm{End}_G A} A$. Any chief factor of G is an irreducible G-module via the conjugation action, and it is central precisely when it is a trivial G-module. By a refined version of the Theorem of Jordan-Hölder [**1**, p. 33] the number $\delta_G(A)$ of complemented chief factors of G, which are G-isomorphic to a given A, is constant for any chief series of G. We say that A is *complemented, as a G-module*, if $\delta_G(A) > 0$. Let

$$\Omega(G) := \{\text{non-isomorphic, irreducible, complemented } G\text{-modules}\}.$$

The following formula, for the minimal number $d(G)$ of generators of G, can be deduced from the work of Gaschütz [**2**]:

$$d(G) = \max_{A \in \Omega(G)} h_G(A),$$

where

$$h_G(A) := \left[\frac{\delta_G(A) - 1 - \theta_G(A)}{\varphi_G(A)}\right] + 2$$

and $\theta_G(A) := 1$ if A is trivial, $\theta_G(A) := 0$ otherwise.

For what follows our reference is [**1**]. Let $\mathfrak{X}$ be a Schunck class of characteristic π in the universe $\mathfrak{S}$ of finite soluble groups. A π-group G is generated by its $\mathfrak{X}$-projectors, which are all conjugate. We let $\eta_{\mathfrak{X}}(G)$ be the minimal number of $\mathfrak{X}$-projectors which generate G. In a similar way, if $\mathfrak{F}$ is a saturated formation in $\mathfrak{S}$ and the characteristic of $\mathfrak{F}$ is the set $\mathbb{P}$ of all primes, G is generated by its $\mathfrak{F}$-normalizers. Again, they are all conjugate. We denote by $\tilde{\eta}_{\mathfrak{F}}(G)$ the minimal number of $\mathfrak{F}$-normalizers which generate G. The aim of this paper is to obtain formulas for the functions $\eta_{\mathfrak{X}}$ and $\tilde{\eta}_{\mathfrak{F}}$ similar to the one of Gaschütz for the function d.

Let H be an $\mathfrak{X}$-projector of G and let $A \in \Omega(G)$. We show that, if M_1/N_1 and M_2/N_2 are complemented chief factors of G that are G-isomorphic to A, then

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$M_1 \cap H \leq N_1$ if and only if $M_2 \cap H \leq N_2$. In this case we say that H *avoids* A and define

$$\Omega_{\mathfrak{X}}(G) := \{A \in \Omega(G) | H \text{ avoids } A\}.$$

For a π-group G, we obtain the formula:

$$\eta_{\mathfrak{X}}(G) = \max\left\{\max_{A \in \Omega_{\mathfrak{X}}(G)} \{h_G(A)\}, 1\right\}.$$

In particular, when the Schunck class is a saturated formation $\mathfrak{F}$, $\Omega_{\mathfrak{F}}(G)$ actually consists of those A's in $\Omega(G)$ for which every H-chief factor of A is $\mathfrak{F}$-eccentric.

Now assume, more generally, that H is a subgroup of G such that $H^G = G$. For each $\alpha \in \mathrm{End}_G A$, $\alpha(C_A(H)) \leq C_A(H)$. It follows that $C_A(H)$ is a subspace of A, as a vector space over $\mathrm{End}_G A$, and we put $\varphi_{G,H}(A) := \dim_{\mathrm{End}_G A} C_A(H)$. If A is non-trivial, $C_A(H) < A$ as $H^G = G$. Hence $\varphi_G(A) - \varphi_{G,H}(A) \neq 0$ and, for such an A, we define

$$h_{G,H}(A) := \left[\frac{\delta_G(A) - 1 + \varphi_G(A)}{\varphi_G(A) - \varphi_{G,H}(A)}\right] + 1.$$

In order to compute $\tilde{\eta}_{\mathfrak{F}}(G)$, we let

$$\tilde{\Omega}_{\mathfrak{F}}(G) := \{A \in \Omega(G) \big| A \text{ is } \mathfrak{F}\text{-eccentric}\},$$

and note that any $A \in \tilde{\Omega}_{\mathfrak{F}}(G)$ is non-trivial. We let H be an $\mathfrak{F}$-normalizer and show that

$$\tilde{\eta}_{\mathfrak{F}}(G) = \max\left\{\max_{A \in \tilde{\Omega}_{\mathfrak{F}}(G)} \{h_{G,H}(A)\}, 1\right\}.$$

