Bull. Aust. Math. Soc. **84** (2011), 40–43
doi:10.1017/S0004972711002358

DENSE SETS OF INTEGERS WITH A PRESCRIBED REPRESENTATION FUNCTION

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(Received 18 August 2010)

Abstract

A set $A \subseteq \mathbb{Z}$ is called an asymptotic basis of $\mathbb{Z}$ if all but finitely many integers can be represented as a sum of two elements of A. Let A be an asymptotic basis of integers with prescribed representation function, then how dense A can be? In this paper, we prove that there exist a real number $c > 0$ and an asymptotic basis A with prescribed representation function such that $A(-x, x) \geq c\sqrt{x}$ for infinitely many positive integers x.

2010 *Mathematics subject classification*: primary 11B13; secondary 11B34.
Keywords and phrases: prescribed representation function, Sidon sets.

1. Introduction

For $A \subseteq \mathbb{Z}$ and $n \in \mathbb{Z}$, let

$$r_A(n) = \sharp\{(a, b) \in A \times A : a \leq b, a + b = n\},$$
$$A(y, x) = \sharp\{a \in A : y \leq a \leq x\}.$$

Let $A(x) = A(1, x)$. We call $A \subseteq \mathbb{Z}$ an asymptotic basis of $\mathbb{Z}$ if all but finitely many integers can be represented as a sum of two elements of A, and a unique representation basis if $r_A(n) = 1$ for all $n \in \mathbb{Z}$. A set B of integers is called a Sidon set if $r_B(n) \leq 1$ for all $n \in \mathbb{Z}$. In 2003, Nathanson [5] proved that a unique representation basis of $\mathbb{Z}$ can be arbitrarily sparse. It is natural to ask if there exist unique representation bases that are dense in the sense that their counting functions tend rapidly to infinity. In 2007, Chen [1] proved that for any $\varepsilon > 0$, there exists a unique representation basis A of $\mathbb{Z}$ such that $A(-x, x) \geq x^{1/2-\varepsilon}$ for infinitely many positive integers x. Recently, Lee [4] improved this result by proving that for any increasing function $\phi(x)$ tending to infinity, there exists a unique representation function A of $\mathbb{Z}$ such that $\limsup_{x\to\infty} A(-x, x)\phi(x)/\sqrt{x} > 0$. In 2007, Cilleruelo and Nathanson [3] showed that there exists a unique representation basis A such that $\limsup_{x\to\infty} A(x)/\sqrt{x} \geq 1/\sqrt{2}$.

Supported by the National Natural Science Foundation of China, Grant No 10901002.

In this paper, we obtain the following result.

THEOREM 1.1. *Let* $f : \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\}$ *be a function where* $f^{-1}(0)$ *is a finite set. Then there exists an asymptotic basis* A *of* $\mathbb{Z}$ *such that* $r_A(n) = f(n)$ *for all* n *and*

$$\limsup_{x\to\infty} \frac{A(-x, x)}{\sqrt{x}} \geq \frac{1}{\sqrt{2}}.$$

2. Lemmas

From now on, f will denote a function $f : \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\}$ such that $f^{-1}(0)$ is a finite set. Then there exists a positive integer d_0 such that $f(n) \geq 1$ for every integer n with $|n| \geq d_0$.

LEMMA 2.1 [6]. *Given a function* f *as above, there exists a sequence* $U = \{u_k\}_{k=1}^{\infty}$ *of integers such that, for every* $n \in \mathbb{Z}$,

$$f(n) = \sharp\{k \geq 1 : u_k = n\}.$$

LEMMA 2.2 [4]. *Let* A *be a finite set of integers with* $r_A(n) \leq f(n)$ *for every integer* n. *Also assume that* $0 \notin A$, *and for every integer* n,

$$r_A(n) \geq \sharp\{i \leq m : u_i = n\}$$

for some integer m *which depends only on the set* A. *Then there exists a finite set* B *of integers such that* $A \subseteq B$, $0 \notin B$, $r_B(n) \leq f(n)$ *for every integer* n, *and*

$$r_B(n) \geq \sharp\{i \leq m + 1 : u_i = n\}$$

for every integer n.

LEMMA 2.3 [2]. *For each odd prime* p *there exists a Sidon set* B_p *such that:*

(i) $B_p \subseteq [1, p^2 - p]$;
(ii) $(B_p - B_p) \cap [-\sqrt{p}, \sqrt{p}] = \emptyset$;
(iii) $|B_p| > p - 2\sqrt{p}$.

3. Proof of Theorem 1.1

We shall use induction to construct an ascending sequence $A_1 \subseteq A_2 \subseteq \cdots$ of finite sets of integers such that for any positive integer k:

(i) $r_{A_k}(n) \leq f(n)$ for all $n \in \mathbb{Z}$;
(ii) $r_{A_{2k-2}}(n), r_{A_{2k-1}}(n) \geq \sharp\{i \leq k : u_i = n\}$ for all $n \in \mathbb{Z}$;
(iii) $0 \notin A_k$.