Since a saturated formation $\mathfrak{F}$ is a Schunck class and an $\mathfrak{F}$-projector contains an $\mathfrak{F}$-normalizer, $\eta_{\mathfrak{F}}(G) \leq \tilde{\eta}_{\mathfrak{F}}(G)$. Our formulas give $\eta_{\mathfrak{X}}(G) \leq d(G)$. The functions d, $\eta_{\mathfrak{X}}$, $\tilde{\eta}_{\mathfrak{F}}$ and the gaps in the above inequalities have no upper bounds. For example let G be the semidirect product $(C_2 \times C_2)^n \mathrm{Sym}\,(3)$, where $\mathrm{Sym}\,(3)$ acts on each direct factor in the natural way. In the final section of the paper, we show that, if $\mathfrak{U}$ is the formation of supersoluble groups,

$$\eta_{\mathfrak{U}}(G) = d(G) = \tilde{\eta}_{\mathfrak{U}}(G) = \left[\frac{n-1}{2}\right] + 2;$$

on the other hand, if $\mathfrak{N}$ is the formation of nilpotent groups,

$$\eta_{\mathfrak{N}}(G) = 2, \quad d(G) = \left[\frac{n-1}{2}\right] + 2, \quad \tilde{\eta}_{\mathfrak{N}}(G) = n + 2.$$

2. Preliminary results. We shall make repeated use of the fact that a minimal normal subgroup N of G is abelian. It follows immediately that, if N has a supplement $L \neq G$, then L is a complement of N and L is a maximal subgroup of G.

LEMMA 2.1. *Let N be a minimal normal subgroup of G and let $\langle H_1, \dots, H_r\rangle$ be a complement of N, where each H_i is a subgroup. Then the set*

$$M := \{(n_1, \dots, n_r) \in N^r \mid \langle H_1^{n_1}, \dots, H_r^{n_r}\rangle \text{ is a complement of } N\}$$

is a union of cosets of $C_N(H_1) \times \dots \times C_N(H_r)$. Moreover, for $(m_1, \dots, m_r)$, $(m'_1, \dots, m'_r) \in M$ we have

$$\langle H_1^{m_1}, \dots, H_r^{m_r}\rangle = \langle H_1^{m'_1}, \dots, H_r^{m'_r}\rangle \Longleftrightarrow m_i \equiv m'_i \operatorname{mod} C_N(H_i), \text{ for each } i = 1, \dots, r.$$

Proof. We note that $[N_N(H_i), H_i] \leq N \cap H_i = \{1\}$ forces $N_N(H_i) = C_N(H_i)$, for each i. Now let $(n_1, \dots, n_r) \in N^r$ be such that $\langle H_1, \dots, H_r\rangle = \langle H_1^{n_1}, \dots, H_r^{n_r}\rangle$ and assume $H_i \neq H_i^{n_i}$, for some i. It follows that $H_i < \langle H_i, H_i^{n_i}\rangle \leq H_iN$, $\langle H_i, H_i^{n_i}\rangle \cap N \neq \{1\}$, a contradiction. We conclude that $n_i \in N_N(H_i) = C_N(H_i)$, for each i. □

In the following H denotes a subgroup of G such that $H^G = G$ and, for each homomorphism ϵ, $\eta(\epsilon(H), \epsilon(G))$ denotes the minimal number of conjugates of $\epsilon(H)$ that generate $\epsilon(G)$. We recall that, for a complemented minimal normal subgroup N of G, $|\mathrm{Der}(G/N, N)|$ coincides with the number of complements of N in G.

LEMMA 2.2. *Let N be a minimal normal subgroup of $G = H^G$ and let $r := \eta(NH/N, G/N)$. We have*

(i) *$r \leq \eta(H, G) \leq r + 1$;*

(ii) *if $\eta(H, G) = r + 1$, H is contained in a complement of N and*

$$|N/C_N(H)|^r \leq |\mathrm{Der}(G/N, N)|;$$

(iii) *if N is complemented and every complement of N contains a conjugate of H,*

$$|N/C_N(H)|^r \geq |\mathrm{Der}(G/N, N)|$$

and

$$\eta(H, G) = r + 1 \Longleftrightarrow |N/C_N(H)|^r = |\mathrm{Der}(G/N, N)|.$$

Proof. (i) Clearly $r \leq \eta(H, G)$. Let $(1, g_2, \dots, g_r) \in G^r$ be such that $G = \langle N, H, H^{g_2}, \dots, H^{g_r}\rangle$, and assume that $r < \eta(H, G)$. Then $L := \langle H, H^{g_2}, \dots, H^{g_r}\rangle$ is a complement of N. In particular N does not normalize H, for otherwise N would normalize L, contrary to the assumption that G is generated by the conjugates of H. Hence there exists $n \in \mathbb{N}$ such that $H < \langle H, H^n\rangle \leq HN$. It follows that $\langle H, H^n\rangle \cap N \neq \{1\}$ and $G = \langle H, H^n, H^{g_2}, \dots, H^{g_r}\rangle$. We conclude that $\eta(H, G) = r + 1$.

(ii) In the previous notation, L is a complement of N that contains H. Moreover, for each $(n_1, n_2, \dots, n_r) \in N^r$, $\langle H^{n_1}, H^{g_2 n_2}, \dots, H^{g_r n_r}\rangle$ is a supplement and hence a complement of N. By Lemma 2.1 the complements of this form are exactly $|N/C_N(H)|^r$.

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(iii) Let $\ell_1 = 1$ and let $L = \langle H^{\ell_1}, H^{\ell_2}, \dots, H^{\ell_r}\rangle$ be a complement of N that contains H. The first part of the statement follows from Lemma 2.1 if we show that each complement Y of N is of the form $Y = \langle H^{y_1}, H^{y_2}, \dots, H^{y_r}\rangle$, where $y_i \equiv \ell_i \pmod N$, for each i. For this purpose, we may assume that $H \leq Y$. Denote by $\psi : G \to Y$ the projection such that $y_i := \psi(\ell_i) \equiv \ell_i \pmod N$, for each i, and $\langle H^{y_1}, H^{y_2}, \dots, H^{y_r}\rangle$ is a supplement of N contained in Y. We conclude that $Y = \langle H^{y_1}, H^{y_2}, \dots, H^{y_r}\rangle$. Combining this with (ii) we have that $\eta(H, G) = r + 1$ forces $|N/C_N(H)|^r = |\mathrm{Der}(G/N, N)|$. Conversly let $|N/C_N(H)|^r = |\mathrm{Der}(G/N, N)|$ and assume $G = \langle H, H^{g_2}, \dots, H^{g_r}\rangle$, for some $(1, g_2, \dots, g_r) \in G^r$. If L is a complement of N that contains H and $\lambda : G \to L$ is the projection, we have $L = \lambda(G) = \langle H, H^{\lambda(g_2)}, \dots, H^{\lambda(g_r)}\rangle$. By what has been proved above, each complement of N is of the form $\langle H^{n_1}, H^{\lambda(g_2)n_2}, \dots, H^{\lambda(g_r)n_r}\rangle$, for some $(n_1, n_2, \dots, n_r) \in N^r$. On the other hand, by Lemma 2.1 and our hypothesis, the subgroup $\langle H^{n_1}, H^{\lambda(g_2)n_2}, \dots, H^{\lambda(g_r)n_r}\rangle$ must be a complement of N, for each $(n_1, n_2, \dots, n_r) \in N^r$. From $g_i \equiv \lambda(g_i) \pmod N$, it follows that $G = \langle H, H^{g_2}, \dots, H^{g_r}\rangle$ is a complement of N, a contradiction. □

As above we let

$$\Omega(G) := \{\text{non-isomorphic, irreducible, complemented } G\text{-modules}\}$$

and, for each non-trivial G-module $A \in \Omega(G)$, we let

$$h_{G,H}(A) := \left[\frac{\delta_G(A) - 1 + \varphi_G(A)}{\varphi_G(A) - \varphi_{G,H}(A)}\right] + 1.$$

Moreover we say that a complemented chief factor M_1/N_1 of G *avoids* H when $M_1 \cap H \leq N_1$.

THEOREM 2.3. *Let $G = H^G$ and assume that H satisfies the following conditions*:

(i) *if M_1/N_1 is a complemented chief factor of G that avoids H, then every complement of M_1/N_1 in G/N_1 contains a conjugate of N_1H/N_1*;

(ii) *if M_1/N_1 and M_2/N_2 are G-isomorphic complemented chief factors of G, then M_1/N_1 avoids H if and only if M_2/N_2 avoids H.*

Then the set $\Omega_H(G) := \{A \in \Omega(G) | H \text{ avoids } A\}$ is well defined and

$$\eta(H, G) = \max\left\{\max_{A \in \Omega_H(G)}\{h_{G,H}(A)\}, 1\right\}.$$

Proof. We note that $\Omega_H(G)$ is well defined in virtue of (ii). The result is clear if G has prime order, and so we argue by induction on the order of G. Let N be a minimal normal subgroup of G and let $\overline{G} := G/N, \overline{H} := NH/N$. As $\overline{H}$ satisfies the hypothesis above as a subgroup of $\overline{G}$, we may assume that

$$\eta(\overline{H}, \overline{G}) = \max\left\{\max_{A \in \Omega_{\overline{H}}(\overline{G})}\left\{h_{\overline{G},\overline{H}}(A)\right\}, 1\right\}.$$

Each $A \in \Omega_{\overline{H}}(\overline{G})$ is, by inflation, an irreducible, complemented G-module which avoids H. Moreover, if A_1 and A_2 are distinct elements of $\Omega_{\overline{H}}(\overline{G})$, they are not isomorphic as G-modules. It follows that $\Omega_{\overline{H}}(\overline{G})$ can be considered as a subset of $\Omega_H(G)$. A chief series of G that includes N gives rise, in a natural way, to a chief series of $\overline{G}$. Considering this fact, it follows easily that for each $A \in \Omega_{\overline{H}}(\overline{G})$ that is not G-isomorphic to N, $\delta_G(A) = \delta_{\overline{G}}(A)$ and $h_{G,H}(A) = h_{\overline{G},\overline{H}}(A)$. On the other hand, if A is G-isomorphic to N, then $\delta_G(A) - 1 \leq \delta_{\overline{G}}(A) \leq \delta_G(A)$.