If $u_1 \geq 0$, take $c = d_0 > 0$. If $u_1 < 0$, take $c = -d_0 < 0$. Let

$$A_1 = \{-c, c + u_1\}.$$

Then $2A_1 = \{-2c, u_1, 2c + 2u_1\}$ and

$$r_{A_1}(n) = \begin{cases} 1 & \text{if } n \in 2A_1, \\ 0 & \text{if } n \notin 2A_1. \end{cases}$$

If $n \in 2A_1 \setminus \{u_1\}$ then $|n| \geq d_0$, so $f(n) \geq 1$. And if $n = u_1$, then by the definition of $\{u_k\}$, we have $f(u_1) = \sharp\{k : u_k = u_1\} \geq 1$. Thus, for all $n \in 2A_1$, $r_{A_1}(n) = 1 \leq f(n)$. If $n \notin 2A_1$, $r_{A_1}(n) = 0 \leq f(n)$. Hence, for all $n \in \mathbb{Z}$, $r_{A_1}(n) \leq f(n)$. And, we have $1 = r_{A_1}(u_1) \geq \sharp\{i \leq 1 : u_i = u_1\}$. For other $n \neq u_1$, $r_{A_1}(n) \geq \sharp\{i \leq 1 : u_i = n\} = 0$. Thus $r_{A_1}(n) \geq \sharp\{i \leq 1 : u_i = n\}$ for all n. Therefore, A_1 satisfies all of the conditions (i)–(iii) above.

By Lemma 2.2 there exists a finite set A_{2k} of integers such that $A_{2k-1} \subseteq A_{2k}$, with $r_{A_{2k}}(n) \leq f(n)$ for all $n \in \mathbb{Z}$, and

$$r_{A_{2k}}(n) \geq \sharp\{i \leq k + 1 : u_i = n\}$$

for all n, and $0 \notin A_{2k}$. Then A_{2k} satisfies (i), (ii), (iii) and $A_{2k-1} \subseteq A_{2k}$.

For $k = 1, 2, \ldots,$ let $x_k = \max\{|a| : a \in A_{2k}\}$, and let p_k denote the least prime greater than $4x_k^2$. Thus we have $x_1 = 1$ and $p_1 = 5$. Suppose that we have $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_{2k}$. Let

$$A_{2k+1} = A_{2k} \cup (B_{p_k} + p_k^2 + 2x_k).$$

Then $2A_{2k+1}$ has three parts:

$$2A_{2k}, \quad A_{2k} + B_{p_k} + p_k^2 + 2x_k, \quad 2B_{p_k} + 2p_k^2 + 4x_k.$$

It is easy to see that these sets are pairwise disjoint.

We now prove that $A_{2k} + B_{p_k} + p_k^2 + 2x_k$ is also a Sidon set. For if there exist $(a_1, b_1), (a_2, b_2) \in A_{2k} \times B_{p_k}$ such that $a_1 + b_1 = a_2 + b_2$, then $a_1 - a_2 = b_2 - b_1$. Note that $a_1 - a_2 \in [-2x_k, 2x_k] \subseteq [-\sqrt{p_k}, \sqrt{p_k}\,]$ and $b_2 - b_1 \in B_{p_k} - B_{p_k}$, by Lemma 2.3(ii), we have $(B_{p_k} - B_{p_k}) \cap [-\sqrt{p_k}, \sqrt{p_k}\,] = \{0\}$. Thus $a_1 + b_1 = a_2 + b_2$ if and only if $(a_1, b_1) = (a_2, b_2)$, hence $A_{2k} + B_{p_k} + p_k^2 + 2x_k$ is a Sidon set.

Therefore, we have $r_{A_{2k+1}}(n) \leq f(n)$ for all $n \in \mathbb{Z}$.

Also, $r_{A_{2k+1}}(n) \geq r_{A_{2k}}(n) \geq \sharp\{i \leq k + 1 : u_i = n\}$ for all n. Thus, A_{2k+1} satisfies (i), (ii) and (iii).

Now, let $A = \bigcup_{l=1}^{\infty} A_k$. By conditions (i) and (ii), we have $r_A(n) = f(n)$ for all n and, by Lemma 2.3,

$$\begin{aligned}
\limsup_{x\to\infty} \frac{A(-x, x)}{\sqrt{x}} &\geq \limsup_{k\to\infty} \frac{A(1, 2p_k^2 - p_k + x_k)}{\sqrt{2p_k^2 - p_k + x_k}} \\
&\geq \limsup_{k\to\infty} \frac{|B_{p_k}|}{\sqrt{2p_k^2 - p_k + x_k}} \\
&\geq \limsup_{k\to\infty} \frac{p_k - 2\sqrt{p_k}}{\sqrt{2p_k^2 - p_k + x_k}} \\
&= \frac{1}{\sqrt{2}}.
\end{aligned}$$

This completes the proof of the theorem.

https://doi.org/10.1017/S0004972711002358 Published online by Cambridge University Press

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https://doi.org/10.1017/S0004972711002358 Published online by Cambridge University Press