Case 1. N is not complemented or $N \cap H \neq \{1\}$.

Clearly $\Omega_H(G) = \Omega_{\overline{H}}(\overline{G})$ and, for each $A \in \Omega_H(G)$, we have $h_{G,H}(A) = h_{\overline{G},\overline{H}}(A)$. Hence, by Lemma 2.2 (ii)

$$\eta(H, G) = \eta(\overline{H}, \overline{G}) = \max\left\{ \max_{A \in \Omega_{\overline{H}}(\overline{G})} \left\{ h_{\overline{G},\overline{H}}(A) \right\}, 1 \right\} = \max\left\{ \max_{A \in \Omega_H(G)} \left\{ h_{G,H}(A) \right\}, 1 \right\}.$$

Case 2. N is complemented and $N \cap H = \{1\}$.

Each complement of N contains a conjugate of H. In particular, N is not central, as $H^G = G$. By Lemma 2.2, $\eta(H, G) = \eta(\overline{H}, \overline{G}) := r$, or $\eta(H, G) = r + 1$. Also, we have

$$\left|\mathrm{Der}\left(\overline{G}, N\right)\right| \leq \left|N/C_N(H)\right|^r = |\mathrm{End}_G N|^{(\varphi_G(N) - \varphi_{G,H}(N))r}$$

and equality holds if and only if $\eta(H, G) = r + 1$. Now, by [**2**, Satz 3],

$$\left|\mathrm{Der}\left(\overline{G}, N\right)\right| = |N||\mathrm{End}_G N|^{\delta_{\overline{G}}(N)} = |\mathrm{End}_G N|^{\varphi_G(N) + \delta_G(N) - 1}.$$

It follows that

$$\frac{\varphi_G(N) + \delta_G(N) - 1}{\varphi_G(N) - \varphi_{G,H}(N)} \leq r,$$

with equality if and only if $\eta(H, G) = r + 1$. Hence either $h_{G,H}(N) \leq r = \eta(H, G)$ or $h_{G,H}(N) = \eta(H, G) = r + 1$. In both cases we have

$$\eta(H, G) = \max\left\{ h_{G,H}(N), r \right\}.$$

We may assume that $\Omega_H(G) = \Omega_{\overline{H}}(\overline{G}) \cup \{N\}$. As $h_{G,H}(N) \geq h_{\overline{G},\overline{H}}(N)$ and, for each $A \in \Omega_H(G) - \{N\}$, $h_{G,H}(A) = h_{\overline{G},\overline{H}}(A)$, we obtain

$$\begin{aligned} \eta(H, G) &= \max\left\{ h_{G,H}(N), r \right\} = \max\left\{ h_{G,H}(N), \max_{A \in \Omega_{\overline{H}}(\overline{G})} \left\{ h_{\overline{G},\overline{H}}(A) \right\}, 1 \right\} \\ &= \max\left\{ h_{G,H}(N), \max_{A \in \Omega_{\overline{H}}(\overline{G}) - \{N\}} \left\{ h_{\overline{G},\overline{H}}(A) \right\}, 1 \right\} = \max\left\{ \max_{A \in \Omega_H(G)} \left\{ h_{G,H}(A) \right\}, 1 \right\}. \quad \square \end{aligned}$$

3. The function $\eta_{\mathfrak{X}}$. Let $\mathfrak{S}$ be the universe of finite soluble groups. A class $\mathfrak{X}$ in $\mathfrak{S}$ is said to be a *Schunck class* if it consists precisely of those groups whose primitive epimorphic images are in $\mathfrak{X}$. Here, by a primitive group, we mean a group P with a maximal subgroup M such that $\mathrm{Core}_P(M) = \{1\}$. A subgroup H of G is an $\mathfrak{X}$-pro-

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jector if $\epsilon(H)$ is $\mathfrak{X}$-maximal in $\epsilon(G)$, for any homorphism ϵ. In particular $\epsilon(H)$ is an $\mathfrak{X}$-projector of $\epsilon(G)$. The $\mathfrak{X}$-projectors of G form a unique conjugacy class, denoted by $\mathrm{Proj}_{\mathfrak{X}}(G)$. See [**1**, 3.21]

LEMMA 3.1. *Let M_1/N_1 be a complemented chief factor of G. For any $H \in \mathrm{Proj}_{\mathfrak{X}}(G)$, the following conditions are equivalent*:

(i) *every complement of M_1/N_1 in G/N_1 contains a conjugate of HN_1/N_1*;

(ii) *H avoids M_1/N_1.*

Proof. We show that (ii) implies (i), the converse being obvious. Since $HN_1/N_1 \in \mathrm{Proj}_{\mathfrak{X}}(G/N_1)$, we may replace G by G/N_1, H by HN_1/N_1 and assume that $N_1 = \{1\}$, M_1 is a minimal normal subgroup of G. Let L_1 be a complement of M_1 and let K be an $\mathfrak{X}$-projector of L_1. Then HM_1/M_1 and KM_1/M_1 are $\mathfrak{X}$-projectors of G/M_1, so that up to conjugation, $HM_1 = KM_1$. It follows that H is an $\mathfrak{X}$-projector of KM_1, by 3.22 (a) of [**1**]. As M_1 is nilpotent, $KM_1/M_1 \simeq K$ is in $\mathfrak{X}$ and H avoids M_1; from 3.23 (c) of [**1**] we have $\{K\} = \mathrm{Proj}_{\mathfrak{X}}(K) \subseteq \mathrm{Proj}_{\mathfrak{X}}(KM_1)$. Hence K is an $\mathfrak{X}$-projector of KM_1. We conclude that H and K are conjugate in KM_1. □

LEMMA 3.2. *Assume that M_1/N_1 and M_2/N_2 are G-isomorphic complemented chief factors of G. For any $H \in \mathrm{Proj}_{\mathfrak{X}}(G)$, H avoids M_1/N_1 if and only if it avoids M_2/N_2.*

Proof. Let $C := C_G(M_1/N_1) = C_G(M_2/N_2)$ and consider the following semidirect products. Relative to the conjugation action, we have

$$E_1 := (M_1/N_1)(G/C) \simeq E_2 := (M_2/N_2)(G/C).$$

Note that M_i/N_i is the unique minimal normal subgroup of E_i as it is self-centralizing ($i = 1, 2$). Let L_i/N_i be complements of M_i/N_i in G/N_i, and consider the homomorphisms

$$\epsilon_i : G = M_iL_i \to E_i \text{ such that } m_i\ell_i \mapsto (N_im_i, C\ell_i), \quad (i = 1, 2).$$

Clearly

$$\epsilon_i(M_i) = M_i/N_i \text{ and } \epsilon_i(L_i) = CL_i/C = G/C \text{ as } M_i \leq C.$$

In particular ϵ_1 and ϵ_2 are epimorphisms. Suppose that $H \cap M_1 \leq N_1$. By the previous lemma we may assume that $H \leq L_1$ and hence $\epsilon_1(H) \leq G/C$. It follows that $\epsilon_1(H)$ intersects trivially the unique minimal normal subgroup M_1/N_1 of E_1. As $\epsilon_i(H)$ is an $\mathfrak{X}$-projector of E_i ($i = 1, 2$), $\epsilon_2(H)$ also intersects trivially the unique minimal normal subgroup M_2/N_2 of E_2. On the other hand, $\epsilon_2(H \cap M_2) \leq M_2/N_2$. Hence $H \cap M_2 \leq \ker\epsilon_2 = C \cap L_2$. We conclude that $H \cap M_2 \leq M_2 \cap L_2 = N_2$. □

We recall that, for a class $\mathfrak{X}$, the set π of prime numbers p such that $\mathbb{Z}_p$ is in $\mathfrak{X}$ is called the *characteristic* of the class.

THEOREM 3.3. *Let $\mathfrak{X}$ be a Schunck class of characteristic π and let G be a π-group.*

(i) *G is generated by the* $\mathfrak{X}$*-projectors*;

(ii) *For* $H \in \mathrm{Proj}_{\mathfrak{X}}(G)$, *the set* $\Omega_{\mathfrak{X}}(G) := \{A \in \Omega(G) | H \text{ avoids } A\}$ *is well defined. Also*

$$\eta_{\mathfrak{X}}(G) = \max\left\{\max_{A\in\Omega_{\mathfrak{X}}(G)}\{h_G(A)\}, 1\right\}.$$

Moreover, for each $A \in \Omega_{\mathfrak{X}}(G)$, $\theta_G(A) = 0$. *Hence* $h_G(A) = \left[\dfrac{\delta_G(A) - 1}{\varphi_G(A)}\right] + 2.$

Proof. The image of H in G/H^G is the identity subgroup and it is $\mathfrak{X}$-maximal. It follows that G/H^G is a π'-group; i.e. $G = H^G$. Combining this observation with 3.1 and 3.2, we see that H satisfies the hypothesis of Theorem 2.3. For $A \in \Omega_{\mathfrak{X}}(G)$, let M_1/N_1 be a complemented chief factor of G that is G-isomorphic to A. Now $HN_1/N_1 \in \mathrm{Proj}_{\mathfrak{X}}(G/N_1)$ is selfnormalizing in G/N_1, by 4.8 of [**1**]. From this fact and the condition $H \cap M_1 \leq N_1$, it follows easily that $C_{M_1/N_1}(H) = \{1\} = C_A(H)$. Hence $\varphi_{G,H}(A) = \theta_G(A) = 0$, for each $A \in \Omega_{\mathfrak{X}}(G)$. The result is now a special case of 2.3. □

Comparing this Theorem with the result of Gaschütz for the minimal number $d(G)$ of generators for G, one has immediately the following result.

Corollary 3.4. $\eta_{\mathfrak{X}}(G) \leq d(G)$.

4. The function $\tilde{\eta}_{\mathfrak{F}}$. We need some technical definitions and results: for consistency and proofs we refer to [**1**]. Let $\mathfrak{F}$ be a saturated formation in $\mathfrak{S}$; i.e. a non-empty class of finite soluble groups, closed with respect to epimorphic images and subdirect products, with the following additional property: whenever $F/\Phi(F)$ is in $\mathfrak{F}$, then also F is in $\mathfrak{F}$ ($\Phi(F)$ being the Frattini subgroup). We assume further that $\mathfrak{F}$ has characteristic the set $\mathbb{P}$ of all primes. Under these assumptions, $\mathfrak{F}$ is a Schunck class and there exists a function $f : \mathbb{P} \to \{\text{formations}\}$ with the following properties. For each prime p, (1) $f(p) \subseteq \mathfrak{F}$ consists of those groups which have a normal p-subgroup with quotient in $f(p)$; (2) a group F is in $\mathfrak{F}$ if and only if $F/C_F(L/K) \in f(p)$, for each chief factor L/K of F such that $p \,\big|\, |L/K|$.

A chief factor M_1/N_1 of G is called $\mathfrak{F}$-*central* if and only if $p \,\big|\, |M_1/N_1| \Rightarrow G/C_G(M_1/N_1) \in f(p)$. If this is not the case, then M_1/N_1 is $\mathfrak{F}$-*eccentric*. Since $\{1\}$ is in $f(p)$, for each p, any central chief factor is $\mathfrak{F}$-central.

Let Σ be a Hall system of G and, for each prime p dividing the order of G, denote by $G_{p'}$ the Hall p'-subgroup of G in Σ. An $\mathfrak{F}$-normalizer H of G can thus be defined by

$$H := \bigcap_{p||G|} N_G\big(G_{p'} \cap G^{f(p)}\big),$$

where $G^{f(p)}$ denotes the unique normal subgroup of G minimal with respect to $G/G^{f(p)}$ in $f(p)$. The $\mathfrak{F}$-normalizers of G form a unique conjugacy class. Moreover, if H is an $\mathfrak{F}$-normalizer of G, then $\epsilon(H)$ is an $\mathfrak{F}$-normalizer of $\epsilon(G)$, for each homomorphism ϵ.

Lemma 4.1. *Let H be an* $\mathfrak{F}$*-normalizer of G and let* M_1/N_1 *be a complemented chief factor of G. Then the following conditions are equivalent*:

(i) M_1/N_1 *is* $\mathfrak{F}$*-eccentric;*
(ii) H *avoids* M_1/N_1;
(iii) *every complement of* M_1/N_1 *in* G/N_1 *contains a conjugate of* HN_1/N_1.

Proof. (i) $\Longleftrightarrow$ (ii). See [**1**, p. 401].

(ii) $\Rightarrow$ (iii). As HN_1/N_1 is an $\mathfrak{F}$-normalizer of G/N_1, as usual we may assume that $N_1 = \{1\}$ and M_1 is a minimal normal subgroup of G. We need a definition. A maximal subgroup K of a group F is called $\mathfrak{F}$*-critical* if $F/K_F \notin \mathfrak{F}$ and $F = K\,\mathrm{Fit}(F)$. Now let L be a complement of M_1 and let T be an $\mathfrak{F}$-normalizer of L. Then $T \in \mathfrak{F}$ and there exists a chain

$$T = L_r < \ldots < L_1 = L,$$

where each L_i is maximal in L_{i-1} and $\mathfrak{F}$-critical by [**1**, 3.8]. Since the p-group M_1 is $\mathfrak{F}$-eccentric, $G/C_G(M_1) \notin f(p)$. Now $L_G \leq C_G(M_1)$ gives

$$\frac{G/L_G}{C_{G/L_G}(L_G M_1/L_G)} \simeq \frac{G}{C_G(M_1)} \notin f(p).$$

As $L_G M_1/L_G \simeq M_1$ is a minimal normal subgroup of G/L_G, it follows that $G/L_G \notin \mathfrak{F}$ and L is $\mathfrak{F}$-critical in G. Hence the chain $T = L_r < \ldots < L < G$ is $\mathfrak{F}$-critical and, again by [**1**], T is an $\mathfrak{F}$-normalizer in G. We conclude that T is conjugate to H.

(iii)$\Rightarrow$(ii). This is clear. □

THEOREM 4.2. *Let* $\mathfrak{F}$ *be a saturated formation of characteristic* $\mathbb{P}$.
(i) G *is generated by the* $\mathfrak{F}$*-normalizers;*
(ii) $\tilde{\eta}_{\mathfrak{F}}(G) = \max\left\{\max_{A\in\tilde{\Omega}(G)}\{h_{G,H}(A)\}, 1\right\}$,

where $\tilde{\Omega}_{\mathfrak{F}}(G) := \left\{A \in \Omega(G) | A \text{ is } \mathfrak{F}\text{-eccentric}\right\}$ *and* H *is an* $\mathfrak{F}$*-normalizer.*

Proof. Since $\mathfrak{F}$ has characteristic $\mathbb{P}$, $H^G = G$ by [**1**, p. 401]. By the previous Lemma we can apply Theorem 2.3, with $\Omega_H(G) = \tilde{\Omega}_{\mathfrak{F}}(G)$. □

In the case of a saturated formation, the set $\Omega_{\mathfrak{F}}(G)$ of Theorem 3.3 is better characterized in the following way.

LEMMA 4.3. *Let* H *be an* $\mathfrak{F}$*-projector of* G. *Then*

$$\Omega_{\mathfrak{F}}(G) = \left\{A \in \Omega_H(G) \middle| \text{every } H\text{-chief factor of } A \text{ is } \mathfrak{F}\text{-eccentric in } AH\right\}.$$

Proof. Let $A \simeq M_1/N_1$, a complemented chief factor of G. The H-chief factors of A coincide with the HM_1/N_1-chief factors of the normal subgroup M_1/N_1 of G/N_1. Now HN_1/N_1 is an $\mathfrak{F}$-projector of G/N_1 and hence of HM_1/N_1. As M_1/N_1 is a normal nilpotent subgroup of HM_1/N_1, with quotient $H/(H \cap M_1)$ in $\mathfrak{F}$, it follows that HN_1/N_1 is an $\mathfrak{F}$-normalizer of HM_1/N_1. (See [**1**, 4.2].) Hence HN_1/N_1 covers the $\mathfrak{F}$-central chief factors of HM_1/N_1 and avoids the $\mathfrak{F}$-eccentric ones. By definition

$$A \in \Omega_{\mathfrak{F}}(G) \Longleftrightarrow M_1 \cap H \leq N_1 \Longleftrightarrow \left|\frac{HN_1}{N_1} \cap \frac{M_1}{N_1}\right| = 1.$$

We conclude that $A \in \Omega_{\mathfrak{F}}(G)$ if and only if all the H-chief factors of A are $\mathfrak{F}$-eccentric. □

5. Examples. Denote by $\mathfrak{N}$ the saturated formation of *nilpotent* groups. $\mathfrak{N}$ is local with respect to the formation function f such that $f(p) = \{1\}$, for each prime p. It follows that a chief factor is central if and only if it is $\mathfrak{N}$-central. The $\mathfrak{N}$-projectors are the *Carter subgroups* and the $\mathfrak{N}$-normalizers are the system normalizers. If $A = M_1/N_1$ is a chief factor and H is a Carter subgroup of G, then H avoids $M_1/N_1 \Longleftrightarrow C_A(H) = \{1\}$. As a matter of fact, $H \cap M_1 \leq N_1$ implies that $C_A(H) = \{1\}$, since HN_1/N_1 is selfnormalizing in G/N_1. On the other hand,

$$H \cap M_1 \not\leq N_1 \Rightarrow \{1\} < Z\left(\frac{HN_1}{N_1}\right) \cap \frac{M_1}{N_1} \leq C_A(H).$$

Hence, in this case, we have

$$\Omega_{\mathfrak{N}}(G) = \{A \in \Omega(G) | C_A(H) = \{1\}\},$$

where H is a Carter subgroup of G, and

$$\tilde{\Omega}_{\mathfrak{N}}(G) = \{A \in \Omega(G) | A \text{ non trivial } G\text{-module}\}.$$

Remark. Let $\mathfrak{F}$ be a saturated formation of characteristic π, G a π-group and H an $\mathfrak{F}$-projector of G. Then $N_G(H) = H$ so that, for each minimal normal subgroup N of G, we have

$$H \cap N = \{1\} \Rightarrow C_N(H) = \{1\}.$$

However, the converse is not true in general. For example, if $\mathfrak{U}$ is the formation of supersoluble groups, G is the symmetric group Sym (3) and N is the alternating group Alt (3), then

$$H = G, C_N(H) = \{1\}, H \cap N = N.$$

Denote by $\mathfrak{U}$ the saturated formation of *supersoluble* groups. $\mathfrak{U}$ is local with respect to the formation function f such that $f(p) = \{$abelian groups of exponent dividing $(p-1)\}$, for each prime p. A chief factor is $\mathfrak{U}$-eccentric if and only if it is not cyclic. By Lemma 4.3

$$\Omega_{\mathfrak{U}}(G) = \{A \in \Omega(G) | A \text{ has no cyclic } H\text{-chief factor}\},$$

where H is a $\mathfrak{U}$-projector. On the other hand we have

$$\tilde{\Omega}_{\mathfrak{N}}(G) = \{A \in \Omega(G) | A \text{ non cyclic}\}.$$

1. Let G be the symmetric group Sym (4). Consider the chief series

$$N_4 = \{1\} < N_3 = C_2 \times C_2 < N_2 = \text{Alt}(4) < N_1 = \text{Sym}(4),$$

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and let A_i be a G-module G-isomorphic to the chief factor N_i/N_{i+1}, $1 \leq i \leq 3$. The Carter subgroups of G are the Sylow 2-subgroups and the system normalizers are the subgroups generated by a 2-cycle. It is easy to see that

$$\Omega(G) = \{A_1, A_2, A_3\}, \quad \Omega_{\mathfrak{N}}(G) = \{A_2\}, \quad \tilde{\Omega}_{\mathfrak{N}}(G) = \{A_2, A_3\},$$

with $h_G(A_1) = 1, h_G(A_2) = h_G(A_3) = 2$ and $h_{G,H}(A_2) = 2, h_{G,H}(A_3) = 3$, where H is a system normalizer. It follows that

$$d(G) = \eta_{\mathfrak{N}}(G) = 2, \quad \tilde{\eta}_{\mathfrak{N}}(G) = 3.$$

2. Let G be the semidirect product $(C_2 \times C_2)^n \mathrm{Sym}\,(3)$, where $\mathrm{Sym}\,(3)$ acts on each direct factor in the natural way. In this case $\Omega(G) = \{A_1, A_2, A_3\}$, where

$$A_1 \text{ is } G\text{-isomorphic to } ((C_2 \times C_2)^n \mathrm{Sym}\,(3))/((C_2 \times C_2)^n \mathrm{Alt}\,(3)),$$

$$A_2 \text{ is } G\text{-isomorphic to } ((C_2 \times C_2)^n \mathrm{Alt}\,(3))/(C_2 \times C_2)^n$$

and A_3 is G-isomorphic to $C_2 \times C_2$. Since $\delta_G(A_1) = \delta_G(A_2) = 1$ and $\delta_G(A_3) = n$ we have

$$h_G(A_1) = 1, \quad h_G(A_2) = 2, \quad h_G(A_3) = \left[\frac{n-1}{2}\right] + 2.$$

Again the Carter subgroups of G are the Sylow 2-subgroups while the subgroup H_1 generated by a 2-cycle of $\mathrm{Sym}\,(3)$ is a system normalizer. In this case we have

$$\Omega_{\mathfrak{N}}(G) = \{A_2\}, \quad \tilde{\Omega}_{\mathfrak{N}}(G) = \{A_2, A_3\},$$

and $h_{G,H_1}(A_2) = 2, h_{G,H_1}(A_3) = n + 2$. It follows that

$$\eta_{\mathfrak{N}}(G) = 2, \quad d(G) = \left[\frac{n-1}{2}\right], \quad \tilde{\eta}_{\mathfrak{N}}(G) = n + 2.$$

On the other hand the $\mathfrak{U}$-projectors and the $\mathfrak{U}$-normalizers coincide and are precisely the complements in G of the normal subgroup $(C_2 \times C_2)^n$. We have

$$\Omega_{\mathfrak{U}}(G) = \tilde{\Omega}_{\mathfrak{U}}(G) = \{A_3\}$$

and $h_G(A_3) = h_{G,H_2}(A_3) = \left[\frac{n-1}{2}\right] + 2$, where H_2 is a $\mathfrak{U}$-normalizer. It follows that

$$\eta_{\mathfrak{U}}(G) = d(G) = \tilde{\eta}_{\mathfrak{U}}(G) = \left[\frac{n-1}{2}\right] + 2.$$

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